Fourth Chapter

Some Space Biased Aspects of Near–rings and Near–ring Groups

4.1 Prerequisites.

4.2 \((S_X-S_X)\) maps and associated concepts.

4.3 Quasi–regular cyclic \(S_{Ne}\)-open \(N\)-groups.

4.4 Quasi–regular cyclic \(S_{Ne}\)-closed \(N\)-groups.

4.5 Pseudo nilpotent \(S_{Ne}\)-open \(N\)-groups.
The continuity in a topological space is carried out by internal and external compositions of a group or a near-ring group respectively and in some cases restrictions related to boundedness, connectedness and the Hausdorff properties lead us to some effective results.

Extending the idea of boundedness already available with the help of the notion of topological nilpotent set we got some elegant results in case of topological near-ring groups with acc on annihilators [[11]].

It is observed that in case of a topological group, where the binary operation is continuous in the product space, the corresponding co-ordinate wise continuity is an obvious character in so-called one sheet space! But, the converse needs a hard work in the real sense if it happens in so-called broken two sheets space. Of course, Beidleman and Cox in [3] are of view that coordinate-wise continuity is all that is necessary in many cases.
What is stated above may be verified (at least intuitively) with the help of the following graphs with respect to the usual topologies on $\mathbb{R}$ and $\mathbb{R}^2$. 

\[ H(x, y) = x + \frac{1}{y} \]
\[ t(x, y) = x + y \]
\[ r = x + \frac{1}{2} \]

The diagrams show 3D plots of the functions \( t(x, y) \) and \( 2x+10y \).
We recall that in the definition of topological ring, Kaplansky [32] insisted that addition and multiplication be continuous on the product space; however, as defined by Beidleman et al. the authors
found that co-ordinate wise continuity is all that is necessary in many cases.
Some careful observations have elegantly revealed what we have attempted and carried out with some sort of rare and alarming beauty, hitherto the so-called continuity of such structures are concerned.

Keeping aside the concrete so-called topological aspect of what has been explained above, we dare to review this aspect of above type of algebraic structure from more or less algebraic point of view in a broaden court-yard with a view to play the same game in a more sophisticated country of algebra.

For the moment we leave available topological nomenclature, however instead, embrace some abstract familiar algebraic way of approach. Undoubtedly everything would be justified with sufficient examples if and when necessary.

The supposed pseudo continuity in such a so-called algebraic space is carried out by internal and external compositions of a group or a near-ring group respectively to give some general view of some topological properties including boundedness, connectedness and Hausdorff character etc.

Extending the idea of boundedness already available with the help of the notion of (so-called) $S_k$-nilpotent set, we get some elegant results in case of such a space biased near-ring groups with acc on annihilators. It is interesting to note the relevancy and elegance of the results obtained, as the same may be determined with accommodating justification on such a space biased near-ring groups that their so-called ps(pseudo)-discrete character is in
association with the so-called algebraic boundedness with zero-radical or so-called ps-openness of the radical with same character.

At this juncture we want to review some results which may be termed, in some sense, an extension of what Chowdhury et al. has carried out [10, 54, 21]. The structure of semi-prime ring [28, 29] seems to be still relevant due to its elegance. At the same time the authors are with sweet remembrance of what Meldrum has remarked about the importance and the intricacy of what has already been carried out by this group along the line of acc on annihilators in case of near-ring groups.

It is observed that near-ring groups with acc on annihilators of subsets of the group in the attached near-ring found to be well behaved so far the so-called space biased algebraic structure is concerned with some so-called pseudo quality on nilpotency as well as strongly semi-prime character. This has involved for the proper development of what we have attempted for.

The notion of boundedness of Beidleman and Cox are playing some shaman character with the algebraic space as the authors are claiming for! Together with these, a near-ring group with so-called Goldie character has been playing an interesting and elegant worthwhile game where the group having finite number of elements (e’s) belonging to the group with zero annihilator, which occurs as a necessity of such a near-ring group. The justification has properly been accommodated with a sufficient number of examples so as to congregate our endeavor.
The discrete character of such an algebraic biased space is playing a generously subjective role with a very deep insight, which seems to include so many aspects even in some cases, the orientable and non orientable space relating to Klein's example.

The main results of this chapter are the outcome of our paper [[12]]. This chapter is divided into five sections. The first section contains some definitions and preliminary results what are necessary in further discussion. The second section deals with some results on \((S_x - S_y)\) maps and associated concepts. The results on quasi-regular cyclic \(S_{N_b}\)-open \(N\)-groups are discussed in third section. The fourth section deals with quasi-regular cyclic \(S_{N_b}\)-closed \(N\)-groups. The last section consists of the results of pseudo nilpotent \(S_{N_b}\)-open \(N\)-groups.

**4.1 Prerequisites:**

We notice, at first, the following observations and discuss what we have attempted for so far our two sided as well as unbalanced (non-symmetric) continuity problems are concerned.

**4.1.1 Observation [[12]]**: Let \(X = \{a, b, c\}\) and \(Y = \{\alpha, \beta, \gamma, \delta\}\) be two sets where \(S_x = \{\{a, b\}, \{c\}\}(\subseteq P(X))\) and \(S_y = \{\{\alpha\}, \{\beta, \gamma, \delta\}\}(\subseteq P(Y))\).

Consider a mapping \(f : X \to Y\) defined by

\[
    f(a) = \alpha, \quad f(b) = \alpha, \quad f(c) = \beta.
\]

Now we have \(\{\alpha\}(\in S_y)\) containing \(f(a)\) and \(\{a, b\}(\in S_x)\) containing \(a(\in X)\) such that \(f(\{a, b\}) = \{\alpha\}\).
Here we note that for any $W \in S_Y$ containing $f(a)$ there exists $V \in S_X$ containing $a \in X$ such that $f(V) \subseteq W$.

4.1.2 Observation [[12]]: For a set $X = \{a, b, c\}$ having $S_X = \{\{a, b\}, \{b, c\}\}$, consider a mapping $f : X \times X \to X$ defined by
$$
f(a, a) = b, f(a, b) = c, f(a, c) = a, f(b, a) = c, f(b, b) = b, f(b, c) = b,
f(c, a) = a, f(c, b) = b, f(c, c) = b.
$$
We notice here $\{b, c\} \in S_X$ containing $f(a, b)$ and $\{a, b\} \times \{a, b\} \in S_{X \times X}$ containing $(a, b)$ such that $f(\{a, b\} \times \{a, b\}) = \{b, c\}$.
Thus for any $W \in S_X$ containing $f(a, b)$ there exists an $U \times V \in S_{X \times X}$ containing $(a, b)$ such that $f(U \times V) \subseteq W$.

Now we begin with two arbitrary non-empty sets $X$ and $X'$ where $S_X \subseteq P(X)$ and $S_{X'} \subseteq P(X')$.

4.1.3 Definition: A map $f : X \to X'$ is an $(S_X-S_{X'})$ map at a point say, $a \in X$ provided for each $S' \in S_{X'}$ containing $f(a)$ we have $S \in S_X$ containing $a \in X$ such that $f(S) \subseteq S'$.

If $f$ is $(S_X-S_{X'})$ map at each $a \in X$, then $f : X \to X'$ is an $(S_X-S_{X'})$ map.

It is not difficult to follow that what has been stated above is only a re-statement of our familiar idea of topological continuity.

4.1.4 Definition: A set $G \in S_X$ is called an $S_X$-set and a set $H \in S_{X'}$ is an $S_{X'}$-set provided $f : X \to X'$ is an $(S_X-S_{X'})$ map.

4.1.5 Definition: If $f : X \times X \to X$ is a binary $(S_X \times S_X-S_X)$ map then a set $G \in S_X$ is called an $S_X$-open set.
4.1.6 **Example:** In the symmetric group \((S_3, +)\) w.r.t. the addition defined in Table 1.4, if we consider \(S_{S_3} = \{\{0, x, y\}, \{a, b, c\}\}\), then \(\{0, x, y\}\) and \(\{a, b, c\}\) are \(S_{S_3}\)-open sets.

4.1.7 **Definition:** A subset, say \(B\) of \(X\) is called an \(S_X\)-closed if \(B^c\) (complement of \(B\)) is \(S_X\)-open.

4.1.8 **Example:** In Example 4.1.6, the \(S_{S_3}\)-closed sets are \(\{0, x, y\}\) and \(\{a, b, c\}\).

We note here that the intersection of two \(S_{S_3}\)-closed sets need not be again \(S_{S_3}\)-closed.

4.1.9 **Definition:** The intersection of all \(S_X\)-closed subsets of \(X\) containing a subset; say \(B\) of \(X\) is called the \(S_X\)-closure of \(B\). And it is denoted by \(\overline{B}\).

Thus in contrast to what we have in case of a topological space the \(S_X\)-closure of a set need not be again \(S_X\)-closed one.

Now suppose \(x \in \overline{B}\).

If possible let for some \(F \in S_X\) with \(x \in F\) such that \(F \cap B = \emptyset\).

Hence \(B \subseteq F^c\) such that \(x \notin F^c\) giving thereby \(x \notin \overline{B}\) which is not true.

