CHAPTER VI

FREE CONVECTIVE HEAT TRANSFER IN A VISCOUS INCOMPRESSIBLE FLUID BETWEEN A UNIFORMLY MOVING LONG VERTICAL WAVY WALL AND A PARALLEL FLAT WALL.
6.1. INTRODUCTION:

The free convective flow of a viscous fluid over a uniformly moving long vertical wavy wall is a problem which is very worth mentioning here in recent years. The eyes of many research scholars have been attracted to this very problem, in view of its applications in transpiration cooling of re-entry vehicles and rocket boosters, cross-hatching on ablative surfaces and film vaporization in combustion chambers.

In view of these applications, a linear analysis of compressible boundary layer flows over a wavy wall has been discussed by Lekoudis, Nayfeh and Saric (1976). The Rayleigh problem for a wavy wall has been studied in detail by Shankar and Sinha (1976). They have been drawn some interesting conclusions, namely that at low Reynolds numbers the waviness of the wall quickly ceases to be of importance as the liquid is dragged along by the wall, while at large Reynolds numbers the effects of viscosity are confined to a thin layer close to the wall. An interesting effects of small amplitude wall waviness upon the stability of the laminar boundary layer has been discussed by Lessen and Gangwani (1976). The object of the present authors is to extend the problem studied by Vajravelu and Sastri (1978) to the case when the wavy wall moves uniformly. It is found that the plate velocity has significant effect on the flow and heat transfer.

6.2. BASIC EQUATIONS:

Let us consider the two-dimensional steady laminar free convective flow along the vertical channel as shown in Flg. -1, in which the $x$ - axis is taken parallel to the flat wall while the $y$ - axis perpendicular to it in such a way that their respective equations are $\bar{y} = \bar{x} \cos \bar{kx}$ and $\bar{y} = d$ at constant temperatures $\bar{T}_w$ and $\bar{T}_1$, respectively.

In view of these, we consider that:

1. All the fluid properties except density in the buoyancy force term are constant.
2. The viscous dissipative effects and work done by the pressure are neglected.
in the energy equation.

3. The flow is laminar, steady and two-dimensional.

4. The volumetric heat source/skin term in the energy equation is constant.

6. The wave length of the wavy wall, which is proportional to $K^1$, is large.

Therefore, the equations governing the flow and heat transfer are:

$$
\rho \left( \frac{\partial \bar{u}}{\partial x} + v \frac{\partial \bar{u}}{\partial y} \right) = -\frac{\partial \bar{p}}{\partial x} + \mu \left( \frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} \right),
$$

$$
\rho \left( \frac{\partial \bar{v}}{\partial x} + v \frac{\partial \bar{v}}{\partial y} \right) = -\frac{\partial \bar{p}}{\partial y} + \mu \left( \frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2} \right),
$$

where $v = \mu / \rho$ is the kinematic viscosity

$$
\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0,
$$

$$
\rho C_p \left( \frac{\partial \bar{T}}{\partial x} + v \frac{\partial \bar{T}}{\partial y} \right) = \kappa \left( \frac{\partial^2 \bar{T}}{\partial x^2} + \frac{\partial^2 \bar{T}}{\partial y^2} \right) + Q,
$$

where $\bar{u}, \bar{v}$ are velocity components, $\bar{p}$ fluid pressure, $\rho g$ the buoyancy force, $Q$ the constant heat addition/absorption and the other symbols have their usual meanings.

The appropriate boundary conditions for the present problem are:

$$
\bar{y} = \varepsilon \cos Kx : \bar{u} = \bar{u}_0, \bar{v} = 0, \bar{T} = \bar{T}_w
$$

$$
\bar{y} = d : \bar{u} = 0, \bar{v} = 0, \bar{T} = \bar{T}_1
$$

Let us consider the non-dimensional quantities defined as below:

$$x = \frac{\bar{x}}{d}, \quad y = \frac{\bar{y}}{d}, \quad u = \frac{\bar{u} d}{\nu}, \quad v = \frac{\bar{v} d}{\nu}, \quad p = \frac{\bar{p}}{\rho (\nu / d)^2}, \quad \theta = (\bar{T} - \bar{T}_S) / (\bar{T}_w - \bar{T}_S),$$
\[ \xi = \frac{u_0 d}{v}, \]
\[ m = \frac{\overline{T}_1 - \overline{T}_s}{(\overline{T}_w - \overline{T}_s)}, \]
\[ \alpha = \frac{Qd^2}{\kappa (\overline{T}_w - \overline{T}_s)}, \]
\[ P = \frac{\mu C_p}{\kappa}, \]
\[ \varepsilon = \frac{\varepsilon}{d}, \]
\[ \lambda = \frac{K d}{d}, \]
\[ G = \frac{d^2 \beta (\overline{T}_w - \overline{T}_s)}{\nu^2}, \]

where the subscript 's' denote quantities in the static fluid condition.

