CHAPTER 5

SOME PERMUTATION GROUPS OF SMALL DEGREE AS LARGE AUTOMORPHISM GROUPS OF COMPACT RIEMANN SURFACES.

The main purpose of this chapter is to show that a number of symmetric groups of small degree are $M_3$-groups. We also prove the existence of some other $M_3$-groups of comparatively small genus. Since the groups of marks 1 and 6 are homomorphic images of the triangle groups $(2,3,7)$ and $(2,3,11)$ respectively, they are perfect. Since no non-trivial symmetric group is perfect, no group of mark 1 or 6 is symmetric. The groups of marks 2, 3, 4, 5 are respectively homomorphic images of the triangle groups $(2,3,8), (2,4,5), (2,3,9), (2,3,10)$. Thus the groups of marks 2 to 5 are not necessarily perfect indicating the possibility that some of these may be representable as symmetric groups. It was proved by Chetiya [17] that no symmetric group of degree $\leq 17$ can be a large group of mark 2. It is therefore worthwhile to examine if some symmetric groups of small degree occur as large groups of mark 3. We discover that the symmetric group $S_n$ does occur as a large group of mark 3 for $n = 5, 11, 15, 16$. From this it follows that $S_5$ is the smallest symmetric group having the smallest
mark 3. Meanwhile, in a recent paper Conder [60] has completely classified the alternating groups into those which are $M_1$-groups and those which are not.

5.1 \( S_5, S_{11}, S_{15}, S_{16} \) as $M_2$-groups.

In order to find out the possible cycle structures of the generators of $M_2$-groups of degree $N$ we consider a finite permutation group which acts transitively on $N$ points and then we make use of the following result due to Singerman [64]:

Let $G$ be a finite permutation group which acts transitively on $N$ points and which is a homomorphic image of the triangle group $(1,m,n)$. Let $G$ be generated by $x, y, z$ satisfying $x^1 = y^m = z^n = xyz = 1$. Let the permutation $x$ have $x_{i}$ $i$-cycles ($i = 1, 2, \ldots, l-1$), $y$ have $y_{j}$ $j$-cycles ($j = 1, 2, \ldots, m-1$) and $z$ have $z_{k}$ $k$-cycles ($k = 1, 2, \ldots, n-1$). Then there exists an integer $g \geq 0$ such that

\[
2g - 2 + \sum_{i=1}^{l-1} x_{i} (1 - \frac{i}{l}) + \sum_{j=1}^{m-1} y_{j} (1 - \frac{j}{m}) + \sum_{k=1}^{n-1} z_{k} (1 - \frac{k}{n}) = N(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n})
\]

(5.1)

Since an $M_2$-group is a homomorphic image of $(2,4,5)$ we put $l = 2$, $m = 4$, $n = 5$. Then we easily see that
\[ x_i = 0, \quad i \neq 1; \]
\[ y_j = 0, \quad j \neq 1, 2; \]
\[ z_k = 0, \quad k \neq 1. \]

Therefore formula (5.1) becomes

\[ 10x_1 + 15y_1 + 10y_2 + 16z_1 = N + 40(1-g) \quad \cdots (5.2) \]

The values of \( N \) are restricted here to be less than or equal to 16. We then have \( g = 0 \) or \( g = 1 \), for otherwise the right-hand side of (5.2) becomes negative. We thus ultimately get the following formulae for our purpose, which hold for \( N < 40 \):

\[ 10x_1 + 15y_1 + 10y_2 + 16z_1 = N + 40 \quad (5.3) \]
\[ 10x_1 + 15y_1 + 10y_2 + 16z_1 = N \quad (5.4) \]

Now for a fixed \( N \) the solutions of (5.3) and (5.4) give the possible cycle-structures of the generators of \( M_3 \)-groups of degree \( N \). We shall say that an \( M_3 \)-group is of type

\[ \{N|x_1; y_1, y_2; z_1 \} , \]
if \( N = \) degree of \( G \),

\[
\begin{align*}
&x_1 = \text{number of fixed points of } x, \\
&y_1 = \text{number of fixed points of } y, \\
&y_2 = \text{number of 2-cycles of } y, \\
z_1 = \text{number of fixed points of } z,
\end{align*}
\]

where \( x, y, z \) are the generators of \( G \) satisfying

\[
x^2 = y^4 = z^5 = xyz = 1.
\]