On the contrary we assume that for all \(V \in S_X\) with \(x \in V\) such that \(V \cap B \neq \emptyset\).

Suppose if possible let \(x \notin \overline{B}\), then for some \(F^c \in S_X\) with \(B \subseteq F\) such that \(x \in F\).

Therefore, \(B \cap F^c = \emptyset\) for some \(F^c \in S_X\) with \(x \in F^c\) which contradicts the assumption. ♦
Thus we get the following note:

4.1.10 **Note:** For $B \subseteq X$, $x \in \overline{B}$ if and only if $V \cap B \neq \emptyset$, for all $V \in S_X$ with $x \in V$.

Here we study some characteristics of minimal $S_X$-closed set.

4.1.11 **Example:** Consider the near-ring $D_8$ of dihydral group w.r.t. the addition defined by the Table 2.3 and multiplication defined by Table 3.1 having $S_{D_8} = \{\{0, 2a\}, \{a, 3a\}, \{b, 2a+b\}, \{3a+b, a+b\}\}$.

For $B = \{0, 2a+b\}$, we have $S_{D_8}$-closed subsets containing $B$ are $\{0, 2a, a, 3a, b, 2a+b\}$ and $\{0, 2a, b, 2a+b, 3a+b, a+b\}$ which are minimal as $S_{D_8}$-closed subset of $D_8$ containing $B$. Now $\{0, 2a, b, 2a+b\}$ is the intersection of all such minimal $S_{D_8}$-closed subsets of $D_8$ containing $B$.

This enables us to define the following:

4.1.12 **Definition:** The intersection of all minimal $S_E$-closed subset of $E$ containing a subset; say $B$ of $E$ is called the **minimal $S_E$-closure** of $B$ and is denoted by $\overline{B}^m$.

Thus in symbol,

$$\overline{B}^m = \cap \{F \mid F \text{ is a minimal } S_E\text{-closed set containing } B (\subseteq E)\}.$$ 

Obviously, 4.1.11 clears it that the minimal $S_E$-closure of a set may not be $S_E$-closed.

It is noted that in case $S_E$ is a topology then we get only one minimal $S_E$-closed set containing $B (\subseteq E)$ which is $\overline{B}$.

Here we study [[12]] some characteristics of minimal $S_X$-closed set.
The result 2.4.22 what we have established in second chapter plays an important role with the so-called minimal $S_X$-closure of a set, say $X$ as follows.

1. Suppose $E_1$ is an Artinian $N$-subgroup of $E$ and
   \[ S_E = \{ A^c \mid A \text{ is an } N \text{-subgroup of } E_1 \} \]

2. If $E$ satisfies the acc on annihilators of subsets of $E$ in $N$ and
   \[ S_E = \{ \text{Ann}(S)^c \mid S \subseteq N \} \]

3. If $E$ is of 2.4.22 and
   \[ S_E = \{ \text{Ann}_E(S)^c \mid S \in S_N \} , \]
   where $S_N = \{ (S =) \text{Ann}(E_1) \mid E_1(\subseteq E) \}$. In all these cases, we have $\overline{B} = \overline{B^*}$ (here, each of the collections of the $S_E$-closed sets containing $B$ either empty or only a singleton set from the collection $S_E$. Hence in the first case $\overline{B} = \overline{B^*} = E$ and in the other case we have $\overline{B} = \overline{B^*} = \text{only one set in the collection of all the } S_E \text{-closed sets}.$

4.1.13 Example: In the group $(\mathbb{Z}_8, +)$ w.r.t. addition modulo 8, consider $S_{\mathbb{Z}_8} = \{ \{0, 4\}, \{1, 5\}, \{2, 6\}, \{3, 7\} \}$. The $S_{\mathbb{Z}_8}$-closed sets containing $\{0, 2\}$ are $\{0, 2, 3, 4, 6, 7\}$ and $\{0, 1, 2, 4, 5, 6\}$ and hence the $S_{\mathbb{Z}_8}$-closure of $\{0, 2\}$ is $\{0, 2, 4, 6\}$. Again it is observed here that the $S_{\mathbb{Z}_8}$-closure of $\{0, 2, 4, 6\}$ is $\{0, 2, 4, 6\}$.

And this leads us to define what we now tempt for:

4.1.14 Definition: A subset $B$ of $X$ is a ps-closed when $\overline{B} = B$. 

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Therefore \(\{0, 1, 4, 5\}\) in Example 4.1.13, is \(ps\)-closed.

As a note we say that \(S_x\)-closeness of a set implies the \(ps\)-closeness of the same but the converse need not be true.

Now we suppose, \(f_1 + f_2 \in \overline{F_1 \oplus F_2}\) for two \(ps\)-closed subsets \(F_1\) and \(F_2\) of two \(N\)-groups, say \(E_1\) and \(E_2\) respectively.

We have, by 4.1.10, \((G_1 \oplus G_2) \cap (F_1 \oplus F_2) \neq \emptyset\) for any \(S_{E_1 \oplus E_2} - open\) subset \(G_1 \oplus G_2\) of \(E_1 \oplus E_2\) containing \(f_1 + f_2\).

\[\Rightarrow G_1 \cap F_1 \neq \emptyset\text{ and } G_2 \cap F_2 \neq \emptyset.\]

\[\Rightarrow f_1 \in \overline{F_1} \text{ and } f_2 \in \overline{F_2}, \text{ [by 4.1.10]}.\]

\[\Rightarrow f_1 \in F_1 \text{ and } f_2 \in F_2, \text{ as } F_1 \text{ and } F_2 \text{ being two } ps\text{-closed subsets of } E_1 \text{ and } E_2 \text{ respectively}.\]

\[\Rightarrow f_1 + f_2 \in F_1 \oplus F_2.\]

Hence \(F_1 \oplus F_2\) is a \(ps\)-closed subset of \(E_1 \oplus E_2\). ♦

Thus we note the following

**4.1.15 Note:** If \(F_1\) and \(F_2\) are \(ps\)-closed subsets of the \(N\)-groups \(E_1\) and \(E_2\) respectively then \(F_1 \oplus F_2\) is a \(ps\)-closed subset of \(E_1 \oplus E_2\) ♦

As the \(S_x\)-closure, we define the following:

**4.1.16 Definition:** The union of all \(S_x\)-open subsets of \(X\) contained in a subset, say \(D\) of \(X\) is called the \(S_x\)-interior (denoted by \(D^0\)) of \(D\).

**4.1.17 Example:** In the dihedral group \((D_8, +)\) w.r.t. the addition defined Table 2.3, let \(S_{D_8} = \{\{0, 2a\}, \{a, 3a\}, \{b, 2a + b\}\),
\{3a + b, a + b\}, the $S_{B_{8}}$–interior of a set \{0, a, 2a, 3a, b\} is \{0, a, 2a, 3a\} which is not $S_{B_{8}}$–open.

But in case of a topology $S_{X}$, the $S_{X}$–interior of a set is $S_{X}$–open.

In the above example if we take $D = \{0, a, 2a, 3a\}$, then $D^{0} = D$ and the like cases lead us to define the following:

**4.1.18 Definition:** A subset $D$ of $X$ is ps–open when $D^{0} = D$.

And hence from 4.1.10 we have the following note:

**4.1.19 Note:** For $B \subseteq X$, $x \in \overline{B}$ if and only if $V \cap B \neq \emptyset$, for all ps–open subset $V$ of $X$ containing $x$ ($\in X$).

Following results are obvious from the definition of ps–open set

**4.1.20. Note (i)** $x \in D^{0}$ if and only if there exists a $S_{X}$–open subset $U$ of $X$ containing $x$ such that $x \in U \subseteq D$.

(ii) $x \in D^{0}$ if and only if there exists a ps–open subset $V$ of $X$ containing $x$ such that $x \in V \subseteq D$.

For an $S_{X}$–open subset $V$ of a (additive) group $X$ and a subset $B$ of $X$, we have

$(B + V)^{0} \subseteq B + V$.

Again for $x \in B + V$ with $x \notin (B + V)^{0}$ we get $x = a + v$, for some $a \in B$ and $v \in V$.

But $v (= - a + x) \in V$, as + being $(S_{X} \times S_{X} - S_{X})$ map, so there exist an $S_{X}$–open subset $W$ of $X$ containing $- a (\in X)$ and an $S_{X}$–open subset $U$ of $X$ containing $x (\in X)$ such that $W + U \subseteq V$.

Hence $- a + U \subseteq W + U \subseteq V$.

$\Rightarrow x \in U \subseteq a + V \subseteq B + V$.
\[ x \in (B + V)^o \] which is not true.

\[ (B + V)^o = B + V. \]

Thus we note the following:

4.1.21 Note

(i) If \( V \) is an \( S_x \)-open subset of a (additive) group \( X \), then \( B + V \) is a ps–open subset of \( X \), for any subset \( B \) of \( X \).

(ii) If \( W \) is an ps–open subset of a (additive) group \( X \), then \( D + W \) is a ps–open subset of \( X \), for any subset \( D \) of \( X \).

