In this thesis we are going to present some new results about groups of biholomorphic self-transformations, usually called automorphisms, of compact Riemann surfaces. The study of automorphisms of Riemann surfaces has acquired a new importance from its relation with problems of moduli and Teichmüller space [10, 29, 30, 49, 56, 57]. In the late nineteenth century, fundamental results were obtained by Hurwitz [34] who showed that the group of automorphisms of a compact Riemann surface of genus \( g \) is finite if \( g \geq 2 \). If \( N(g) \) is the order of the largest group of automorphisms that a Riemann surface of genus \( g \) can admit, Hurwitz proved that \( N(g) \leq 84\,(g-1) \). About the same time Wiman improved this bound for a cyclic group, by showing that the maximum possible order for an automorphism is \( 2(2g+1) \). Macbeath [43] showed that there are infinitely many values of \( g \) for which \( N(g) = 84(g-1) \). Maclachlan [52] obtained a sharp lower bound for the order of such a group. He proved that \( N(g) \geq 8g+8 \), with equality if \( g \) is of the form \( 89p + 1 \), where \( p \) is a prime number satisfying rather a large list of congruences.
An infinite family of $g$ is also found for each of which $N(g) = 8g+24$. These results have also been obtained by Accola [1].

It is known [15, 25] that every finite group can be represented as a group of automorphisms of a compact Riemann surface of some genus not less than $2$. The bounds of Hurwitz and others may be considered as particular cases of the following general problem: Given a finite group, what is the minimum genus of a surface for which this is a group of automorphisms? This problem has completely been solved for cyclic groups by Harvey [27, 28]. The case of a non-cyclic abelian group has been studied by Maclachlan [50, 51]. An explicit minimum value of $g$, in the general case, is obtained in terms of the prime decomposition of the order of the group. The approach used by him is based on the concept of Fuchsian groups, and is essentially equivalent to the combinatorial methods by Hurwitz and Wiman. A detailed account of the basic theory motivating it is found in the proceedings of Dundee Summer School given by Macbeath [43].

The upper bound $84(g-1)$ for the order of the group of automorphisms of a compact Riemann surface of genus $g$ is of course not attained for all values of $g \geq 2$. Macbeath[43] proved that for each integer $m$ there is an algebraic curve (equivalent to a compact Riemann surface) of genus $g = 2m^6+1$ with $84(g-1)$ birational self-transformations (the maximum
number possible). This was previously known [34] only for \( m = 1 \), i.e. \( g = 3 \), the curve being Klein's quartic
\[ x^3 y + y^3 z + z^3 x = 0. \]
The smallest value of \( g \) for which the upper bound \( 84(g-1) \) is attained is 3 and the corresponding Riemann surface is in fact equivalent to the famous quartic curve of genus 3 discovered by Klein [36], the automorphism group being the simple group of order 168 i.e. PSL(2,7).

A finite group of order \( 84(g-1) \) which occurs as an automorphism group of a compact Riemann surface of genus \( g \) is usually referred to as a maximal automorphism group or a Hurwitz group of genus \( g \). Since the publication of Hurwitz's result in 1892, mathematicians gave little attention to the problem of finding the other possible values of \( g \) for which there exist Hurwitz groups, until the early sixties of this century when Macbeath [43, 44, 45] revived the problem, and Lehner and Newman [39] followed him. They proved the existence of an infinite number of values of \( g \) for which there exists a Hurwitz group of genus \( g \). Macbeath [43, 46] further showed that many of the linear fractional groups are Hurwitz groups. Also Higman has shown in an unpublished note that a number of alternating groups, the smallest being \( A_{15} \), are Hurwitz groups.