Obviously \( N \geq 5 \). Since our object here is to show that \( S_5, S_{11}, S_{15}, S_{16} \) are \( M_3 \)-groups, we solve (5.3) and (5.4) for \( N = 5, 11, 15, 16 \) respectively. We then get the following types of \( M_3 \)-groups:

\[
\begin{align*}
(i) \ &\{5|3 \ ; \ 1,0 \ ; \ 0\}, \\
(ii) \ &\{11|1 \ ; \ 1,1 \ ; \ 1\}, \\
(iii) \ &\{15|1 \ ; \ 1,3 \ ; \ 0\}, \\
(iv) \ &\{15|1 \ ; \ 3,0 \ ; \ 0\}, \\
v) \ &\{15|3 \ ; \ 1,1 \ ; \ 0\}, \\
(vi) \ &\{16|0 \ ; \ 0,4 \ ; \ 1\}, \\
vii) \ &\{16|0 \ ; \ 2,1 \ ; \ 1\}, \\
viii) \ &\{16|2 \ ; \ 0,2 \ ; \ 1\}, \\
(ix) \ &\{16|4 \ ; \ 0,0 \ ; \ 1\}, \\
x) \ &\{16|0 \ ; \ 0,0 \ ; \ 1\}.
\end{align*}
\]

We shall show that there exists a symmetric group of each of the types (i), (ii), (iv) and (viii). For this we first find the generators of such a group of a given degree and
then show that it is the symmetric group of that degree. The generators are obtained by using the "diagram technique" which we illustrate for the first possibility \( \{5|3; 1,0; 0\} \). In this case \( x \) has three fixed points, \( y \) has one fixed point, \( y \) has no 2-cycles, \( z \) has no fixed points. We draw a pentagonal figure [Fig. 1]

```
\begin{figure}[h]
\centering
\begin{tikzpicture}[>=latex]
    \node (1) at (0,0) {1};
    \node (2) at (1,1) {2};
    \node (3) at (1,-1) {3};
    \node (4) at (0,-2) {4};
    \node (5) at (-1,-1) {5};

    \draw (1) -- (2);
    \draw (2) -- (3);
    \draw (3) -- (4);
    \draw (4) -- (5);
    \draw (5) -- (1);

    \draw [dashed] (1) -- (2);
    \draw [dashed] (2) -- (3);
    \draw [dashed] (3) -- (4);
    \draw [dashed] (4) -- (5);
    \draw [dashed] (5) -- (1);
\end{tikzpicture}
\caption{Fig. 1}
\end{figure}
```

whose vertices (taken in a certain order, say anticlockwise) denote the 5-cycle of \( z \). A fixed point of \( x \) is denoted by a heavy dot in the figure and a 2-cycle in \( x \) is denoted by two vertices joined by a curly line. Two fixed points of \( x \) may occur at consecutive points (vertices), one point apart, two points apart, and so on. We start at a point, traverse along a curly line (if it is a fixed point, we do not move at this instance, but still we count it as a move) and then go along an edge in the preassigned direction and stop at the end point. Now if we see that we have arrived at the starting point, then that point is a fixed point of \( xz = \alpha \), say. If it is not so we repeat this process once more and see if we have arrived back at
the starting point and if we have, then the starting point and the point we stopped at represent in that order the points of a 2-cycle of \(\alpha\). If we have not arrived back at the starting point we are to repeat the process twice more and see if we have arrived back at the starting point, and if we have, then the points we stopped at will represent in that order, the points of a 4-cycle of \(\alpha\). In our present case \(x = (1)(2)(3)(4,5)\), \(\alpha = xz = (1,2,3,4)(5)\), and \(z = (1,2,3,4,5)\). If \(\langle x, y, z \rangle\) denotes the group generated by \(x, y, z\), then \(x\) and \(\alpha\) are sufficient to generate \(\langle x, y, z \rangle\), for \(z = xa\) and \(y = xa^{-1}x\) where \(x^2 = a^4 = (xa)^5 = 1\). From \(y = xa^{-1}x\), it follows that the cycle structure of \(y\) is same as that of \(\alpha\). If the cycle structure of \(\alpha\) obtained from the figure is not according to our requirement, then the placing of the fixed points of \(x\) in the diagram and joining the remaining vertices into pairs by curly lines need re-adjustment. For \(N > 5\), the generator \(z\) may have fixed points which are denoted by cross marks lying outside the pentagon (s) and each of these points is joined with a vertex of the pentagon(s) by a curly line. The rest of the procedure is then similar to that explained above.