But in case \( S_x \) is a topology then \( D + W \) is an \( S_x \)-open subset of \( X \).

It is interesting to note that by 4.1.20(i) when \( X \) is a topological group, then such a \( D + W \) is open, and the role of the union of subsets leads to such an obvious result.

Absence of the closeness w.r.t. the union of the set elements in \( S_x \) seems to be responsible for this finer unusual characteristic.

Henceforth we assume that the binary operations on \( N \) and \( E \) are \((S_N \times S_N - S_N)\) and \((S_E \times S_E - S_E)\) maps respectively. Moreover the scalar multiplication of \( E \) over \( N \) is \((S_N \times S_E - S_E)\) map.

So trivially we get the following:

For given \( r \ (\in N \), the mappings \( \phi_r^1, \phi_r^2, \phi_r^3 \) and \( \phi_r^4 \) defined by

(i) \( \phi_r^1(x) = x + r \)

(ii) \( \phi_r^2(x) = r + x \)

(iii) \( \phi_r^3(x) = rx \)

(iv) \( \phi_r^4(x) = xr \)

from \( N \) into \( N \), for all \( x \in N \), are \((S_N - S_N)\) maps.
Again for given $e \in E$ and $m \in N$, the mappings $\mu^1, \mu^2, \mu^3$ and $\mu^4$, defined by

(i) $\mu^1(x) = x + e$

(ii) $\mu^2(x) = e + x$, (for all $x \in E$) are $(S_E-S_E)$ maps

(iii) $\mu^3(n) = ne$, (for all $n \in N$) is a $(S_N-S_E$ map,

(iv) $\mu^4(x) = mx$, (for all $x \in E$) is a $(S_E-S_E)$ map. ♦

The identity $1$ of the near–ring $N$ enables us about the $(S_E-S_E)$ map of the mapping $(-1)\mu^5 : E \to E$, defined by $(-1)\mu^5(x) = -x$, for all $x \in E$.

Same thing also occurs in case of $N$, i.e. the mapping $(-1)\phi^5 : N \to N$, defined by $(-1)\phi^5(x) = -x$, for all $x \in N$ is a $(S_N-S_N)$ map.

As $1 \in N$ and $-x = (-1)x$, for all $x \in E$, so we have the following result.

4.1.22 Lemma [[12]]: For a given $e \in E$, the mapping $\mu^6:E \to E$, defined by $\mu^6(x) = e - x$ is $(S_E-S_E)$ map, for all $x \in E$.

Proof: Let $x \in E$ and $V \in S_E$ containing $\mu^6(x) (\in E)$.

Now, $\mu^2$ being $(S_E-S_E)$ map so there exists $U \in S_E$ containing $-x (\in E)$ such that

$e + U \subseteq V$.  \hspace{1cm} (i)$

Again we get, because of $(-1)\mu^5$, a $W \in S_E$ containing $x (\in E)$ such that $-W \subseteq U$. \hspace{1cm} (ii)

Thus $e - W \subseteq e + U$, [by (ii)]

$\subseteq V$, [by (i)].
As \( x \in E \) is arbitrary so the mapping \( \mu^E : E \to E \) is \((S_E - S_E)\) map, for all \( x \in E \).

4.1.23 Corollary: For a given \( n \in N \), the mapping \( \phi^N : N \to N \) defined by \( \phi^N(x) = n - x \) is \((S_N - S_N)\) map, for all \( x \in N \).

4.1.24 Lemma [[12]]: For any \( S_E\)-open subset \( V \) of \( E \), \( -V \) is a ps–open subset of \( E \).

Proof: Let \( x \in -V \) then \( x = -v \), for some \( v \in V \).

Hence \( v (= - x) \in V \).

As \((-) \mu^E\) being a \((S_E - S_E)\) map, so there exists \( U \in S_E \) containing \( x \) \((\in E)\) such that
\[ -U \subseteq V. \]

\[ \Rightarrow U \subseteq -V. \]

Thus, by 4.1.20, \( x \in (-V)^o \).

As \( x \) being arbitrary, so \(-V\) is a ps–open subset of \( E \). ♦♦♦

4.1.25 Corollary: For any \( S_N\)-open subset \( U \) of \( N \), \( -U \) is a ps–open subset of \( N \).

We have the following referring to 4.1.20 (i).

4.1.26 Lemma [[12]]: If \( V \) is an \( S_E\)-open subset of \( E \) and \( e \in E \) then \( e + V, \ V + e \) are ps–open subsets of \( E \).

4.1.27 Corollary: If \( U \) is an \( S_N\)-open subset of \( N \) and \( n \in N \) then \( n + U, \ U + n \) are ps–open subsets of \( N \).

4.1.28 Lemma [[12]]: If \( B \) is an \( N\)-subgroup (ideal) of \( E \) then the \( S_E\)-closure \( 
\bar{B} \) of \( B \) is also an \( N\)-subgroup (ideal) of \( E \).

Proof: It is obvious for \( \bar{B} = E \).
Suppose if possible let \( \overline{B} \neq E \).

Let \( a, b \in \overline{B} \) and \( W \) be an \( S_E \)-open subset of \( E \) containing \( a - b \) (\( \in E \)). As the mapping \( (x, y) \to x - y \) of \( E \times E \) into \( E \) is an \( (S_E \times S_E - S_E) \) map so, there exist \( S_E \)-open subset \( U \) of \( E \) containing \( a \) (\( \in E \)) and \( V \) of \( E \) containing \( b \) (\( \in E \)) such that

\[
U - V \subseteq W \quad \text{...(i)}.
\]

Now \( a, b \in \overline{B} \), so by 4.1.10, \( U \cap B \neq \emptyset \) and \( V \cap B \neq \emptyset \).

Again \( (U \cap B) - (V \cap B) \subseteq (U - V) \cap B \). \( \text{...(ii)} \)

Hence, by (i), \( W \cap B \) \( \supseteq (U \cap B) - (V \cap B) \) \( \neq \emptyset \)

\( \phi \neq (U \cap B) - (V \cap B) \), (as \( U \cap B \neq \emptyset \) and \( V \cap B \neq \emptyset \)).

\( \subseteq (U - V) \cap B \), [by (ii)].

\( \subseteq W \cap B \).

Thus by 4.1.10, \( a - b \in \overline{B} \), for all \( a, b \in \overline{B} \).

In other words \( \overline{B} \) is a subgroup when \( B \) is so.

Suppose \( B \) is an \( N \)-subset of \( E \).

Let \( b \in \overline{B} \) and \( n \in N \).

We are to show that \( nb \in \overline{B} \).

Let \( W \) be an \( S_E \)-open subset of \( E \) containing \( nb \) (\( \in E \)).

Since for given \( m \in N \), \( \mu^*: E \to E \), defined by, \( \mu^*(x) = mx \), (for all \( x \in E \)) is a \( (S_E - S_E) \) map, we get an \( S_E \)-open subset \( V \) of \( E \) containing \( b \) (\( \in E \)) such that

\[
nV \subseteq W \quad \text{...(iii)}.
\]

But \( V \cap B \neq \emptyset \), as \( b \in \overline{B} \) and by 4.1.10.

Let \( x \in V \cap B \), then \( nx \in nV \) and \( nx \in B \), as \( B \) is an \( N \)-subset of \( E \).

Thus \( \phi \neq nV \cap B \).
which gives, by 4.1.10, nb ∈ B, for all n ∈ N and b ∈ B.

In other words B is an N-subset of E.

Suppose B is a normal subgroup of E.

Let e ∈ E and b ∈ B, we are to show that -e + b + e ∈ B.

Let W be an S_n-open subset of E containing -e + b + e.

Since for given e ∈ E, the mapping \( \mu^2 : E \to E \), defined by \( \mu^2(x) = e + x \) (for all x ∈ E) is (S_n-S_n) map, so we get an S_n-open subset V of E containing b + e (∈ E) such that

- \( -e + V \subseteq W \) ...(iv).

By the same argument we get a S_n-open subset U of N containing b (∈ E) such that \( U + e \subseteq V \) ...(v).

Thus from (iv) and (v) we get

- \( -e + U + e \subseteq -e + V \)

\( \subseteq W \).

Again \( U \cap B \neq \emptyset \), by 4.1.10, as b ∈ B.

Now for x ∈ U ∩ B

we get -e + x + e ∈ -e + U + e

and -e + x + e ∈ B, as B is a normal subgroup of E.

Thus \( \phi \neq (-e + U + e) \cap B \)

\( \subseteq W \cap B \), giving thereby, by 4.1.10,

- e + b + e ∈ B, for all e ∈ E and b ∈ B.

Hence B is a normal subgroup of E.

Finally suppose that B is an ideal of E.

Let n ∈ N, e ∈ E and b ∈ B.
To show that, $n(e + b) - ne \in \overline{B}$.

Let $W$ be an $S_{E}$-open subset of $E$ containing $n(e + b) - ne (\in E')$.