From the famous Riemann-Hurwitz formula one can easily show that the first six possible orders, in order of
magnitude, of the automorphism group of a compact Riemann surface of genus $g$ are as follows:

$$
\begin{align*}
\lambda_1 &= 84(g-1), \\
\lambda_2 &= 48(g-1), \\
\lambda_3 &= 40(g-1), \\
\lambda_4 &= 36(g-1), \\
\lambda_5 &= 30(g-1), \\
\lambda_6 &= \frac{132}{5}(g-1).
\end{align*}
$$

Any finite group representable as a Riemann surface automorphism group of order $\lambda_n$, $n = 1, 2, \ldots, 6$, will be referred to as a large group of automorphisms of genus $g$ and $n$ will be called the mark of the group. These large groups of orders $\lambda_1, \ldots, \lambda_6$ will respectively be denoted by $M_1$-group, $\ldots$, $M_6$-group. Thus $M_1$-groups are the maximal automorphism groups or the Hurwitz groups, $M_2$-groups are the second maximal automorphism groups, $M_3$-groups are the third maximal automorphism groups, and so on.

The following basic theorem connecting Fuchsian groups with Riemann surface with some technical terms to be explained later in Chapter 2, is found in Macbeath [43]:

...
Any finite group $G$ acts as a group of automorphisms of some compact Riemann surface of genus $g \geq 2$, if and only if $G$ is isomorphic to the factor group $\Gamma/K$ where $\Gamma$ is a Fuchsian group with compact orbit space, and $K$ is a Fuchsian surface group with orbit genus $g$.

From above it can be shown that a given finite group $G$ is a group of automorphisms of a compact Riemann surface of genus $g$ if and only if there exists a Fuchsian group $\Gamma$ and a homomorphism $\phi$ of $\Gamma$ onto $G(\cong \Gamma/K)$ such that the kernel of $\phi$ is a Fuchsian surface group of orbit genus $g$. In this case the finite group $G$ is called a surface-kernel factor group. The above theorem may now be restated as follows:

A finite group $G$ is an automorphism group of a compact Riemann surface of genus $g \geq 2$ if and only if it is a surface-kernel factor group of a Fuchsian group $\Gamma$, the orbit genus of the kernel being $g$.

It is known [43] that a finite group $G$ will be a large group of mark 1,2,... or 6 if and only if it is a surface-kernel factor group of the triangle group $(2,3,7)$, $(2,3,8)$, $(2,4,5)$, $(2,3,9)$, $(2,3,10)$ or $(2,3,11)$ respectively.
Moreover it can be shown that a non-trivial finite group \( G \) is an \( M_1 \)-group, an \( M_2 \)-group, \ldots or an \( M_6 \)-group if and only if \( G \) is generated by two elements \( x \) and \( y \) satisfying respectively the relations

\[
\begin{align*}
 x^2 &= y^3 = (xy)^7 = 1, \\
 x^2 &= y^3 = (xy)^8 = 1, \\
 x^2 &= y^4 = (xy)^5 = 1, \\
 x^2 &= y^3 = (xy)^9 = 1, \\
 x^2 &= y^3 = (xy)^{10} = 1, \\
 x^2 &= y^3 = (xy)^{11} = 1.
\end{align*}
\] (1.1)

or

By the term "satisfying" used above, we mean "fulfilling" in Miller's sense. The term "satisfying" will be uniformly used in this sense throughout this thesis. There are three different ways of studying the existence of large groups of automorphisms of different marks. First, starting with any of the triangle groups \((2,3,7), (2,3,8), (2,4,5), (2,3,9), (2,3,10)\) and \((2,3,11)\), we may try to find its finite surface-kernel factor groups. Secondly, starting with a known finite group we may try to examine if it can be generated by two elements satisfying any one of the relations (1.1). Lastly, if a known finite group contains two elements \( x \) and \( y \) satisfying any one of the relations (1.1) we may try to ascertain what this group \( \langle x, y \rangle \) is. In Chapters 3 and 4 we follow the first and the second ways respectively while in Chapter 5 we follow the third way.
We have already mentioned that there are infinitely many values of $g$ for which there exists a Hurwitz group, i.e., an $M_1$-group of genus $g$. Although a number of $M_2$-groups occurring in completely different contexts [8,9,13,21,59,62] had been known for a long time, it was Chetiya [17,18] who made a systematic study of the problem of finding $M_2$-groups. He showed that many of the permutation groups, including some alternating groups, of degree $\leq 17$ are $M_2$-groups. He further proved the existence of an infinite number of soluble $M_2$-groups. In a recent paper Conder [60] has determined exactly which $S_n$ and $A_n$ are factor groups of the triangle group $(2,3,k)$, $k \geq 7$.