We now list below the generators \(x\) and \(\alpha = xz\) of \(M_3\)-groups of types (ii), (iv) and (viii) obtained with the help of Figures 2, 3 and 4 respectively (given at the end of this chapter).
Type (ii) $\{11|1 ; 1, 1 ; 1\}$
\[ x = (2) (1,4) (3,6) (5,7) (8,11) (9,10) \]
\[ a = xz = (1,5,8,7) (2,3,6,4) (9,11) (10) \]

Type (iv) $\{15|1 ; 3, 0 ; 0\}$
\[ x = (1) (2,6) (3,4) (5,11) (7,15) (8,9) (10,12) (13,14) \]
\[ a = xz = (1,2,7,11) (3,5,12,6) (4) (8,10,13,15) (9) (14) \]

Type (viii) $\{16|2 ; 0,2 ; 1\}$
\[ x = (1) (12) (2,9) (3,7) (4,6) (5,15) (8,11) (10,1) (13,16) \]
\[ a = xz = (4,7) (1,2,10,15) (3,8,11,9) (5,16,14,6) (12,13) \]

In the following section $x$ and $a$ will be denoted for the sake of uniformity by $a$ and $b$ respectively.

We now prove that $S_5$, $S_{11}$, $S_{15}$, $S_{16}$ are all $M_3$-groups. We state below two well-known results (see Hall [26]) for future reference:

(A) The order of a $k$-ply transitive group of degree $n$ is divisible by $n(n-1) \ldots (n-(k-1))$.

(B) Let the integer $n = kp + r$, where $p$ is a prime and $p > k$, $r > k$ except $k = 1$, $r = 2$. Then a group of degree $n$ cannot be as much as $(r+1)$-fold transitive unless it is $S_n$ or $A_n$, the alternating group on $n$ symbols.
THEOREM 5.1.1 - \( S_5 \) is an \( M_3 \)-group.

PROOF : We have already seen that \( G = \langle a, b \rangle \), where

\[
\begin{align*}
  a &= (1 \ 2 \ 3 \ 4 \ 5) \\
  b &= (1, 2, 3, 4) (5)
\end{align*}
\]

and so \( a^2 = b^4 = (ab)^5 = 1 \), is an \( M_3 \)-group of type \( \{5|3 ; 1,0 ; 0\} \).

We now prove that \( G = \langle a, b \rangle \) is the symmetric group \( S_5 \). We immediately get \( ab = (1, 2, 3, 4, 5) \). We now have the following permutations in \( G \):

\[
\begin{align*}
  b^2 &= (1,3) (2,4) (5) \\
  b^{-1} &= (1,4,3,2) (5) \\
  (ab)^2 &= (1,3,5,2,4) \\
  ab^2 &= (1,3) (2,4,5) \\
  (ab^2)^2 &= (1) (3) (2,5,4) \\
  b^{-1}(ab)^2 &= (1) (2,3,4,5)
\end{align*}
\]

Now \( (5.7) \implies G_1 \) is transitive,

\( (5.6) \implies G_{1,3} \) is transitive,

\( (5.5) \implies G_{1,3,2} \) is transitive.

Thus \( G \) is 4-transitive. By (A) we know that \( |G| \) is
divisible by $5 \cdot 4 \cdot 3 \cdot 2$ i.e. by 120. Moreover $G$ is a subgroup of $S_5$ and the order of $S_5$ is 120. Hence $|G| \leq 120$. But since $|G|$ is divisible by 120, $|G| = 120$ showing that $G = S_5$.

**Theorem 5.1.2** - $S_{11}$ is an $M_2$-group.

**Proof**: We have already obtained that $G = \langle a, b \rangle$, where

\[
a = (2) (1,4) (3,6) (5,7) (8,11) (9,10)
\]
\[
b = (10) (1,5,8,7) (2,3,6,4) (9,11)
\]

and so $a^2 = b^4 = (ab)^5 = 1$, is an $M_3$-group of type

\[
\{11|1 ; 1,1 ; 1\}
\]

and it is a subgroup of $S_{11}$. We now prove that $G = \langle a, b \rangle$ is a symmetric group $S_{11}$. We see that $ab = (6) (1,2,3,4,5) (7,8,9,10,11)$. We now have the following permutations in $G$:

\[
b^2 = (9)(10)(11)(1,8)(5,7)(2,6)(3,4)
\]
\[
a(ab)^3 = (1,2,5,10,7,3,6)(8,9)(4)(11)
\]
\[
\{a(ab)^3\}^2 = (1,5,7,6,2,10,3)(4)(8)(9)(11)
\]
\[
ab^2 = (1,3,2,6,4,8,11)(9,10)(5)(7)
\]
\[
(ab^2)^{-1}(ab)^{-1} = (1,10,8,3,5,4,6)(7,11)(2)(9)
\]
\[
u = \{(ab^2)^{-1}(ab)^{-1}\}^2 = (1,8,5,6,10,3,4)(2)(7)(9)(11)
\]
\[
x = a^{-1}b^{-1}ab = (1,6,3,4,8,10,7,2,5,9,11)
\]
\[
y = bab^2 = (1,7,6,11,9,4,5)(5,8)(2)(10)
\]
\[
\alpha = x^{-2}y^{-1} = (1,11,8,4,7,5)(2,10,9)(3)(6)
\]
\[
\alpha = (1,10,8,2,3,7,4)(5,11)(6)(9)
\]
\[
u(ab\alpha)^4 = (1,4,3,2)(5,6,7,8)(9)(10)(11)
\]
\[
u(ab\alpha)^4b^2 = (1,3,6,5,2,8,7)(4)(9)(10)(11)
\]
Now (5.8) and (5.9) \( \Rightarrow \) \( G_{11} \) is transitive,
(5.8) and (5.10) \( \Rightarrow \) \( G_{11,9} \) is transitive,
(5.8) and (5.11) \( \Rightarrow \) \( G_{11,9,10} \) is transitive,
(5.12) \( \Rightarrow \) \( G_{11,9,10,4} \) is transitive.

Thus \( G \) is 5-transitive. We have \( 11 = 1.7 + 4 \) and so
by (B) \( G \) is either \( S_{11} \) or \( A_{11} \). But \( G \not\in A_{11} \), since
both the generators of \( G \) are odd. Hence \( G = S_{11} \).

THEOREM 5.1.3 - \( S_{15} \) is an \( M_3 \)-group.

PROOF: We have seen that \( G = \langle a, b \rangle \),

where

\[
a = (1) \ (2,6) \ (3,4) \ (5,11) \ (7,15) \ (8,9) \ (10,12) \ (13,14)
\]

\[
b = (1,2,7,11) \ (3,5,12,6) \ (8,10,13,15) \ (4) \ (9) \ (14)
\]

and so \( a^2 = b^4 = (ab)^5 = 1 \), is an \( M_3 \)-group of type
\[\{15|1; 3,0; 0\}\] and it is a subgroup of \( S_{15} \). We now
prove that \( G \) is the symmetric group \( S_{15} \). We have the
following permutations in \( G \):

\[
\begin{align*}
&\begin{pmatrix} 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15 \end{pmatrix} \\
&\begin{pmatrix} 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14 \ 15 \end{pmatrix}
\end{align*}
\]
\[ x = a^{-1}b^{-1}ab = (1,12,9,11,4,7,14,6,2,13,15,3,5,8,10) \]
\[ x^2 = (1,9,4,14,2,15,5,10,12,11,7,6,13,3,8) \]
\[ x^3 = (1,11,14,13,5) (12,4,6,15,8) (9,7,2,3,10) \]
\[ y = bab^2a = (1,11,15,8,4,10,13) (2,12,5,7) (3,6) (9,14) \]
\[ y^3 = (1,8,13,15,10,11,4) (2,7,5,12) (3,6) (9,14) \]
\[ y^7 = (1)(4)(8)(10)(11)(13)(15)(3,6)(9,14)(2,7,5,12) \]
\[ x^2y = (1,14,12,15,7,3,4,9,10,5,13,6) (2,8,11) \]
\[ (x^2y)^3 = (2)(8)(11)(1,15,4,5)(14,7,9,13)(12,3,10,6) \]
\[ x^2y^3 = (1,14,7,3,13,6,15,12,4,9) (2,10) (5,11)(8) \]
\[ (x^2y^3)^2 = (1,7,13,15,4) (14,3,6,12,9) (2)(5)(8)(10)(11) \]
\[ x^3y = (1,15,4,3,13,7,12,10,14) (2,6,8,5,11,9) \]
\[ (x^3y)^6 = (2)(5)(6)(8)(9)(11)(1,12,3)(15,10,13)(4,14,7) \]

Now (5.14) and (5.15) \( \Rightarrow \) \( G_8 \) is transitive,

(5.13) and (5.14) \( \Rightarrow \) \( G_{8,11} \) is transitive,

(5.14) and (5.16) \( \Rightarrow \) \( G_{8,11,2} \) is transitive,

(5.16) and (5.17) \( \Rightarrow \) \( G_{8,11,2,5} \) is transitive.

Thus \( G \) is 5-transitive. Now, since 15 = 1.11 + 4, by (8) \( G \) is either \( S_{15} \) or \( A_{15} \). But both the generators of \( G \) are odd. Therefore \( G \) is \( S_{15} \).
THEOREM 5.1.4 - \( S_{16} \) is an \( M_3 \)-group.