Since for given $e \in E$, the mapping $\mu_{e}^{1}: E \to E$, defined by $\mu_{e}^{1}(x) = x + e$, (for all $x \in E$) is a $(S_{E}-S_{E})$ map, so we get a $S_{E}$-open subset $V$ of $E$ containing $n(e + b) (\in E')$ such that

$$V - ne \subseteq W. \quad \ldots \text{(vi)}$$

Again for some $m \in N$, the mapping $\mu_{m}^{1}: E \to E$, defined by $\mu_{m}^{1}(x) = mx$, (for all $x \in E$) is a $(S_{E}-S_{E})$ map, so there exists a $S_{E}$-open subset $U$ of $N$ containing $e + b (\in E)$ such that

$$nU \subseteq V. \quad \ldots \text{(vii)}$$

Similarly by using the same argument we get an $S_{E}$-open subset $X$ of $E$ containing $b (\in E)$ such that

$$e + X \subseteq U. \quad \ldots \text{(viii)}$$

Thus from (vi), (vii), (viii) we get $n(e + X) - ne$

$$\subseteq nU - ne$$

$$\subseteq V - ne$$

$$\subseteq W.$$

Again $X \cap B \neq \phi$, by 4.1.10 [as $b \in \overline{B}$].

Let $x \in X \cap B$,

then we get $n(e + x) - ne \in B$, as $B$ being an ideal

and $n(e + x) - ne \in n(e + X) - ne$.

Hence $\phi \neq (n(e + X) - ne) \cap B$

$$\subseteq W \cap B.$$ 

$\Rightarrow n(e + b) - ne \in \overline{B}$, [by 4.1.10], for all $n \in N$, $e \in E$ and $b \in \overline{B}$. 

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Thus $\overline{B}$ is an ideal of $E$ provided $B$ is so. ♦♦♦

4.1.29 Corollary: If $B$ is an $N$-subgroup (ideal) of $NN$ then the $S_N$-closure $\overline{B}$ of $B$ is also an $N$-subgroup (ideal) of $NN$.

4.1.30 Definition: A subset $B$ of $X$ is $S_X$-dense in a subset, say $D$ of $X$ when $\overline{B} = D$.

If $\overline{B} = X$, then $B$ is $S_X$-dense in $X$.

In 4.1.13, the set $\{0, 2\}$ is $S_X$-dense in $\{0, 2, 4, 6\}$.

4.1.31 Definition: The $X$ is ps-discrete when every nonempty subset of $X$ is ps-open.

4.1.32 Example: The symmetric group $(S_3, +)$ of 4.1.6 w.r.t. $S_{S_3} = \{\{0\}, \{a\}, \{b\}, \{c\}, \{x\}, \{y\}\}$ is ps-discrete.

4.1.33 Definition: A subset $Y$ of $X$ is called a ps-compact if for any class $C = \{A_i\}_{i \in \Lambda}$ of ps-open subsets of $X$ with $Y \subseteq \bigcup_{i \in \Lambda} A_i$ there exist $A_1, A_2, A_3, \ldots, A_n \in C$ such that $Y \subseteq A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$.

If the members of $C$ are $S_X$-open then $X$ is called $S_X$-compact.

In particular, if $Y = X$, then $X$ is correspondingly called ps-compact and $S_X$-compact.

Suppose $Y$ is a ps-compact subset of $X$.

Let $C = \{A_i\}_{i \in \Lambda}$ be a class of $S_X$-open subsets of $X$ such that $Y \subseteq \bigcup_{i \in \Lambda} A_i$.

But every $S_X$-open subset of $X$ being ps-open, each $A_i$ is ps-open.

Now there exist $A_1, A_2, A_3, \ldots, A_n \in C$ such that $Y \subseteq A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$, as $Y$ being ps-compact.
Thus $Y$ is $S_x$-compact.

On the contrary, assume that $Y$ is a $S_x$-compact subset of $X$. Let $D = \{D_i\}_{i \in A}$ be a class of ps-open subsets of $X$ such that $Y \subseteq \bigcup_{i \in A} D_i$.

Now, for each $i, D_i = \bigcup_{j \in A} A_{ij}$, where $A_{ij}$ is an $S_x$-open subset of $X$ contained in $D_i$.

Thus we get a class $P = \{A_{ij}\}_{i, j \in A}$ of $S_x$-open subsets of $X$ such that $Y \subseteq \bigcup_{i \in A} \bigcup_{j \in A} A_{ij}$.

But $X$ being $S_x$-compact, we get $A_{1j_1}, A_{2j_2}, \ldots, A_{nj_n} \in P$ such that

\[
Y \subseteq A_{1j_1} \cup A_{2j_2} \cup \ldots \cup A_{nj_n} \subseteq D_{i_1} \cup D_{i_2} \cup \ldots \cup D_{i_n}.
\]

Thus $Y$ is ps-compact and hence we note the following:

4.1.34 Note [[12]]: The two notions of $S_x$-open and ps-open are compact invariant.

We know that $X$, (topological) space is disconnected if and only if there is a nonempty proper subset of $X$, which is both open and closed.

We are now at a position to describe our result in terms of so-called topological sense as follows:

4.1.35 Definition: $X$ is ps-disconnected if it has a non-empty proper subset, which is both ps-open and ps-closed.
For instance in section five of this chapter we have the radical \( J(E) \) of the \( N \)-group \( E \) which is ps-open as well as ps-closed and hence \( E \) is ps-disconnected.

The concept so-called \( E \)-bounded which may be termed as a generalization [[11]] of what Beidleman has dealt with in the discussion of boundedness in case of a topological near-ring [3] can be extended to so-called \((S_E - E)\)-bounded.

**4.1.36 Definition:** A subset \( B \) of \( N \) is called \((S_E - E)\)-bounded if for any \( S_E \)-open subset \( U \) of \( E \) containing \( 0 \) (zero of \( E \)) there exists an \( S_E \)-open subset \( V \) of \( E \) such that \( BV \subseteq U \).

If \( B = N \), then \( N \) is itself \((S_E - E)\)-bounded.

**4.1.37 Example:** In the near-ring \( N \) of Klein’s four group w.r.t the multiplication defined in Table 2.11 with \( S_N = \{ \{0, a\}, \{b, c\} \} \), we note that, for any subset \( L(\neq 0) \) of \( N \), \( L\{ b,c \} = \{ a, a \} \) an \( S_N \)-open subset of \( N \) containing \( 0 \) (zero of \( N \)).

It is important to note for our purpose that for obvious reason, we should follow the subsequent extension of what has been described above as follows:

A subset \( B \) of \( N \) is \((S_E - E)\)-bounded where \( E = \bigoplus_{i=1}^{n} E_i \) each \( E_i \) is an \( N \)-group, if for any \( S_E \)-open subset \( \bigoplus_{i=1}^{n} V_i \) of \( \bigoplus_{i=1}^{n} E_i \) containing \( 0 \) (zero of \( E \)), there exists an \( S_E \)-open subset \( \bigoplus_{i=1}^{n} U_i \) of \( \bigoplus_{i=1}^{n} E_i \) such that \( \bigoplus_{i=1}^{n} BU_i \subseteq \bigoplus_{i=1}^{n} V_i \).
It is clear that if a subset $B$ of $N$ is $(S_E - E)$-bounded where $E = \bigoplus_{i=1}^{n} E_i$, then $B$ is $(S_{E_i} - E_i)$-bounded for each $i$.

The notion of topologically nilpotent subset of the near-ring due to Beidleman and Cox has been extended to so-called topologically nilpotent subset of the $N$-group [[11]] and same would be done in $S_E$ with the help of the $S_E$-open set, so-called $S_E$-nilpotent set as follows:

4.1.38 Example: We observe the near-ring $N (= Z_8)$ of 2.2.4 with $S_N = \{\{0, 4\}, \{1, 5\}, \{2, 6\}, \{3, 7\}\}$ that the left $N$-subsets of the near-ring $N$ are $\{0, 4\}$ and $\{0, 2, 4, 6\}$.

Now for the subset $L (= \{0, 2\})$ of $N$ we have $\{0, 2, 4, 6\}L = \{0, 4\}$ and $\{0, 2, 4, 6\}^2L = 0$, which belongs to every $S_N$-open subset of $N$ (containing 0).

This enables us to define the following:

4.1.39 Definition: A subset $\bigoplus_{i=1}^{n} D_i$ of $E$ where $E = \bigoplus_{i=1}^{n} E_i$ is $S_E$-nilpotent if for any $S_E$-open subset $\bigoplus_{i=1}^{n} U_i$ of $E$ containing 0, there exists a left $N$-subset $C_1 \cup C_2 \cup \ldots \cup C_n$ of $N$ such that $\bigoplus_{i=1}^{n} C_i D_i \subseteq \bigoplus_{i=1}^{n} U_i$, for some $t \in \mathbb{Z}^+.$

4.2 $(S_X-S_X)$ maps and associated concepts:

In this section we discuss the answer to the question that is at the beginning put in case of a two sided continuity and one sided
continuity separately in terms of so-called \((S_x \times S_x - S_x)\) and \((S_x - S_x)\) maps:

**Observations:**

4.2.1 [[12]]: We observe that if \((G, +)\) is a topological group, \(A\) is the binary operation and \(T\) is a topology such that \(A : G \times G \to G\) is continuous at \((a, b)\) then the maps \(_aA : G \to G\) where \(_aA(x) = a + x\) and \(_bA : G \to G\) where \(_bA(x) = x + b\) for all \(x \in G\)
are both \(T\)-continuous at \(b\) and \(a\) respectively.
But the converse may not be true.
In other words if \(_aA\) and \(_bA\) are both \(T\)-continuous then \(A : G \times G \to G\) need not be continuous.