Therefore, the equations (6.2.1) to (6.2.4) reduce to:

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{gd^3}{v^2} \]  
\[ (6.2.6) \]

\[ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \]  
\[ (6.2.7) \]

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]  
\[ (6.2.8) \]

\[ P \left( u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} \right) = \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \alpha \]  
\[ (6.2.9) \]

The non-dimensional boundary conditions are:

\[ y = \varepsilon \cos \lambda x : u = \xi, v = 0, \theta = 1 \]  
\[ y = 1 : u = 0, v = 0, \theta = m \]  
\[ (6.2.10) \]

Using the well known Boussinesq approximation

\[ (\rho - \rho_s) / \rho = -\beta (T_w - T_s) \theta \]
On the static fluid, the equation (6.2.6) becomes

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{\partial}{\partial x} (p - p_s) + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + G \theta$$  \hspace{1cm} (6.2.11)

6.3. METHOD OF SOLUTION:

To solve the equations (6.2.11) and (6.2.7) to (6.2.9), we assume that the flow field and the temperature field are as follows:

$$u(x, y) = u_0(y) + u_1(x, y), \quad v(x, y) = v_1(x, y)$$

$$p(x, y) = p_0(x) + p_1(x, y), \quad \theta(x, y) = \theta_0(y) + \theta_1(x, y)$$  \hspace{1cm} (6.3.1)

where the perturbation parts $u_1, v_1, p_1$ and $\theta_1$ are small compared with the mean or the zeroth-order quantities.

On using (6.3.1) into the equations (6.2.11) and (6.2.7) to (6.2.9), we have:

For zeroth order:

$$\frac{d^2 u}{dy^2} + G \theta_0 = C, \quad \frac{d^2 \theta_0}{dy^2} = -\alpha$$  \hspace{1cm} (6.3.2)

and for the first order:

$$u_0 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} = - \frac{\partial p_1}{\partial x} + \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} + G \theta_1$$  \hspace{1cm} (6.3.3)

$$u_0 \frac{\partial v_1}{\partial x} = - \frac{\partial p_1}{\partial y} + \frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2}$$  \hspace{1cm} (6.3.4)

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial x} = 0$$  \hspace{1cm} (6.3.5)

$$P \left( u_0 \frac{\partial \theta_0}{\partial x} + v_1 \frac{\partial \theta_0}{\partial y} \right) = \frac{\partial^2 \theta_1}{\partial x^2} + \frac{\partial^2 \theta_1}{\partial y^2}$$  \hspace{1cm} (6.3.6)

where the constant pressure gradient term $C = \frac{\partial}{\partial x} (p_0 - p_s)$ has been taken equal to zero by Ostrach (1952).
In view of (6.3.1), the boundary conditions (6.2.10) can be divided into two groups:

For zeroth order,
\[
y = 0: \quad u_0 = \xi, \quad \theta_0 = 1 \quad \text{and} \quad y = 1: \quad u_0 = 0, \quad \theta_0 = m
\]
and for the first order,
\[
y = 0: \quad u_1 = -\text{Re} \{ \varepsilon u_0' e^{i\lambda x} \}, \quad v_1 = 0, \quad \theta_1 = -\text{Re} \{ \varepsilon \theta_0' e^{i\lambda x} \}
\]
\[
y = 1: \quad u_1 = 0, \quad v_1 = 0, \quad \theta_1 = 0
\]
where the prime denotes differentiation w.r.t. \( y \).

### 6.4. ZEROTh-ORDER SOLUTION (MEAN PART):

The solutions of the equations (6.3.2) subject to the boundary conditions (6.3.7) are:
\[
\theta_0 = 1 + A_1 y - \frac{1}{2} \alpha y^2
\]
\[
u_0 = \xi + A_2 y - \frac{1}{2} G y^2 - \frac{1}{6} G A_1 y^3 - \frac{1}{24} G \alpha y^4
\]

### 6.5. FIRST-ORDER SOLUTION (PERTURBED PART):

To solve the equations (6.3.3) to (6.3.6) for the first order quantities, we introduce the stream function \( \overline{\Psi}_1 \), defined by
\[
u_i = -\overline{\Psi}_1, \quad v_i = \overline{\Psi}_1, x
\]
Now, eliminating \( p_1 \) from (6.3.3) and (6.3.4) and using (6.5.1), we have
\[
u_0 (\overline{\Psi}_1, xxx + \overline{\Psi}_1, yxy) - u_0' \overline{\Psi}_1, x = 2 \overline{\Psi}_1, xxyy + \overline{\Psi}_1, xxx + \overline{\Psi}_1, yyy - G \theta_1, y
\]
and the equation (9.3.6) becomes
\[
P (u_0 \theta_1, x + \overline{\Psi}_1, x \theta_0') = \theta_1, xx + \theta_1, yy
\]
Again, we assume
\[ \Psi_1(x, y) = e^{i\lambda x} \Psi(y), \quad \theta_1(x, y) = e^{i\lambda x} t(y) \] (6.5.4)