PROOF: We have already seen that \( G = \langle a, b \rangle \), where

\[
\]
\[
b = (4,7)(1,2,10,15)(3,8,11,9)(5,16,14,6)(12,13)
\]

so that \( a^2 = b^4 = (ab)^5 = 1 \), is an \( M_3 \)-group of type

\[\{16|2 ; 0,2 ; 1\}\] and it is a subgroup of \( S_{16} \). We now prove that \( G = \langle a, b \rangle \) is the symmetric group \( S_{16} \). We have the following permutations in \( G \):

\[
b^2 = (4)(7)(12)(13)(1,10)(2,15)(3,11)(8,9)(5,14)(6,16)
\] (5.18)

\[
ab^2 = (1,10,5,2,8,3,7,11,9,15,14)(4,16,13,6)(12)
\] (5.19)

\[
(ab^2)^4 = (1,8,9,10,3,15,5,7,14,2,11)(4)(6)(12)(13)(16)
\] (5.20)

\[
(ab)^3 = (11)(1,4,2,5,3)(6,9,7,10,8)(12,15,13,16,14)
\]

\[
a(ab)^3 = (1,4,9,5,13,14,8,11,6,2,7)(3,10,12,15)(16)
\]

\[
\alpha = \{a(ab)^3\}^4 = (1,13,6,4,14,2,9,8,7,5,11)(3)(10)(12)(15)(16)
\] (5.21)

\[
(ab)^{-1} = (11)(1,5,4,3,2)(6,10,9,8,7)(12,16,15,14,13)
\]

\[
a(ab)^{-1} = (1,5,14,9)(2,8,11,7)(3,6)(4,10,13,15)(12,16)
\]

\[
\beta = \{a(ab)^{-1}\}^2 = (1,14)(5,9)(2,11)(8,7)(4,13)(10,15)(3)(6)(12)(16)
\]

\[
\] (5.22)
Now (5.19) and (5.21) \( \Rightarrow \) \( G_{12} \) is transitive,
(5.18), (5.20) and (5.22) \( \Rightarrow \) \( G_{12,13} \) is transitive,
(5.20) and (5.22) \( \Rightarrow \) \( G_{12,13,16} \) is transitive.

Thus \( G \) is 4-transitive. By (B) we see that \( G = S_{16} \).

We have proved above that each of \( S_5, S_{11}, S_{15}, S_{16} \)
can be generated by only two elements \( a \) and \( b \) satisfying
\( a^2 = b^4 = (ab)^5 = 1 \). In each case we have explicitly written
down the permutations \( a \) and \( b \). This enables us to
compute any finite product of powers of \( a \) and \( b \), and
determine its order. Now by using the coset enumeration
method ([21], p 12), one can proceed to find a presentation
of each of these groups. An electronic computer may perhaps
be used for this purpose. We hope, this may lead one to
discover some new and simpler presentations of these groups.

5.2 Some \( M_3 \)-Groups of Small Genus

In this section we prove by using the same method as
in \( \S \) 5.1 the existence of an \( M_3 \)-group of comparatively small
genus belonging to each of the following types which are
different from the types (i) to (x) mentioned in \( \S \) 5.1:

\( (xi) \) \( \{6|2 ; 0, 1 ; 1\} \) \( (xii) \) \( \{10|0 ; 2, 2 ; 0\} \)
\( (xiii) \) \( \{12| 0 ; 0 , 2; 2\} \) \( (xiv) \) \( \{12| 0 ; 0, 2 ; 2\} \)
\( (xv) \) \( \{12| 2 ; 0, 0 ; 2\} \).
We now list below the generators $x$ and $\alpha = xz$ of the $M_3$-groups of above five types obtained with the help of Figures 5, 6, 7, 8 and 9 respectively (given at the end of this chapter).

Type (xi) \{6 | 2 ; 0, 1 ; 1\}
$$x = (1,6) (2) (3,5) (4)$$
$$\alpha = xz = (1,6,2,3) (4,5).$$

Type (xii) \{ 10 | 0 ; 2, 2 ; 0\}
$$x = (1,10) (2,9) (3,8) (4,5) (6,7)$$
$$\alpha = xz = (1,6,8,4) (2,10) (3,9) (5) (7)$$

Type (xiii) \{12 | 0 ; 0, 2 ; 2\}
$$x = (1,8) (2,7) (3,6) (4,11) (5,10) (9,12)$$
$$\alpha = xz = (1,9,12,10) (2,8) (3,6,4,7) (5,11)$$