4.2.2 [[12]]: It is obvious that if \(f\) is an \((S_x \times S_x - S_x)\) map at a point say, \((a, b) \in X \times X\), then the mappings \(_a f : X \to X\) and \(_b f : X \to X\) defined by \(_a f(x) = f(a, x)\) and \(_b f(x) = f(x, b)\) are \((S_x - S_x)\) maps at \(b\) and \(a\) respectively.
But the converse may not be true.

4.2.3 Example: We consider \(X = \{a, b, c\}, S_x = \{\{a, b\}, \{c\}\}\) and a mapping \(f : X \times X \to X\)
defined by \(f(a, a) = a, f(a, b) = b, f(a, c) = c, f(b, a) = c, f(b, b) = b, f(b, c) = a, f(c, a) = a, f(c, b) = c, f(c, c) = c.\)
Here we get \(_a f\) and \(_b f\) are \((S_x - S_x)\) maps at \(b\) and \(a\) respectively but \(f\) is not an \((S_x \times S_x - S_x)\) map at \((a, b)\).
As for any \(\{a, b\} \in S_X\) containing \(af(b)\) we get \(\{a, b\} \in S_X\) containing \(b\) (\(\in X\)) such that \(af(\{a, b\}) = \{a, b\}\).

In other words \(af\) is an \((S_X - S_2)\) map at \(b\).

But \(f(\{a, b\} \times \{a, b\}) = \{a, b, c\} \subset \{a, b\}\).

In other words \(f\) is not an \((S_X \times S_X - S_X)\) map at \((a, b)\).

4.2.4 At first, it was noticed that the converse of the above appears as false at least when \(S\) is a topology.

Each of the following examples reveals some characteristics of the binary operation of a group w.r.t. the respective subclasses of the power set of the group.

4.2.5 Example: In the symmetric group \((S_3, +)\) w.r.t. addition defined in Table 1.4 let \(S_{S_3} = \{(a, c), (b, c), (x, y)\}\). Here for \(\{x, y\}\) (\(\in S_{S_3}\)) containing \(y = a + b\) we have \(\{b, c\} \in S_{S_3}\) containing \(b\) and \(\{a, c\} \in S_{S_3}\) containing \(a \in S_3\) such that

\[a + \{b, c\} = \{x, y\}\]
\[\text{and } \{a, c\} + b(= \{y\}) \subset \{x, y\}\]

but \(\{a, c\} + \{b, c\} (= \{0, x, y\}) \not\subset \{x, y\}\).

4.2.6 Example: In the group \((Z_8, +)\) w.r.t. the addition modulo 8, consider \(S_{Z_8} = \{\{2, 3\}, \{2, 4\}, \{5, 6, 7\}\}\).

Now for \(\{5, 6, 7\} \in S_{Z_8}\) containing \(7 = 3 + 4\) (\(\in Z_8\)) there exists \(\{2, 4\} \in S_{Z_8}\) containing \(4 \in Z_8\) such that

\[3 + \{2, 4\} (= \{5, 7\}) \subset \{5, 6, 7\}\]

and also there exists \(\{2, 3\} \in S_{Z_8}\) containing \(3\) such that
\{2, 3\} + 4 (= \{6, 7\}) \subset \{5, 6, 7\} but
\{2, 3\} + \{2, 4\} (= \{4, 5, 6, 7\}) \not\subset \{5, 6, 7\}.

4.2.7 Example: In the dihedral group \((D_8, +)\) w.r.t. the addition defined by the Table 2.3 with
\[ S_{D_8} = \{\{a, 2a\}, \{b, 3a + b\}, \{a + b, 2a + b\}\} \]
we see that for \{a + b, 2a + b\}(\in S_{D_8}) containing 2a + b (\in D_8) there exists \{b, 3a + b\}(\in S_{D_8}) containing b (\in D_8) such that
\[ 2a + \{b, 3a + b\} = \{a + b, 2a + b\} \text{ and} \]
also there is \{a, 2a\}(\in S_{D_8}) containing 2a (\in D_8) such that
\[ \{a, 2a\} + \{b, 3a + b\} = \{a + b, 2a + b\} \]
but \{a, 2a\} + \{b, 3a + b\} (= \{a + b, b, 2a + b\}) \not\subset \{a + b, 2a + b\}.

The above examples bear out our inquiry about the two sided as well as unbalanced (non-symmetric) continuity problems and reveal that we require some condition(s) that may lead us to what we are looking for.

4.2.8 Example: Let us consider the symmetric group \((S_3, +)\) with
\[ S_{S_3} = \{\{0, b\}, \{a, y\}, \{b, c, x, y\}\}. \]
Here we note that \(a + \{0, b\} (= \{a, y\}) \in S_{S_3}\) but
\[ \{0, b\} + c (= \{c, y\}) \not\in S_{S_3} ; \]
\[ \{0, b\} + \{0, b\} = \{0, b\}; \]
for \{b, c, x, y\}(\in S_{S_3}) containing \(c (= 0 + c) (\in S_3)\)
there exists \{0, b\}(\in S_{S_3}) containing 0 (\in S_3) such that
\[ \{0, b\} + c (= \{c, y\}) \subset \{b, c, x, y\}; \]
also for \(\{b, c, x, y\}(\in S_{S_3})\) containing \(x (= a + c) \(\in S_3)\) we get
\(a + \{b, c, x, y\} = \{b, c, x, y\}\) but \(\{a, y\} + \{b, c, x, y\} \not\subset \{b, c, x, y\}\).

4.2.9 Example: Let us consider
\(S_{Z_8} = \{0, 6\}; \{0, 7\}; \{2, 4\}; \{3, 5\}; \{3, 4\}; \{4, 5\}; \{1, 7\}\) in the group \((Z_8, +)\).

Here for any \(W(\in S_{Z_8})\) containing \(0 (= 0 + 0) \(\in Z_8)\)
we get \(4 + W, W + 5 \in S_{Z_8}\);
but no \(U\) and \(V(\in S_{Z_8})\) containing \(0 \(\in Z_8)\) such that \(U + V \subseteq W\).

Again for \(\{3, 5\}\) and \(\{4, 5\}(\in S_{Z_8})\) containing \(5 (= 0 + 5) \(\in Z_8)\)
there exists \(\{0, 6\}\) or \(\{0, 7\}(\in S_{Z_8})\) containing \(0 \(\in Z_8)\) such that
\(\{0, 6\} + 5 = \{3, 5\}\) and \(\{0, 7\} + 5 = \{4, 5\}\).
And we have for \(\{1, 7\}(\in S_{Z_8})\) containing \(1 (= 4 + 5) \(\in Z_8)\)
there exists \(\{3, 5\}(\in S_{Z_8})\) containing \(5 \(\in Z_8)\) such that
\(4 + \{3, 5\} = \{1, 7\}\)
but no \(U\) and \(V(\in S_{Z_8})\) containing \(4\) and \(5 \(\in Z_8)\) respectively
such that \(U + V \subseteq \{1, 7\}\).

4.2.10 Example: Let \(S_{D_8} = \{0, b\}, \{a, 3a + b\}, \{2a, 2a + b\},\)
\(\{3a, 3a + b\}\) in the dihedral group \((D_8, +)\).

It is clear that
\(3a + \{0, b\} (= \{3a, 3a + b\})\); \(\{0, b\} + 3a + b (= \{a, 3a + b\})\in S_{D_8}\);
\(\{0, b\} + \{0, b\} = \{0, b\}\);
for \{3a, 3a + b\}(\in S_D^g) containing 3a + b (= 0 + 3a + b) \(\in D_8\)

there exists no \(V(\in S_D^g)\) containing 0(\(\in D_8\)) such that
\(V + 3a + b \subseteq \{3a, 3a + b\}\).

Again for \{2a, 2a + b\} \(\in S_D^g\) containing 2a + b (= 3a + 3a + b) \(\in D_8\) there exists \{3a, 3a + b\}(\(\in S_D^g\)) containing 3a + b(\(\in D_8\)) such that
\(3a + \{3a, 3a + b\} = \{2a, 2a + b\}\)

but no \(U\) and \(V(\in S_D^g)\) containing 3a and 3a + b(\(\in D_8\)) respectively such that \(U + V \subseteq \{2a, 2a + b\}\).

4.2.11 Example: Taking \(S_S^3 = \{\{0, x\}, \{a, c\}, \{b, c\}, \{x, y\}\}\) in the symmetric group \((S_3, +)\), we see that
\(a + \{0, x\}(=\{a, c\}), \{0, x\} + b(=\{b, c\})\in S_S^3; \) for \{0, x\}(\(\in S_S^3\)) containing
0 (= 0 + 0) \(\in S_3\) there exist no \(U\) and \(V(\in S_S^3)\) containing 0(\(\in S_3\)) such that \(U + V \subseteq \{0, x\}\);

for \{b, c\}(\(\in S_S^3\)) containing b(= 0 + b) there exists \{0, x\}(\(\in S_S^3\)) containing 0(\(\in S_3\)) such that \{0, x\} + b = \{b, c\}.