In view of (6.5.4), the equations (6.5.2) and (6.5.3) reduce to

\[ \Psi^{IV} - \Psi''(2\lambda^2 + i\lambda u_0) + \Psi(\lambda^4 + i\lambda u_0'' + iu_0\lambda^3) = G\, t' \] (6.5.5)
\[ t'' - t(\lambda^2 + u_0 - P\lambda) = i\, P\lambda\, \Psi\, \theta' \] (6.5.6)

and the boundary conditions (6.3.8) become

\[
\begin{align*}
y = 0 & : \quad \Psi' = u_0', \quad \Psi = 0, \quad t = -\theta_0' \\
y = 1 & : \quad \Psi' = 0, \quad \Psi = 0, \quad t = 0
\end{align*}
\] (6.5.7)

If we consider only small values of \( \lambda \) (or \( K \ll 1 \)), then substituting

\[ \Psi(\lambda, y) = \sum_{i=0}^{2} \lambda^i \Psi_i, \quad t(\lambda, y) = \sum_{i=0}^{2} \lambda^i t_i \]

into (6.5.5) to (6.5.7) gives, to the order of \( \lambda^2 \), the following sets of ordinary differential equations and corresponding boundary conditions:

\[
\begin{align*}
\Psi_0^{IV} &= G\, t_0', \quad t_0'' = 0 \quad \text{(6.5.8)} \\
\Psi_1^{IV} &= i(u_0\Psi_0'' - \Psi_0 u_0''') + G\, t_1' \quad \text{(6.5.9)} \\
t_1'' &= i\, P(u_0 t_0 + \Psi_0 \theta_0) \quad \text{(6.5.10)} \\
\Psi_2^{IV} &= 2\Psi_0''' + i(u_0\Psi_1'' - \Psi_1 u_0''') + G\, t_2' \quad \text{(6.5.11)} \\
t_2'' &= t_0 + i\, P(u_0 t_1 + \Psi_1 \theta_0') \quad \text{(6.5.12)}
\end{align*}
\]

and

\[
\begin{align*}
y = 0 & : \quad \Psi'_0 = u'_0, \quad \Psi_0 = 0, \quad t_0 = -\theta_0' \\
y = 1 & : \quad \Psi'_0 = 0, \quad \Psi_0 = 0, \quad t_0 = 0 \quad \text{(6.5.13)}
\end{align*}
\]

\[
\begin{align*}
y = 0 & : \quad \Psi'_1 = 0, \quad \Psi_1 = 0, \quad t_1 = 0 \quad \text{for } i \geq 1 \quad \text{(6.5.14)}
\end{align*}
\]

\[
\begin{align*}
y = 1 & : \quad \Psi'_1 = 0, \quad \Psi_1 = 0, \quad t_1 = 0 \quad \text{(6.5.14)}
\end{align*}
\]
The solutions of the equations (6.5.8) to (6.5.12) subject to the boundary conditions (6.5.13) and (6.5.14) are:

\[ t_0 = -A_1 + (A_1 + \alpha) y - \alpha y^2 \] (6.5.13)

\[ \Psi_0 = A_2 y - \frac{1}{2} GA_1 y^2 + A_3 y^3 - \frac{1}{24} GA_1 y^4 + A_4 y^5 - \frac{1}{160} \alpha G y^6 \] (6.5.14)

\[ t_1 = i P \left[ A_{12} - \frac{1}{2} \xi A_1 y^2 + \frac{1}{6} A_6 y^3 + \frac{1}{12} A_7 y^4 + \frac{1}{20} A_8 y^5 \right. \]
\[ + \frac{1}{30} A_9 y^6 + \frac{1}{42} A_{10} y^7 + \frac{1}{50} A_{11} y^8 + \frac{1}{160} \alpha^2 G y^9 \] \[ \left. \right] \] (6.5.15)

\[ \Psi_1 = i \left[ -A_{22} y^3 - \frac{1}{24} (PA_{12} - A_1 \xi) Gy^4 + \frac{1}{120} A_{14} y^5 \right. \]
\[ + \frac{1}{360} A_{15} y^6 + \frac{1}{840} A_{16} y^7 + \frac{1}{1680} A_{17} y^8 + \frac{1}{3024} A_{18} y^9 \]
\[ + \frac{1}{5040} A_{19} y^{10} + \frac{1}{7920} A_{20} y^{11} + \frac{1}{11880} A_{21} y^{12} + By^{13} \] \[ \right] \] (6.5.16)

\[ t_2 = A_{37} y + \frac{1}{2} A_{23} y^2 + \frac{1}{6} A_{24} y^3 + \frac{1}{12} A_{25} y^4 + \frac{1}{20} A_{26} y^5 \]
\[ + \frac{1}{30} A_{27} y^6 + \frac{1}{42} A_{28} y^7 + \frac{1}{56} A_{29} y^8 + \frac{1}{72} A_{30} y^9 \]
\[ + \frac{1}{90} A_{31} y^{10} + \frac{1}{110} A_{32} y^{11} + \frac{1}{132} A_{33} y^{12} + \frac{1}{156} A_{34} y^{13} \]
\[ + \frac{1}{192} A_{35} y^{14} + \frac{1}{216} A_{36} y^{15} + \frac{1}{480} B \alpha y^{16} \] \[ \right] \] (6.5.17)

\[ \Psi_2 = A_{45} y^3 + \frac{A_{38}}{24} y^4 + \frac{A_{39}}{120} y^5 + \frac{A_{40}}{360} y^6 + \frac{A_{41}}{840} y^7 \]
\[ + \frac{A_{42}}{1680} y^8 + \frac{A_{43}}{3024} y^9 + \frac{A_{44}}{5040} y^{10} + \frac{A_{45}}{7920} y^{11} + \frac{A_{46}}{11880} y^{12} \]
\[ + \frac{A_{47}}{17160} y^{13} + \frac{A_{48}}{24024} y^{14} + \frac{A_{49}}{32760} y^{15} + \frac{A_{50}}{43680} y^{16} + \frac{A_{51}}{57120} y^{17} \]
\[ + \frac{A_{52}}{73440} y^{18} + \frac{A_{53}}{93024} y^{19} \] \[ \right] \] (6.5.18)