Type (xiv) \{12| 0 ; 0, 2 ; 2\}
$$x = (1,10) (2,8) (3,6) (4,7) (5,11) (9,12)$$
$$\alpha = xz = (1,11) (2,9,12,10) (3,6,4,8) (5,7)$$

Type (xv) \{12|2 ; 0, 0 ; 2\}
$$x = (1,11) (2,6) (3,10) (4,7) (5,9) (8) (12)$$
$$\alpha = xz = (1,12,8,9) (2,6,3,11) (4,7,5,10)$$

In what follows $x$ and $\alpha$ will be denoted for the sake of uniformity by $a$ and $b$ respectively.
We now proceed to find the groups whose generators are given above.

THEOREM 5.2.1 - Let $G = \langle a, b \rangle$ where $a = (1, 6)(2)(3, 5)(4)$, $b = (1, 6, 2, 3) (4, 5)$. Then $G = A_6$ and it is an $M_3$-group of genus 10.

PROOF: Since $a^2 = b^4 = (ab)^5 = 1$, $G$ is an $M_3$-group of type $(xi)$. $G$ contains the following elements:

$$ab^2 = (1, 3, 5, 6, 2) (4) \quad (5.23)$$

$$(ab^{-1}a^{-1})^2 = (1, 5) (2, 6) (3) (4) \quad (5.24)$$

$$(a^{-1}b^{-1}ab)(bab^2a)^{-1} = (1) (3) (4) (2, 6, 5) \quad (5.25)$$

Now $(5.23) \Rightarrow G_4$ is transitive,

$(5.24), (5.25) \Rightarrow G_{4,3}$ is transitive,

$(5.25) \Rightarrow G_{4,3,1}$ is transitive.

So $G$ is 4-transitive. By Theorem 5.7.2 of Hall ([26], p 69), $G$ is either $S_6$ or $A_6$. But both the generators of $G$ are even permutations, so that $G = A_6$, and it is of genus 10, since $|G| = |A_6| = 360 = 40 (10-1)$. 
THEOREM 5.2.2 - Let $G = \langle a, b \rangle$ where

\[ a = (1,10)(2,9)(3,8)(4,5)(6,7), \quad b = (1,6,8,4)(2,10)(3,9)(5)(7). \]

Then $G$ is an $M_3$-group of genus 5, having a presentation

\[ R^4 = S^5 = (RS)^2 = (R^{-1}S)^4 = 1. \]

PROOF: Since $a^2 = b^4 = (ab)^5 = 1$, $G$ is an $M_3$-group of type (xii).

Set $x = b$, $y = (ab)^{-1}$. Then

\[ x = (1,6,8,4)(2,10)(3,9)(5)(7), \]
\[ y = (1,5,4,3,2)(6,10,9,8,7), \]
\[ xy = (1,10)(2,9)(3,8)(4,5)(6,7), \]
\[ x^{-1}y = (1,3,8,10)(2,9)(4,7,6,5), \]

so that $x^4 = y^5 = (xy)^2 = (x^{-1}y)^4 = 1$. We know from Coxeter and Moser ([21], p 134) that there is a group $\langle R, S \rangle$ of order 160 having the presentation $R^4 = S^5 = (RS)^2 = (R^{-1}S)^4 = 1$.

So it follows that $\langle x, y \rangle$ is a homomorphic image of this group of order 160. Since $a$ and $b$ are expressible in terms of $x$ and $y$, i.e. $a = (xy)^{-1}$, $b = x$, therefore $\langle x, y \rangle = \langle a, b \rangle$.

Hence $\langle a, b \rangle$ is a homomorphic image of $\langle R, S \rangle$. Since $\langle a, b \rangle$ is an $M_3$-group, its order is a multiple of 40. Therefore $|\langle a, b \rangle| = 40, 80$ or 160. That $|\langle a, b \rangle|$ is unequal to 40 or 80 follows from Coxeter and Moser ([21], p 140) and Sherk ([62], p 475) respectively. So $|\langle a, b \rangle| = 160 = 40(5-1)$, i.e. $G$ is of genus 5. Moreover $G$ is isomorphic to the group of
order 160 having a presentation $R^4 = S^5 = (RS)^2 = (R^{-1}S)^4 = 1$, which is usually denoted by $(4,5|2,4)$ ([21], p 139).

The above group is soluble in view of Theorem 9.3.2 of Hall ([26], p 143). We conjecture that this is the smallest soluble group occurring as an $M_3$-group.