This is a summary of the results known so far about the large groups of automorphisms of compact Riemann surfaces. Let us now indicate what new results about this topic we are going to present in this thesis.

To facilitate easy reading of, and ready reference in the latter chapters we discuss in Chapter 2 in some detail the theory of Fuchsian groups and establish the relation between Fuchsian groups and automorphisms of Riemann surfaces.

In Chapter 3 we present a number of general theorems which prove the existence of an infinite number of infinite families of soluble groups of automorphisms of compact Riemann surfaces which are factor groups of certain triangle groups. From these results it follows immediately as special cases that there exist infinite number of soluble large groups of automorphisms of marks 2, 3, 4, and 5. To be precise, we prove the following theorems:

**THEOREM 3.1.1** - If (i) $p = 5$, $m = 2$ or (ii) $p$ is an odd prime,
\( p \not\mid m, m > p \), then for each positive integer \( n \) there is a soluble automorphism group \( G \) of order \( 2pm^{p-1}n^{2g} \) on a compact Riemann surface of genus \( \frac{1}{2} pm^{p-1}n^{2g}(1 - \frac{1}{m} - \frac{2}{p}) + 1 \) where \( g = \frac{1}{2} pm^{p-1}(1 - \frac{1}{m} - \frac{2}{p}) + 1 \) such that

\[
G \triangleright G' \triangleright G'' \triangleright G''' = \{1\}, \quad G''' = G''''
\]
only when \( n = 1 \).

**Corollary 3.1.1.1** - The triangle group \((2,5,4)\) i.e. \((2,4,5)\) has, for each positive integer \( n \), a soluble surface-kernel factor group of order \( 160n^{10} \) acting on a compact Riemann surface of genus \( 4n^{10} + 1 \).

**Corollary 3.1.1.2** (Chetiya [17]) - The triangle group \((2,3,8)\) has, for each positive integer \( n \), a soluble surface-kernel factor group of order \( 96n^6 \) acting on a compact Riemann surface of genus \( 2n^6 + 1 \).

**Corollary 3.1.1.3** - For each positive integer \( n \), the triangle group \((2,3,10)\) has a soluble surface-kernel factor group of order \( 150n^{12} \) acting on a compact Riemann surface of genus \( 5n^{12} + 1 \).

**Theorem 3.1.2** - Let \( m \) be an odd integer \( \geq 3 \). Then for every positive integer \( n \), the Fuchsian triangle group \((2,3,3m)\) has a soluble surface-kernel factor group of order \( 12m^3n^{2g} \) acting on a compact Riemann surface of genus \( m^2(m-2)n^{2g} + 1 \), where

\[
g = m^3 - 2m^2 + 1
\]
only when \( n = 1 \).

**Corollary 3.1.2.1** - For each positive integer \( n \), the triangle group \((2,3,9)\) has a soluble surface-kernel factor group of order \( 324n^{20} \) acting on a compact Riemann surface of genus \( 9n^{20} + 1 \).
In Chapter 4 we give the complete solution of the problem of separating all the projective linear groups, PGL(2, k) and PSL(2, k), where k is any finite field, into those which are large groups and those which are not. The following theorems are proved:

**THEOREM 4.4.1 (Macbeath [46])** No PGL(2, q), q \neq 2, is an $M_1$-group, and PSL(2, q) is an $M_1$-group if

(i) \( q = p = 7 \),

(ii) \( q = p \equiv \pm 1 \pmod{7} \),

(iii) \( q = p^3, p \equiv \pm 2, \pm 3 \pmod{7} \),

and for no other values of \( p \).