In view of these solutions (6.5.13) to (6.5.18), the first order quantities can be written as:
\[ u_1 = \varepsilon [\Psi'_1 \sin \lambda x - \Psi'_2 \cos \lambda x] \] (6.5.19)

\[ v_1 = -\varepsilon \lambda [\Psi_1 \sin \lambda x + \Psi_2 \cos \lambda x] \] (6.5.20)

\[ \theta_1 = \varepsilon [t_1 \cos \lambda x - t_1 \sin \lambda x] \] (6.5.21)

where \( \Psi = \Psi_r + i \Psi_i \), \( t = t_r + i t_i \), \( \Psi_r = \text{Re} (\Psi) = \Psi_0 + \lambda^2 \Psi_2 \),

\[ \Psi_i = \lambda \text{Im} (\Psi_i), \quad t_r = \text{Re} (t) = t_0 + \lambda^2 t_2, \quad t_i = \lambda \text{Im} (t_i) \]

Due to waviness of the wall, the expressions for \( u_1 \), \( \theta_1 \) and \( v_1 \) are called the first order solutions or the disturbed parts.

6.6. SKIN-FRICTION AND HEAT TRANSFER COEFFICIENT AT THE WALLS:

The non-dimensional shearing stress \( \tau_{xy} \) at any point in the fluid is

\[ \tau_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = u'_1(y) + \varepsilon e^{i\lambda x} [u''_1(y) + i\lambda v'_1(y)] \] (6.6.1)

where \( u_1(y) \) and \( v_1(y) \) are given by

\[ u_1(x,y) = \varepsilon e^{i\lambda x} u_1(y) \quad \text{and} \quad v_1(x,y) = \varepsilon e^{i\lambda x} v_1(y). \]

At the wavy wall \( y = \varepsilon \cos \lambda x \), \( \tau_{xy} \) becomes

\[ \tau_w = \tau^0_0 + \varepsilon \cos \lambda x [u''_0(0) + u'_1(0)] \] (6.6.2)

and at the flat wall \( y = 1 \), \( \tau_{xy} \) takes the form

\[ \tau_1 = \tau^0_0 + \varepsilon u_1(1) \cos \lambda x \] (6.6.3)

where \( \tau^0_0 = u'_0(0) \) and \( \tau^0_0 = u'_0(1) \) are the zeroth-order skin-frictions at the walls and

\[ \overline{u_1} = \Psi_1 \tan \lambda x - \Psi'_2 \].

Again, the non-dimensional Nusselt number \( Nu \) is given by

\[ Nu = \frac{\partial \theta}{\partial y} = \theta'_0(y) + \varepsilon e^{i\lambda x} t'(y) \] (6.6.4)
At the wavy wall $y = \varepsilon \cos \lambda x$, $Nu$ becomes

$$Nu_w = Nu_0^0 + \varepsilon \cos \lambda x \left[ \theta''_0(0) + \tau'(0) \right]$$
$$= Nu_0^0 + \varepsilon \left[ \theta''_0(0) + \tau'(0) \right] \cos \lambda x - \varepsilon \mu \tau'(0) \sin \lambda x$$

and at the flat wall $y = 1$, $Nu$ becomes

$$Nu_1 = Nu_1^0 + \tau'(1) \varepsilon \cos \lambda x ,$$
$$= Nu_1^0 + \varepsilon \tau'(1) \varepsilon \cos \lambda x - \varepsilon \mu \tau'(1) \sin \lambda x$$

where $Nu_0^0 = \theta'_0(0)$ and $Nu_1^0 = \theta'_0(1)$ are the zeroth-order Nusselt numbers at the walls.

6. 7. PRESSURE DROP:

With the help of $p_0(x) = \text{constant}$ and $p(x,y) = p_0(x) + p_1(x,y)$, the fluid pressure $p$ at any point $(x,y)$ is given by

$$p(x,y) = \int dp = \int \left[ \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy \right]$$

On using the equations (6.3.3) and (6.3.4), $p(x,y)$ becomes

$$p(x,y) = Re \left[ i \left( \frac{\varepsilon}{\lambda} \right) e^{i \lambda x} Z(y) \right] + L , \quad (6.7.1)$$

where $L$ is a parameter and

$$Z(y) = (\Psi'' - \lambda^2 \Psi') - i\lambda \left( u_0 \Psi' - u_0' \Psi \right) - Gt$$

The pressure drop (Pd) is the difference between the pressure at any point $y$ in the flow field and that at the flat wall, with $x$ fixed.