**THEOREM 5.2.3** - Let $G = \langle a, b \rangle$ where

\begin{align*}
a &= (1,8)(2,7)(3,6)(4,11)(5,10)(9,12) \\
b &= (1,9,12,10)(2,8)(3,6,4,7)(5,11).
\end{align*}

Then $G \cong S_5$ and it is an $M_3$-group of genus 4.

**PROOF**: Since $a^2 = b^4 = (ab)^5 = 1$, $G$ is an $M_3$-group of type (xiii). Consider the subgroup $G_1 = \langle x, y \rangle$ of $G$ where $x = ba$ and $y = ab$. We see that

\begin{align*}
x^5 &= y^5 = (xy)^2 = (x^{-1}y)^3 = 1, \text{ since} \\
x &= ba = (1,12,5,4,2)(3)(6,11,10,8,7)(9), \\
y &= ab = (1,2,3,4,5)(6)(7,8,9,10,11)(12), \\
xy &= (1,12)(2)(3,4)(5)(6,7)(8)(9,10)(11), \\
x^{-1}y &= (1,3,4)(2,5,12)(6,8,11)(7,9,10)
\end{align*}

We know from Coxeter and Moser ([21], p 137), that $A_5$, the alternating group of degree 5, has a presentation

\begin{align*}
A^5 = B^5 = (AB)^2 = (A^{-1}B)^3 = 1,
\end{align*}
and so $G_1$ is a homomorphic image of $A_5$. Since $A_5$ is simple and $G_1$ is nontrivial, $G_1$ is isomorphic to $A_5$. This shows that $G \neq G_1$ as the order of $G_1$ is 60, a non-multiple of 40. Then by Theorem 3.2.1 it follows that $G_1$ is the derived group of $G$, and that

$$|G| = 2, |G_1| = 2, |A_5| = 120 = 40(4-1).$$

Hence $G$ is of genus 4.

Consider $<ab^{-1}, b>$. Then obviously $<ab^{-1}, b> = <a, b>$. If we set $R = ab^{-1}$, $S = b$, then we see that

$$R^5 = S^4 = (RS)^2 = (R^2S^2)^2 = 1.$$ 

Then by Coxeter and Moser ([21], p 137), $<a, b>$ is a homomorphic image of $S_5$. But $|G| = |<a, b>| = |S_5|$. Hence $G = S_5$.

**THEOREM 5.2.4** - Let $G = <a, b>$ where

$$a = (1,10)(2,8)(3,5)(4,7)(5,11)(9,12)$$

$$b = (1,11)(2,9,12,10)(3,6,4,8)(5,7).$$

Then $G$ is an $M_2$-group of genus 19, and it is $Z_2 \times A_6$.

**PROOF**: Since $a^2 = b^4 = (ab)^5 = 1$, $G$ is an $M_2$-group of type (xiv). Consider the subgroup $G_1 = <x, y>$ of $G$ where $x = ba$ and $y = ab$. We see that
\[ x^5 = y^5 = (xy)^2 = (x^{-1}y)^4 = 1 \quad \text{since} \]
\[ x = ba = (1, 5, 4, 2, 12)(3)(6, 7, 11, 10, 8)(9), \]
\[ y = ab = (1, 2, 3, 4, 5)(7, 8, 9, 10, 11)(6), \]
\[ xy = (1)(2, 12)(3, 4)(5)(6, 8)(7)(9, 10)(11), \]
\[ x^{-1}y = (1, 12, 3, 4)(2, 5)(6, 9, 10, 7)(8, 11). \]

We know from Coxeter and Moser ([21], p 137) that \( A_6 \) has a presentation \( A^5 = B^5 = (AB)^2 = (A^{-1}B)^4 = 1 \).
Therefore \( G_1 \) is a homomorphic image of \( A_6 \). Since \( A_6 \) is simple and \( G_1 \) is non-trivial, \( G_1 \) is isomorphic to \( A_6 \). It is seen from the cycle structures of \( x \) and \( y \) that \( a \) and \( b \) cannot be expressed in terms of \( x \) and \( y \), since no element of \( \{1, 2, 3, 4, 5, 12\} \) is taken to an element of \( \{6, \ldots, 11\} \) by \( x \) or \( y \) whereas \( a \) takes 1 to 10. So \( G \neq G_1 \). By Theorem 3.2.1, \( G_1 \) is the derived group of \( G \) and \( |G| = 2|G_1| = 2|A_6| = 720 = 40(19-1) \).
Hence \( G \) is of genus 19.