Again for \{x, y\}(\(\in S_S^3\)) containing y( = a + b) \(\in S_3\) there exists
\{b, c\}(\(\in S_S^3\)) containing b(\(\in S_3\)) such that
\(a + \{b, c\} = \{x, y\}\) but
\(\{a, c\} + \{b, c\} (= \{0, x, y\}) \not\subseteq \{x, y\}\).

4.2.12 Example: In the dihedral group \((D_8, +)\), consider
\(S_D^g = \{\{0, 2a + b\}, \{3a, a + b\}, \{a, 3a + b\}\}\).
It is seen here that $3a + \{0, 2a + b\} (= \{3a, a + b\}) \in \mathcal{S}_{\mathcal{D}_b}$, but $\{0, 2a + b\} + 3a + b (= \{3a, 3a + b\} \not\in \mathcal{S}_{\mathcal{D}_b}$;
\{0, 2a + b\} + \{0, 2a + b\} = \{0, 2a + b\};
for $\{a, 3a + b\} (\in \mathcal{S}_{\mathcal{D}_b})$ containing $3a + b (= 0 + 3a + b) (\in \mathcal{D}_8)$, there exists no $V (\in \mathcal{S}_{\mathcal{D}_b})$ containing $0 (\in \mathcal{D}_8)$ such that
$V + 3a + b \subseteq \{a, 3a + b\}$. 
Again for $\{0, 2a + b\} (\in \mathcal{S}_{\mathcal{D}_b})$ containing $2a + b (= 3a + 3a + b) (\in \mathcal{D}_8)$ there exists $\{a, 3a + b\} (\in \mathcal{S}_{\mathcal{D}_b})$ containing $3a + b (\in \mathcal{D}_8)$ such that
$3a + \{a, 3a + b\} = \{0, 2a + b\}$
but $\{3a, a + b\} + \{a, 3a + b\}(= \{0, b, 2a, 2a + b\}) \not\subset \{0, 2a + b\}$.

It is of worth mentioning that each of the above examples gives rise some different characteristics of the binary operation of a group, say $X$, but these are not enough to reach our goal. In real sense the binary operation on $X$ is not an $\mathcal{S}_{x \times \mathcal{S}_{x}}$ map at a point of $X \times X$.

**4.2.13 Theorem [[12]]:** If a mapping $f : X \times X \to X$ satisfies the conditions
(a) for some $e (\in X)$, $f(e, x) = x = f(x, e)$.
(b) $f(x, f(y, z)) = f(f(x, y), z)$ for all $x, y, z \in X$ along with the following conditions at $(a, b) (\in X \times X)$.
1) for any $V (\in \mathcal{S}_X)$ containing $e$ we have $a_f(V), f_b(V) \in \mathcal{S}_X$. 
2) $f$ is an $\mathcal{S}_{X \times \mathcal{S}_{X-S_x}}$ map at $(e, e)$. 
3) $f_b$ is an $\mathcal{S}_{X-S_x}$ map at $e$. 
4) $a_f$ is an $\mathcal{S}_{X-S_x}$ map at $b$, 
then $f$ is a $\mathcal{S}_{X \times \mathcal{S}_{X-S_x}}$ map at $(a, b)$.
Proof: For any $W(\in S_X)$ containing $f(a, b)$ ($= a f(b) \in X$) we get by (iv), $V(\in S_X)$ containing $b$ such that

$$a f(V) \subseteq W \quad ...(*).$$

We have, by (a), $f(e, b) = b = f_b(e) \in X$ giving thereby $V(e X)$ contains $f_b(e)$.

Hence, by (iii), there exists $U(\in S_X)$ containing $e$ such that

$$f_b(U) \subseteq V \quad ...(**).$$

And by (b), $f(e, e) = e$ and hence from (ii) we get $A \times B(\in S_X \times S_X)$ containing $(e, e)$ such that

$$f(A \times B) \subseteq U \quad ...(***).$$

Finally we get

$$f(a f(A) \times f_b(B)) = f(a f(A) \times f(A \times f(B \times \{ b \})))$$

$$= f(a f(A) \times f(f(A \times f(B \times \{ b \}))))$$

$$= f(a f(A) \times f(f(A \times f(B) \times \{ b \})))$$

$$\subseteq f(a f(U) \times \{ b \}))$$

$$= f(a f(U))$$

$$\subseteq f(a \times V), \text{ by [***].}$$

$$= a f(V)$$

$$\subseteq W, \text{ by [*].}$$

Therefore $f$ is a $(S_X \times S_X \times S_X)$ map at $(a, b)$. ♦♦♦

It is obvious that in the above examples, if $f$ is a binary operation then the condition (i) does not hold good in Ex 4.2.8 and 4.2.12; condition (ii) does not hold good in Ex 4.2.9 and 4.2.11; also Ex 4.2.10 and 4.2.12 do not follow the condition (iii).

Thus we arrive to conclude that to become $f$ as an $(S_X \times S_X \times S_X)$ map at $(a, b)(\in X \times X)$, $(a)(b),(i),(ii),(iii)$ together with (iv) are the
conditions to be satisfied. And we claim it as a very important and innovatively exciting sufficient exposition!

4.2.14 **Theorem** [[12]]: If two mappings $f : X \times X \to X$ and $g : X \times Y \to Y$ where $X$ is equipped with a binary operation having an identity $e$ (which is also a scalar multiplicative identity with respect to $Y$) and $g(f(n, m), y) = g(n, g(m, y))$ for all $n, m \in X$ and $y \in Y$ satisfy the following conditions (called a two sided $S$-system) at $(x, y) \in X \times Y$.

i) for any $V(e \in S_x)$ containing $e$ we have $x f(V) \in S_x, g_y(V) \in S_y$.

ii) $f$ is an $(S_x \times S_x - S_x)$ map at $(e, e)$.

iii) $g_y$ is an $(S_x - S_y)$ map at $e$.

(iv) $x g$ is an $(S_y - S_y)$ map at $y$, then $g$ is a $(S_x \times S_y - S_y)$ map at $(x, y)$.

**Proof:** Now for any $W(e \in S_y)$ containing $g(x, y) (= x g(y))$ we get, by (iv), $V(e \in S_y)$ containing $y$ such that

$$x g(V) \subseteq W \quad \ldots(*)$$

Again, as $e$ being a scalar multiplicative identity w.r.t $y$, so $g_y(e) \in V$.

Because of $S$-system character, there exists $U(e \in S_x)$ containing $e(e \in X)$ such that

$$g_y(U) \subseteq V \quad \ldots(**)$$

and therefore we get, $A \times B (e \in S_x \times S_x)$ containing $(e, e)(e \in X \times X)$ such that

$$f(A \times B) \subseteq U \quad \ldots(***)$$

At last, $g(x f(A) \times g_y(B)) = g((f(x) \times A) \times g(B \times \{y\}))$

$$= g(x) \times g(A \times g(B \times \{y\})), \text{ (by hypothesis).}$$
\[ g(\{x\} \times g(f( A \times B) \times \{y\})) \text{, (by hypothesis).} \]
\[ \subseteq g(\{x\} \times g( U \times \{y\})) \text{, [by(***)].} \]
\[ = g(\{x\} \times g_U( U)) \]
\[ \subseteq g(\{x\} \times V) \text{, [by(*)].} \]
\[ = xg( V), \]
\[ \subseteq W, [by (**)]. \]

Therefore \( g \) is an \((S_X \times S_Y - S_Y)\) map at \((x, y)\). ♦♦♦

**Table 4.1**

<table>
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<th>(E \rightarrow E)</th>
<th>0</th>
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<th>b</th>
<th>c</th>
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<td>(f_0)</td>
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<td>(f_{15})</td>
<td>0</td>
<td>c</td>
<td>0</td>
<td>c</td>
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</table>

From the following examples (where the last one of the above mentioned system is present in each of them) it is clear that, in case of Klein 4–group \( a \) two–sided \( S \)-system is a sufficient condition for the said makeup, because absence of any one of them nullifies the same.

Let \( E = \{0, a, b, c\} \) be the Klein 4–group w.r.t addition defined in Table 1.2 and consider the mappings on \( E \) defined by the Table 4.1.
It follows that \( N = \{ f_0, f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12}, f_{13}, f_{14}, f_{15} \} \) is a right near-ring with unity \( f_1 \) with respect to the following operations:

\[
(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (f \cdot g)(x) = f(g(x)) \quad \text{for all } f, g \in N, x \in E. \quad \ldots \quad (*)
\]

Here we observe the near-ring group structure of \( E \) over \( N \) with respect to the operation \( N \times E \to E \)

\[
(f_p, e) \mapsto f_p(e) \quad \ldots \quad (**)
\]

**4.2.15 Example:** Consider \( S_N = \{ \{ f_9, f_{10} \} \} \) and \( S_E = \{ \{ a, b \}, \{ b, c \} \} \).

Here for \( \{ b, c \} (\in S_E) \) containing \( c (= f_{10}(a)) (\in E) \) we have \( \{ a, b \}(\in S_E) \) containing \( a(\in E) \) and \( \{ f_9, f_{10} \}(\in S_N) \) containing \( f_{10} (\in N) \) such that \( f_{10}\{a, b\} = \{b, c\} \) and \( \{f_9, f_{10}\}\{a\}(= \{c\}) \subset \{b, c\} \) but \( \{f_9, f_{10}\}\{a, b\}(= \{0, b, c\}) \not\subset \{b, c\} \).