Therefore,

$$Pd = p(x,y) - p(x,1) = (\varepsilon / \lambda) \ Re \left[ i e^{i \lambda x} \left\{ Z(y) - Z(1) \right\} \right] \quad (6.7.2)$$

The pressure drop $Pd$ at $\lambda x = 0$ and $\lambda x = \frac{\pi}{2}$ are denoted by $P_{d_0}$ and $P_{d_1}$.
respectively. Some interesting profiles of $P_d$ have been presented here.

6.8. DISCUSSION OF THE RESULTS:

The behaviour of $u_0$ has been presented in Fig.2 for several values of $\alpha$, $m$, and $\xi$. From this figure it is seen that $u_0$ increases as $\alpha(I, II, III)$ and $m(I, VI, VII)$ decreases, while it increases as $\xi$ (velocity of the wavy wall).

It is observed from Fig.3 that the first order velocity $u_1$ increases as $\alpha(I, II, III)$, whereas it increases when $m(I, VI, VII)$ and $\xi(I, VI, VII, VIII)$. From the same figure it is seen that the effects of the parameters $\alpha$, $m$, and $\xi$ on $u_1$ are negligible for $y < 0.5$. The graph II of Fig.3 indicates that for $\alpha = -1, G = 5, P = 0.7, \lambda = 0.5, \epsilon = 0.01$ and $\lambda x = 0$ indicates that $u_1$ sharply decreases as $y$ increases for $y < 0.5$, whereas it increases as $y$ for $0.5 < y < 1.2$ (nearly) and then again $u_1$ decreases as $y$ increases for $y > 1.2$. Further, it is seen from Fig.3 that $u_1$ becomes negative almost throughout the channel except for a very small region near the wavy wall.

Figure 4 shows that $v_1$ decreases as $m(I, IV, V)$, $\xi(I, VI, VII)$ and $\alpha(I, II, III)$ increase. But from the same figure it is observed that the effects of the parameters $\alpha$, $m$, and $\xi$ are negligible on $v_1$ for small values of $y$ and it is also seen that $v_1$ decreases as $y$ increases for large values of $y$. Also, the normal velocity $v_1$ is negative almost throughout the channel, that means the fluid towards the wavy wall.

From Fig.5 it is seen that $\tau_0^0$ increases as $m(I, II, III, IV)$ and $\xi(I, V, VI, VII)$. However, it is observed that $\tau_0^0$ decreases steadily as $\alpha$ increases. Fig.6 indicates that $\tau_w$ increases as $m(I, II, III, IV)$, whereas it decreases as $\xi(I, V, VI, VII)$ increases. From Fig.7 it is clear that $Nu_0^0$ increases as $m(I, II, III, IV)$ and $\xi(I, V, VI, VII)$ for $y > 0.7$. It is observed from Fig.8 that $Nu_w$ decreases as $\xi(I, V, VI, VII)$ for $y < 1.2$ and $P(I, VIII, IX)$ increase, while $Nu_w$ increases as $m(I, II, III, IV)$.

Figure 9 shows that $\tau_1^0$ increases as $\alpha$ and $m(I, II, III)$, but it decreases as $\xi(I, IV, V, VI)$ increases. Fig.10 indicates that $\tau_1$ increases as $m(I, II, III)$ decreases,
whereas it decreases as $\zeta(I,IV,V,VI)$ increases. It is seen from Fig. 11 that $N_u^0$ decreases as $\zeta(I,IV,V)$, while it increases as $m(I,II,III)$. Further, it is observed from Fig. 11 that $N_u^0$ decreases for large values of $\alpha$. From Fig. 12 we conclude that $N_u$ increases as $m(I,II,III)$ and $\zeta(I,IV,V)$, but it decreases when $P(I,VI,VI)$ is increased.

Finally, the behaviour of the pressure drops $p_{d_0}$ and $p_{d_1}$ at $\lambda x = 0$ and $\lambda x = \pi/2$ have been presented in Figures 13 and 14 against $y$ as abscissa. It is seen from Fig. 13 that $p_{d_0}$ decreases as $\alpha(I,II,III)$, $m(I,IV,V,VI)$ and $\zeta(I,II,III)$ increase, while $p_{d_0}$ is increased steadily for small values of $y$. We observed from Fig. 14 that $p_{d_1}$ increases as $m(I,II,III)$, whereas it decreases as $\zeta(I,IV,V,VI)$ increases for $y < 0.5$. Also, it is seen that $p_{d_1}$ decreases as $p$ and $y$ for $y > 1.2$ (nearly).
FIG. 1. FLOW CONFIGURATION
FIG. 2: Zeroth order velocity profile $u_0$ against $y$ for the parameters:

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</tr>
<tr>
<td>$\xi$</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>15</td>
<td>20</td>
<td>25</td>
<td></td>
</tr>
</tbody>
</table>

For $G = 5$, $P = 0.7$, $\lambda = 0.5$, $\varepsilon = 0.01$
FIG. 3: First order velocity profile $u_1$ against $y$ for the parameters:

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>1</td>
<td>-1</td>
<td>-5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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</tr>
<tr>
<td>$m$</td>
<td>.5</td>
<td>.5</td>
<td>.5</td>
<td>1</td>
<td>1.5</td>
<td>.5</td>
<td>.5</td>
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</tr>
<tr>
<td>$\xi$</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>15</td>
<td>20</td>
<td>25</td>
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</tbody>
</table>

For $G = 5$, $P = 0.7$, $\varepsilon = 0.01$, $\lambda x = 0$
FIG. 4: First order velocity profile $v_1$ against $y$ for the parameters:

<table>
<thead>
<tr>
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<th>I</th>
<th>II</th>
<th>III</th>
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<th>V</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
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</thead>
<tbody>
<tr>
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<td>-5</td>
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<td>1</td>
<td>1</td>
<td>1</td>
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<td>.5</td>
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<td>.5</td>
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<tr>
<td>$\xi$</td>
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<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>15</td>
<td>20</td>
<td>25</td>
</tr>
</tbody>
</table>

For $G = 5$, $\lambda = 0.7$, $\lambda_x = 0.5$, $\varepsilon = 0.01$, $\lambda x = 0$
Zeroth order Skin friction $\tau_{0}^0$ at $y=0$ against $\alpha$ for the parameters:

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
</tr>
</thead>
<tbody>
<tr>
<td>m</td>
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<td>.25</td>
<td>-2</td>
<td>-8</td>
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<td>.5</td>
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<tr>
<td>$\xi$</td>
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<td>10</td>
<td>10</td>
<td>10</td>
<td>15</td>
<td>20</td>
<td>25</td>
</tr>
</tbody>
</table>

For $G=5$, $P=5$, $\lambda=.5$, $\varepsilon=.01$, $\lambda x=0$
First order skin friction $\tau_0$ at $Y=0$ against $\alpha$ for the parameters.

For $G=5$, $P=7$, $\lambda=5$, $\beta=0.1$, $\lambda=0$.

Fig. 6
Zeroth order nusselt number $N_u_0^0$ at $y=0$ against $\alpha$ for the parameters:

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
</tr>
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<tbody>
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<tr>
<td>$\xi$</td>
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<td>10</td>
<td>10</td>
<td>10</td>
<td>20</td>
<td>25</td>
</tr>
</tbody>
</table>

For $G=5$, $P=7$, $\lambda=.5$, $\varepsilon=.01$, $\lambda_x=0$. 

Fig - 7
First order nusselt number $N_u_{xy}$ at $y=0$ against $\alpha$ for the parameters:

<table>
<thead>
<tr>
<th>$m$</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
<th>IX</th>
</tr>
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<tbody>
<tr>
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<td>-8</td>
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</tr>
<tr>
<td>$\xi$</td>
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<td>10</td>
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<td>25</td>
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<tr>
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<td>.7</td>
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<td>.7</td>
<td>.7</td>
<td>.7</td>
<td>4</td>
<td>7</td>
</tr>
</tbody>
</table>

Fig - 8
Zeroth order skin friction $\tau_1^0$ at $y=0$ against for the parameters:

<table>
<thead>
<tr>
<th>$m$</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi$</td>
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<td>.2</td>
<td>.8</td>
<td>.5</td>
<td>.5</td>
<td>.5</td>
</tr>
</tbody>
</table>

For $G=5$, $P=.7$, $\lambda=.5$, $\varepsilon=.01$, $\lambda x=0$

Fig - 9
First order skin friction $\tau$ at $y=1$ against $\alpha$ for the parameters:

For $G=5$, $P=7$, $\lambda=5$, $\xi=0.01$, $\lambda=0$.

Fig - 10
For $G=5$, $P=7$, $\lambda=5$, $\varepsilon=0.1$, $\lambda<0$.

Zeroth order Nusselt number $Nu_0$ at $y=1$ against $\alpha$ for the parameters:

- I
- II
- III
- IV
- V

$\mathrm{Re}=10, 15, 20$
For $G=5$, $\lambda=5$, $S=0.1$, $x=0$.

First order nusselt number $Nu$ at $y=1$ against $\alpha$ for the parameters:
- $x=10$, $10$, $10$, $10$, $10$, $10$, $10$.
- $m=5$, $5$, $5$, $5$, $5$, $5$, $5$.
- $P=7$, $7$, $7$, $7$, $7$, $7$, $7$.