This group \( G \) is \( \mathbb{Z}_2 \times A_6 \). Consider the action of \( G \) on the blocks
\[ 1' = \{1, 7\}, \ 2' = \{2, 8\}, \ 3' = \{3, 9\}, \ 4' = \{4, 10\}, \]
\[ 5' = \{5, 11\}, \ 6' = \{6, 12\}. \]
Then

\[
a = (1', 4') (3', 6') (2') (5'), \\
b = (1', 5') (2', 3', 6', 4'), \\
ba = (1', 5', 4', 2', 6') (3'), \\
ab = (1', 2', 3', 4', 5') (6').
\]

The same argument as above gives that \(ab\) and \(ba\) generate \(A_6\). Thus we have a homomorphism of \(G\) onto \(A_6\). Its kernel must be \(Z_2\), and this must be central. So \(G = Z_2 \times A_6\).

**Theorem 5.2.5** - Let \(G = \langle a, b \rangle\) where

\[
a = (1,11)(2,6)(3,10)(4,7)(5,9)(8)(12), \\
b = (1,12,8,9)(2,6,3,11)(4,7,5,10).
\]

Then \(G = \text{PGL}(2,11)\) and it is an \(M_3\)-group of genus 34.

**Proof**: Since \(a^2 = b^4 = (ab)^5 = 1\), \(G\) is an \(M_3\)-group of type (xv). Consider the subgroup \(G_1 = \langle x, y \rangle\) of \(G\) where \(x = ba\) and \(y = ab\). It is easily verified that

\[
\alpha = (y^{-1}x)^6 = (1,11,3,7,2,9,12,5,6,4,10)(8), \\
\beta = yx^2y^2x = (1,4)(2,11)(3,12)(5,7)(6,9)(8,10), \\
\alpha \beta = (1,2,6)(3,5,9)(4,8,10)(7,11,12), \\
\alpha^4 \beta \alpha^6 \beta = (1,7)(2,4)(3,9)(5,11)(6,8)(10,12),
\]
so that $\alpha^{11} = \beta^2 = (\alpha \beta)^3 = (\alpha^4 \beta \alpha^6 \beta)^2 = 1$. We know from Coxeter and Moser ([211], p 138) that $\text{PSL}(2,11)$ has a presentation

$$A^{11} = B^2 = (AB)^3 = (A^4 BA^6 B)^2 = 1.$$ 

Therefore $\langle \alpha, \beta \rangle$ is homomorphic image of $\text{PSL}(2,11)$ which is simple. Hence $\langle \alpha, \beta \rangle = \text{PSL}(2,11)$. Moreover, can easily be verified that $x = (\alpha^7 \beta \alpha^3 \beta)^{-1}$ and $y = (\alpha^9 \beta \alpha^3 \beta)^{-1}$ so that $G_1 = \langle \alpha, \beta \rangle$. Thus $G_1 = \text{PSL}(2,11)$ which has order 660, a non-multiple of 40. This shows that $G \neq G_1$, and so by Theorem 3.2.1, $G_1$ is the derived group of $G$, and $|G| = 2 |G_1| = 2 |\text{PSL}(2,11)| = 1320 = 40(34-1)$.

Therefore $G$ is of genus 34. We see that

$$\alpha^4 \beta \alpha^6 \beta = (1,7)(2,4)(3,9)(5,11)(6,8)(10,12)$$

$$\alpha \alpha \beta = (1,12,7,8,10,5,3,4,11,2)(6)(9)$$

and so $u = (\alpha^4 \beta \alpha^6 \beta) \alpha \alpha \beta (\alpha^4 \beta \alpha^6 \beta)^{-1}$

$$= (1,6,12,11,9,2,5,4,7,10)(3)(8).$$

Now $u^{-1} \alpha u = (1,6,9,3,10,5,2,11,4,12,7)(8) = \alpha^8$

where 8 is a primitive root of GF(11). Obviously $\langle \alpha, \beta, u \rangle$ contains $G_1$. We claim that $u \notin G_1$. On the
contrary, suppose \( u \in G_1 \). Since \( \alpha \beta \in G_1 \), we then see that \( a \in G_1 \), a contradiction. Hence \( |\langle \alpha, \beta, u \rangle| > |G_1| \).

By Coxeter and Moser ([21], p. 96), \( \langle \alpha, \beta, u \rangle \) is a homomorphic image of \( \text{PGL}(2, 11) \). But since \( |\langle \alpha, \beta, u \rangle| = |G| \) in view of \( |\langle \alpha, \beta, u \rangle| > |G_1| \), \( G = \langle \alpha, \beta, u \rangle \) is isomorphic to \( \text{PGL}(2, 11) \).

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**Fig. 2**

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**Fig. 3**