**THEOREM 4.4.2**

(a) PGL(2, q) is an $M_2$-group if

(i) \( q = p = (2m+1) 8 \pm 1 \),

(ii) \( q = p^2, p = (2m + 1) 8 \pm 3 \),

and for no other values of \( p \).

(b) PSL(2, q) is an $M_2$-group if

(i) \( q = p = 16m \pm 1 \),

(ii) \( q = p^2, p = 16m \pm 3 \),

and for no other values of \( p \).

**THEOREM 4.4.3**

(a) PGL(2, q) is an $M_3$-group if

(i) \( q = p = 5 \),

(ii) \( q = p \equiv \pm 11, \pm 19 \pmod{40} \),

and for no other values of \( p \);

(b) PSL(2, q) is an $M_3$-group if

(i) \( q = p \equiv \pm 1, \pm 9 \pmod{40} \),

(ii) \( q = p^2, p \) an odd prime \( \equiv \pm 2 \pmod{5} \),

and for no other values of \( p \).
THEOREM 4.4.4 No \( \text{PGL}(2,q), p \neq 2 \), is an \( M^- \)-group, and \( \text{PSL}(2,q) \) is an \( M^- \)-group if

1. \( q = p \equiv \pm 1 \pmod{9} \),
2. \( q = p^3, p \equiv \pm 2, \pm 4 \pmod{9} \),

and for no other values of \( p \).

THEOREM 4.4.5 (a) \( \text{PGL}(2,q) \) is an \( M^- \)-group if

1. \( q = p = (2m + 1) 10 + 1 \),
2. \( q = p^2, p = (2m + 1) 10 + 3 \),

and for no other values of \( p \);

(b) \( \text{PSL}(2,q) \) is an \( M^- \)-group if

1. \( q = p = 20m + 1 \),
2. \( q = p^2, p = 20m + 3 \),

and for no other values of \( p \).

THEOREM 4.4.6 No \( \text{PGL}(2,q), p \neq 2 \), is an \( M^- \)-group, and \( \text{PSL}(2,q) \) is an \( M^- \)-group if

1. \( q = p \equiv \pm 1 \pmod{11} \),
2. \( q = p^5, p \equiv \pm 2, \pm 3, \pm 4, \pm 5 \pmod{11} \),

and for no other values of \( p \).

The \( M^- \)-groups being homomorphic images of the triangle group \((2,3,7)\) are perfect. Since no non-trivial symmetric group is perfect an \( M^- \)-group cannot be symmetric. An \( M_2 \)-group and an \( M^- \)-group are homomorphic images of the triangle groups \((2,3,8)\) and \((2,4,5)\) respectively. Thus \( M^- \)-groups and \( M^- \)-groups are not necessarily perfect indicating the possibility that some of these may be representable as symmetric groups. It was proved by Chetiya [17] that no symmetric group of degree \( \leq 17 \) can be an
It is therefore worthwhile to examine if some symmetric groups of small degree occur as $M_3$-groups. This is being done in Chapter 5. We discover that symmetric groups $S_n$ do occur as $M_3$-groups for $n = 5, 11, 15, 16$. From Coxeter and Moser [21, p 140] we know that there is no $(2, 4, 5)$ group of genus 2. Also from Sherk [62] it follows that there is no $M_3$-group of genus 3. We shall see that $S_5$ is an $M_3$-group of genus 4. Hence the smallest value of $g$ for which there is an $M_3$-group of genus $g$ is 4 and the corresponding $M_3$-group is the symmetric group of degree 5. In our search for symmetric groups of small degree occurring as $M_3$-group we find as a by product that a number of known finite groups, viz. $A_6$, $G_{160}$ (a group of order 160 defined by $R^4 = S^5 = (RS)^2 = (R_1S)^4 = 1$), $\text{PGL}(2,11)\mathsmaller{2}\mathsmaller{\times} A_6$ are also $M_3$-groups. In this chapter for each group we find explicitly a pair of generators in terms of permutations. From this we envisage the exciting possibility of finding a new and simpler presentation for each group by using the coset enumeration method, perhaps with the help of an electronic computer, which is explained by Coxeter and Moser [21].