Fig. 12
The pressure drop profile $P_d$ at $\lambda x = 0$ against $y$ for the parameters:

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
<th>IX</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
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<td>1</td>
<td>1</td>
<td>1</td>
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<td>$m$</td>
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<tr>
<td>$\xi$</td>
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<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>15</td>
<td>20</td>
<td>25</td>
<td></td>
</tr>
</tbody>
</table>

For $G=5$, $P=.7$, $\lambda=.5$, $\varepsilon=.01$, $\lambda x=0$
The pressure drop profile $P_d$ at $\lambda x = 0$ against $y$ for the parameters:

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
<th>VI</th>
<th>VII</th>
<th>VIII</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td>.5</td>
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<td>-8</td>
<td>.5</td>
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<td>.5</td>
<td>.5</td>
<td>.5</td>
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<tr>
<td>$\xi$</td>
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<td>10</td>
<td>10</td>
<td>15</td>
<td>20</td>
<td>25</td>
<td>10</td>
<td>10</td>
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<tr>
<td>$P$</td>
<td>.7</td>
<td>.7</td>
<td>.7</td>
<td>.7</td>
<td>.7</td>
<td>.7</td>
<td>4</td>
<td>7</td>
</tr>
</tbody>
</table>

For $G = 5$, $\lambda = 0.5$, $\varepsilon = 0.01$, $\lambda x = \pi/2$
APPENDIX:

\[ A_1 = m + \frac{1}{2} \alpha - 1, \quad A_2 = 1 + \frac{1}{2} G - \xi + \frac{1}{6} GA_1 - \frac{1}{4} G \alpha, \]

\[ A_3 = \frac{17}{60} GA_1 + \frac{5}{48} G \alpha - 1 - \frac{1}{2} G + \xi, \quad A_4 = \frac{1}{120} (A_1 + \alpha), \]

\[ A_5 = A_3 - \frac{G \alpha}{12}, \quad A_6 = (A_1 + \alpha) \xi + A_1 A_5, \]

\[ A_7 = (A_2 - \xi) \alpha + A_1 A_2 - A_5 \alpha - \frac{GA_1^2}{2}, \]

\[ A_8 = -\frac{GA_1}{2} - \frac{G \alpha}{2} - A_2 \alpha + A_1 A_3 + \frac{GA_1 \alpha}{2}, \]

\[ A_9 = -\frac{GA_1 \alpha}{6} - \frac{5 GA_1^2}{24} + \frac{G \alpha}{2} - A_3 \alpha, \]

\[ A_{10} = \frac{GA_1 \alpha}{24} + \frac{GA_1}{4} + A_1 A_4, \quad A_{11} = \frac{GA_1 \alpha}{160} - \frac{G \alpha^2}{4} + \alpha A_4, \]

\[ A_{12} = \frac{A_1 \xi A_6}{2} - A_7 - \frac{A_8}{12} - \frac{A_9}{20} - \frac{A_{10}}{6} - \frac{A_{11}}{56} - \frac{G \alpha^2}{160}, \]

\[ A_{13} = -G \xi A_1 + PGA_{12}, \quad A_{14} = -G A_1 A_2 + 6 \xi A_3 - GA_5, \]

\[ A_{15} = -\frac{GA_1 \xi}{2} - \frac{PGA_1}{2} - GA_1 A_5 + GA_1 + 6 A_2 A_3, \]

\[ A_{16} = 20 \xi A_4 - 4GA_3 + \frac{2G^2 A_1^2}{3} + \frac{G \alpha A_5}{2} - \frac{GA_1 A_2}{2} + \frac{PGA_6}{6}, \]

\[ A_{17} = 20 A_2 A_4 - 2GA_1 A_3 + \frac{7G^2 A_1}{24} - \frac{G^2 \alpha A_1}{24} - \frac{3G \alpha \xi}{16} + \frac{PGA_7}{6}, \]

\[ A_{18} = -11GA_4 + \frac{G^2 A_1^2}{8} + \frac{3 G \alpha A_3}{4} - \frac{3G \alpha A_2}{16} + \frac{PGA_8}{20}, \]