**4.2.16 Example:** Consider \( S_N = \{ \{ f_9, f_{12} \} \} \) and \( S_E = \{ \{ a, b \}, \{ b, c \} \} \).

Here for \( \{ b, c \} (\in S_E) \) containing \( c (= f_{12}(a)) (\in E) \) we have \( \{ a, b \}(\in S_E) \) containing \( a(\in E) \) and \( \{ f_9, f_{12} \}(\in S_N) \) containing \( f_{12} (\in N) \) such that \( f_{12}\{a, b\} = \{b, c\} \) and \( \{f_9, f_{12}\}\{a\} = \{b, c\} \) but \( \{f_9, f_{12}\}\{a, b\}(= \{0,b,c\}) \not\subset \{b, c\} \).

**4.2.17 Example:** Let \( S_N = \{ \{ f_9, f_{10} \}, \{ f_1, f_4 \} \} \) and \( S_E = \{ \{ a, b \}, \{ b, c \} \} \).

Here we note that \( f_{10}\{f_1, f_4\}(= \{f_9, f_{10}\})\in S_N \) but \( \{f_1, f_4\}\{a\}(= \{a\}) \not\in S_E \); \( \{f_1, f_4\}\{f_1, f_4\} = \{f_1, f_4\} \);
for \( \{a, b\}(\in S_E) \) containing \( a (= f_1(a)) \) \( (\in E) \) there exists \( \{f_1, f_3\}(\in S_N) \) containing \( f_1 \) \( (\in N) \) such that \( \{f_1, f_3\}(a) (=\{a\}) \subset \{a, b\} \); also for \( \{b, c\}(\in S_E) \) containing \( c (= f_{10}(a)) \) \( (\in E) \) we get
\[ f_{10}(\{a, b\}) = \{b, c\} \]
but \( \{f_9, f_{10}\}(a, b)(=\{0, b, c\}) \not\subset \{b, c\} \).

4.2.18 Example: Let us consider \( S_N = \{\{f_1, f_3\}, \{f_8, f_{12}\}\} \) and \( S_E = \{\{a, b\}, \{b, c\}\} \). Here we note that \( f_{12}(f_1, f_3)(=\{f_8, f_{12}\})\in S_N \)
and \( \{f_1, f_3\}(a)(=\{a, b\})\in S_E \).
Now for any \( W(\in S_N) \) containing \( f_1 (= f_1.f_1) \) \( (\in N) \)
we get no \( U \) and \( V(\in S_N) \) containing \( f_1(\in N) \) such that \( UV \subseteq W \). Again for \( \{a, b\}(\in S_E) \) containing \( a (= f_1(a)) \) \( (\in E) \) there exists \( \{f_1, f_8\}(\in S_N) \) containing \( f_1(\in N) \) such that \( \{f_1, f_8\}(a)=\{a, b\} \).
And we have for \( \{b, c\}(\in S_E) \) containing \( c (= f_{12}(a)) \) \( (\in N) \) there exists \( \{a, b\}(\in S_E) \) containing \( a(\in E) \) such that \( f_{12}(\{a, b\}) = \{b, c\} \) but \( \{f_8, f_{12}\}(a, b)(=\{0, b, c\}) \not\subset \{b, c\} \).

4.2.19 Example: For \( S_N = \{\{f_1, f_{12}\}, \{f_7, f_{15}\}\} \) and \( S_E = \{\{a, c\}, \{0, a, b\}\} \) it is seen here that
\[ f_7(f_1, f_{12})(=\{f_7, f_{15}\})\in S_N \]
\[ \{f_1, f_{12}\}(\in S_E) \]
for \( \{0, a, b\}(\in S_E) \) containing \( a (= f_1(a)) \) \( (\in E) \), there exists no \( V \) \( (\in S_N) \) containing \( f_1(\in E) \) such that \( Va \subseteq \{0, a, b\} \).
Again for \( \{a, c\}, \{0, a, b\}(\in S_E) \) containing \( a (= f_7(a)) \) \( (\in E) \) we get
\[ f_7(\{a, c\}) = \{a, c\} \]
and \( f_7(\{0, a, b\})(=\{0, a\})(\in \{0, a\}) \subset \{0, a\} \)
but \( \{f_7, f_{13}\}(a, c) = \{a, c\}, \{f_7, f_{13}\}(0, a, b)(=\{0, a, c\}) \not\subset \{0, a, b\} \).
It is noticed in the above Ex 4.2.15, 4.2.16, 4.2.17, 4.2.18 and 4.2.19 that if $f$ is the binary operation (multiplication) on the near-ring $N$ and $g$ is the scalar multiplication on the $N$-group $E$ defined as in (*) and (**) above, the condition (i) does not hold good in Ex 4.2.17; condition (ii) does not hold good in Ex 4.2.18; also Ex 4.2.19 do not follow the condition (iii). Thus all these justify what we have evaluated.

The authors foresee some possible link between what have been presented above and what follows from the results established below (so far the supposed symmetric $(S_X \times S_X - S_X)$ and $(S_X \times S_Y - S_Y)$ maps are concerned).

### 4.3 Quasi-regular cyclic $S_{Ne}$-open $N$-groups:

In what follows we assume that both $Q$ and $Q_e$ are $S_N$-open and $S_{Ne}$-open subsets of $N$ and $Ne$ respectively together with the zero annihilator of the corresponding $e$ which occurs in 2.4.24 as a necessity of near-ring group with so-called Goldie character. In this space biased near-ring group with acc on annihilators we study ps-closeness of direct sum of maximal $N$-subgroups together with the ps-closeness of the direct sum of group sum of ideals related to quasi-regular left ideal of $N$. Moreover the radical of the $N$-group $E\left(=\bigoplus_{i=1}^n Ne_i\right)$ coincides with the radical subgroup if such type of direct sum is $S_\mathbb{E}$-dense in the radical. Also such type of conditions helps us to get the cyclic character of $E$. 

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4.3.1 Theorem [[12]]: If \( \bigoplus_{i=1}^{n} B_{i} e_{i} \) is a maximal \( N \)-subgroup (ideal) of \( \bigoplus_{i=1}^{n} N_{e_{i}} \), then \( \bigoplus_{i=1}^{n} B_{i} e_{i} \) is a ps-closed \( N \)-subgroup (ideal) of \( \bigoplus_{i=1}^{n} N_{e_{i}} \).

**Proof:** One component is sufficient to show the result because of 4.1.15.

Suppose \( B e \) is a maximal \( N \)-subgroup (ideal) of \( N e \).

By 3.1.9, \( B \) is a maximal \( N \)-subgroup (ideal) of \( nN \).

We are to show that \( B e = \overline{B e} \).

As \( B e \) being \( N \)-subgroup (ideal) of \( N e \), so by 4.1.28, \( \overline{B e} \) is also \( N \)-subgroup (ideal) of \( N e \) such that \( B e \subseteq \overline{B e} \subseteq N e \).

Again, \( B e \) being maximal, we get \( B e = \overline{B e} \) or \( \overline{B e} = N e \).

Suppose, \( \overline{B e} = N e \).

\( \Rightarrow e \in \overline{B e} \), as \( e \in N e \) and \( 1 \in N \).

We have, by 4.1.26, \( e - Q e \) as a ps-open subset of \( N e \) containing \( e \).

Hence, by 4.1.19, \( (e - Q e) \cap B e \neq \emptyset \).

\( \Rightarrow (1 - Q)e \cap B e \neq \emptyset , \) as \((e - Q e) = (1 - Q)e \).

Now, as \( \operatorname{Ann}(e) = 0 \), we get, \((1 - Q)e \cap B e = ((1 - Q) \cap B)e \).

So, \( ((1 - Q) \cap B)e \neq \emptyset \).

\( \Rightarrow (1 - Q) \cap B \neq \emptyset , \) as \( \operatorname{Ann}(e) = 0 \), giving thereby \( 1 \in B \)

as, \( 1 = nb = n(1 - q) \), for some \( q \in Q \) and \( b \in B \), which is not true.

Hence \( B e = \overline{B e} \).

In other words, \( B e \) is a ps-closed \( N \)-subgroup (ideal) of \( N e \).