\[ A_{19} = \frac{G^2 \alpha}{10} + 13 G A_1 A_4 - \frac{G^2 \alpha A_1}{24} + \frac{PGA_9}{20}, \]
\[ A_{20} = \frac{3G^2 \alpha A_1}{80} + \frac{11\alpha A_4}{12} + \frac{PG}{42} A_{10}, \quad A_{21} = -\frac{G^2 \alpha^2}{640} + \frac{A_{11}}{56}, \]
\[ A_{22} = \frac{A_{13}}{24} + \frac{A_{14}}{120} + \frac{A_{15}}{360} + \frac{A_{16}}{840} + \frac{A_{17}}{1680} + \frac{A_{18}}{3024} \]
\[ + \frac{A_{19}}{5040} + \frac{A_{20}}{7920} + \frac{A_{21}}{11880} + \frac{A_{22}}{17160}, \]
\[ A_{23} = -A_1 - \xi P^2 A_{22}, \quad A_{24} = A_1 + \alpha - P^2 A_2 A_{22}, \]
\[ A_{25} = -\alpha + \frac{\xi^2 P^2 A_1}{2} + \frac{P^2 GA_{22}}{2}, \]
\[ A_{26} = -\frac{\xi^2 P^2 A_6}{6} + \frac{P^2 \xi A_4 A_2}{2} + P \left( \frac{G}{6} + 1 \right) PA_4 A_{22}, \]
\[ A_{27} = -P^2 \left[ \frac{\xi A_7}{12} + \frac{A_2 A_5}{6} + \frac{\xi GA_1}{4} - \frac{G \alpha A_{22}}{2} \right], \]
\[ A_{28} = -P^2 \left[ \frac{\xi A_8}{20} + \frac{A_2 A_7}{12} + \frac{GA_6}{6} - \frac{G \alpha \xi A_1^2}{4} \right], \]
\[ A_{29} = -P^2 \left[ \frac{\xi A_9}{30} + \frac{A_2 A_8}{20} + \frac{GA_7}{24} - \frac{GA A_6}{36} - \frac{G \alpha A_4}{48} \right], \]
\[ A_{30} = -P^2 \left[ \frac{\xi A_{10}}{42} + \frac{A_2 A_9}{30} + \frac{GA_8}{40} - \frac{GA A_7}{72} + \frac{G \alpha A_5}{144} \right], \]
\[ A_{31} = -P^2 \left[ \frac{\xi A_{11}}{56} + \frac{A_2 A_{10}}{42} + \frac{GA_9}{60} - \frac{GA A_{10}}{120} + \frac{G \alpha A_7}{288} \right], \]
\[ A_{32} = -P^2 \left[ \frac{\xi GA^2}{11520} + \frac{A_2 A_{11}}{56} + \frac{GA_{10}}{84} - \frac{GA A_9}{180} + \frac{G \alpha A_7}{480} \right]. \]
\[ A_{33} = -p^2 \left[ \frac{G^2 A_2}{11520} - \frac{G A_{11}}{112} - \frac{G A_{1A} A_{10}}{252} + \frac{G \alpha A_9}{720} \right], \]
\[ A_{34} = \frac{p^2 G^2 \alpha^2}{11520} - \frac{P A_1 A_{20}}{7920}, \quad A_{35} = -\frac{p^2 G \alpha A_{11}}{1344} - \frac{P A_1 A_{21}}{11880}, \]
\[ A_{36} = p \left[ \frac{G \alpha A_1}{23040} + \frac{A_{21}}{11880} \right], \]
\[ A_{37} = -\left[ \frac{A_{23}}{2} + \frac{A_{24}}{6} + \frac{A_{25}}{12} + \frac{A_{26}}{20} + \frac{A_{27}}{30} + \frac{A_{28}}{42} + \frac{A_{29}}{56} + \frac{A_{30}}{72} \right. \]
\[ +\left. \frac{A_{31}}{90} + \frac{A_{32}}{110} + \frac{A_{33}}{132} + \frac{A_{34}}{156} + \frac{A_{35}}{192} + \frac{A_{36}}{210} \right], \]
\[ A_{38} = G A_{31} - 2 G A_1, \quad A_{39} = 12 A_3 + 6 \xi A_{22} + G A_{23}, \]
\[ A_{40} = -G A_1 - 12 \xi A_{19} + 6 A_2 A_{23}, \]
\[ A_{41} = 40 A_4 - \frac{\xi A_{14}}{6} - \frac{G A_{25}}{12}, \quad A_{42} = -\frac{3 G \alpha}{8} - \frac{\xi A_{15}}{12} - \frac{A_2 A_{25}}{6} + \frac{G A_{26}}{4}, \]
\[ A_{43} = -\frac{\xi A_{16}}{20} - \frac{A_2 A_{15}}{12} - \frac{3 G A_{14}}{40} + \frac{G A_{27}}{5}, \]
\[ A_{44} = -\frac{\xi A_{17}}{30} - \frac{A_2 A_{16}}{20} + \frac{7 G A_{15}}{180} + \frac{G A_{28}}{6}, \]
\[ A_{45} = -\frac{\xi A_{18}}{42} - \frac{A_2 A_{17}}{30} + \frac{G A_{16}}{42} + \frac{G A_{29}}{7}, \]
\[ A_{46} = -\frac{\xi A_{13}}{24} - \frac{A_2 A_{18}}{42} + \frac{9 G A_{17}}{560} + \frac{G A_{30}}{8}, \]
\[ A_{47} = -\frac{\xi A_{20}}{72} - \frac{A_2 A_{19}}{56} + \frac{5 G A_{18}}{432} + \frac{G A_{31}}{9}, \]
\[ A_{48} = -\frac{\xi A_{21}}{90} - \frac{A_2 A_{20}}{72} + \frac{11 G A_{19}}{1260} + \frac{G A_{32}}{10}. \]
\[ A_{49} = -\frac{\xi A_{22}}{110} + \frac{GA_{33}}{11}, \quad A_{50} = -\frac{A_{22}A_2}{110} + \frac{GA_{34}}{14}, \]

\[ A_{51} = \frac{77GA_{22}}{17160} + \frac{GA_{35}}{13}, \quad A_{52} = \frac{5GA_1A_{22}}{3432} + \frac{GA_{35}}{14}, \]

\[ A_{53} = -\frac{3G\alpha A_{42}}{8580} + \frac{GP\alpha A_{22}}{257400}, \]

\[ A_{54} = -\frac{A_{38}}{24} - \frac{A_{39}}{120} - \frac{A_{40}}{360} - \frac{A_{41}}{840} - \frac{A_{42}}{1680} - \frac{A_{43}}{3024} - \frac{A_{44}}{5040} - \frac{A_{45}}{7920} - \frac{A_{46}}{11880}, \]

\[ B = \frac{G\alpha^2}{11520 \times 17160}, \]