Hence, by 4.1.15, \( \bigoplus_{i=1}^{n} B_{i} e_{i} \) is a ps-closed \( N \)-subgroup (ideal) of \( \bigoplus_{i=1}^{n} N_{e_{i}} \).

\[\cdots\cdots\]
The $S_N$-open and $S_{Ne}$-open characteristics of $Q$ and $Qe$ respectively with $\text{Ann } (e) = 0$ and being arbitrary intersection of ps-closed sets are ps-closed, we get the following:

4.3.2 Lemma [[12]]: Both $J(\text{Ne})$ and $Ae$ are ps-closed.

Henceforth we assume that $Se$ denotes the group sum of all ideals of $\text{Ne}$ of the form $Je$ where $I$ is a quasi-regular left ideal of $N$.

4.3.3 Theorem [[12]]: $\bigoplus_{\pi} Se$ is a ps-closed ideal of $\bigoplus_{\pi} Ne$.

Proof: First we see the result for one component.

We have, by 1.1.54 & 1.1.55, $S \subseteq A \subseteq Q$, where $S$ is the group sum of all quasi-regular left ideals of $N$.

Thus we get $Se \subseteq Ae \subseteq Qe$.

So, $Se \subseteq Ae = Ae$, [by 4.3.2].

$\subseteq Qe$.

Now $Se = \{ qe \mid \text{for some } q \in Q \}$ and consider $I = \{ p \mid p \in Q, pe \in Se \}$.

We have by 4.1.28 that $Se$ is an ideal of $\text{Ne}$ such that $Se = Ie$, where $I$ is a quasi-regular left ideal of $N$.

So, $Se \subseteq Se$, by definition of $Se$.

Thus $Se$ is a ps-closed ideal of $Ne$.

Consequently, by 4.1.15, $\bigoplus_{\pi} Se$, is also ps-closed ideal of $\bigoplus_{\pi} Ne$.

\hfill \diamond \diamond \diamond
4.3.4 Theorem [[12]]: If \( \bigoplus_{i=1}^{n} S_{e_i} \) is \( S_e \)-dense in \( J(\bigoplus_{i=1}^{n} Ne_i) \), then
\[
J(\bigoplus_{i=1}^{n} Ne_i) = \bigoplus_{i=1}^{n} Ae_i.
\]

Proof: We have, as in the proof of Theorem 4.3.3, \( Se \subseteq Ae \).

Now \( Ae \subseteq J(Ne) \), as \( A \subseteq J(N) \) and \( J(N)e = J(N)e \) [by 3.1.10].

Also \( J(Ne) = Se \)
\[
\subseteq Ae
\]
\[
\subseteq J(Ne).
\]

\( \Rightarrow J(Ne) = Ae. \)

Thus, by 1.1.49, \( J(\bigoplus_{i=1}^{n} Ne_i) = \bigoplus_{i=1}^{n} Ae_i \).

In the following theorem we assume that our \( N \)-group is of the form \( E \).

4.3.5 Theorem [[12]]: Let \( E \) be an \( N \)-group such that every maximal \( N \)-subgroup of \( E \) is of the form \( Be \) with \( \text{Ann}(e) = 0 \). If \( E \) contains no proper ps-closed \( N \)-subgroup, then \( Ne = E \).

Proof: We have \( Ne \subseteq E \).
Suppose \( Ne \neq E \).
So, \( Ne \) is a non-zero proper \( N \)-subgroup of \( E \).

Let \( H = \{ I \mid I \text{ is an } N \text{-subgroup of } E \text{ such that } Ne \subseteq I \} \).

But \( H \neq \emptyset \), as \( Ne \in H \).

Clearly, \( H \) is a partial ordered set with partial ordering \( \subseteq \). The union of each chain of \( H \) is again an \( N \)-subgroup of \( E \) containing \( Ne \). Hence by Zorn's Lemma 1.1.44, \( H \) has a maximal element, say
such that \( N e \subseteq L \). By hypothesis \( L = Be \) and as in 4.3.1, \( Be \) is a proper ps-closed \( N \)-subgroup of \( E \) which contradicts the assumption. Hence \( Ne = E \). ♦♦♦

We note that the near-ring \( N \) in 2.4.8 of Klein 4-group with \( S_N = \{\{0, a\}, \{b, c\}\} \) has a maximal left \( N \)-subgroup \( B (= \{0, a\}) \). Again \( B b = \{0, a\} \) is a maximal left \( N \)-subgroup of \( N \) with \( \ell(b) = 0 \) such that \( Nb = N \).

Thus the above example is sufficient to explain the point that the converse of the above theorem is not true.

In case of an \( E \), not necessarily of the type \( \bigoplus_{i \in \mathbb{N}} Ne_i \), we have that \( S_E \)-openness of \( Q e \) and \( (S_E - E) \)-boundedness of \( N \) lead us to the following expected results, mainly on ps-openness of \( J(E) \) together with ps-discrete and finiteness of \( E \).

4.3.6 Theorem [[12]]: If \( N \) is \( (S_E - E) \)-bounded and \( E \) is with fully radical character, then \( J(E) \) is ps-open.

Proof: As \( Q e \) is \( S_E \)-open and \( N \) is \( (S_E - E) \)-bounded, there exists an \( S_E \)-open subset \( V \) of \( E \) such that

\[
 NV \subseteq Q e.
\]

Thus we can write \( NV = \{qe \mid \text{for some } q \in Q\} \).

Now \( V_1 = \{q \in Q \mid qe \in NV\} \).

We have \( V_1 \subseteq Q \) \hspace{1cm} ...(i).

And \( NV = V_1 e \), as \( \text{Ann}(e) = 0 \).

Now \( N(V_1 e) = N(NV) \)

\[
= N^2 V.
\]

\[
\subseteq NV = V_1 e.
\]

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\[ \Rightarrow NV_i \subseteq V_i, \text{ as } \text{Ann}(e) = 0. \]
So, \( NV_i = V_i, \text{ as } 1 \in N. \)
Moreover, for each \( x \in V_i, \text{ by } 1.1.26 \text{ and (i), } Nx \text{ is a quasi-regular left } N-\text{subgroup of } N. \)
And by 1.1.56, we get \( Nx \subseteq J(N). \)
Hence, \( x \in J(N), \text{ as } 1 \in N. \)
So, \( V_i \subseteq J(N). \) \hspace{1cm} \ldots(ii)
Thus we have \( V = 1.V. \)
\[ \subseteq NV. \]
\[ = V_i e. \]
\[ \subseteq J(N)e, \text{ [by(ii)].} \]
\[ \subseteq J(N)E. \]
\[ \subseteq J(E). \]
\[ \Rightarrow V \subseteq J(E). \]
If \( y \in J(E), z \in V \text{ then by } 4.1.26, y + ( - z) + V \text{ is a ps–open subset of } E \text{ containing } y(\in J(E)). \)
Moreover \( y + ( - z) + V \subseteq J(E). \)
Hence, by 4.1.19, \( J(E) \text{ is ps–open. } \star\star\star \)

4.3.7 Corollary: If \( N \) is \((S_{b} - E)–\text{bounded, } E \text{ is with fully radical character and } J(E) = 0, \text{ then } E \text{ is an ps–discrete.} \)

Proof: We have \( 0(= J(E)) \text{ is ps–open.} \)
So for \( x \in E; \{ x \} = x + 0 \text{ is ps–open, by } 4.1.26. \)
In other words every singleton subset of \( E \text{ is ps–open and hence } E \text{ is ps–discrete. } \star\star\star\star \).
The following example reveals that the vanishing of radical is essential for the ps-discreteness of $N$-group $E$ when $N$ is $(S^e - E)$-bounded.

The near-ring $N$ ($= \mathbb{Z}_8$) w.r.t. the multiplication defined by the Table 3.5. w.r.t. $S_N$ ($= \{\{0, 4\}, \{1, 5\}, \{2, 6\}, \{3, 7\}\}$ is $(S^e - N)$-bounded, $J(N) = \{0, 4\}$, $\ell(e) = 0$, where $e = 1, 2, 3, 5, 6, 7$ and $N$ is not ps-discrete.

**4.3.8 Corollary:** If $E$ is ps-compact having fully radical character, $N$ is $(S^e - E)$-bounded and $J(E) = 0$, then $E$ is finite.

**Proof:** We have, by above corollary $E$ is ps-discrete.

If $E$ is infinite then $E$ is not ps-compact.

Hence $E$ is finite. ♦♦♦

We now note the following:

(i) Now for a $S^e$-nilpotent subset $\bigoplus_{i=1}^n D_i$ of $E$, we get each $D_i$ is a $S^e$-nilpotent subset of $Ne$.
Again we have $Q e$ as a ps-open subset of $Ne$ containing $0$ (zero of $Ne$), so there exists $n \in \mathbb{Z}^+$ such that $C^n D' e \subseteq Q e$, for some left $N$-subgroup $C$ of $N$.

We get $C^n D' \subseteq Q$, as Ann (e) = 0.

Suppose if possible let $C^n D' \subset J(N)$.

Then $C^n D' \subset B$, for some left ideal $B$ of $N$ maximal as $N$-subgroup.

Hence $Ne c_1 c_2 \ldots c_n d' = B = N$ for some, $c_1 c_2 \ldots c_n d' (\notin B) \in C^n D'$, where $c_1, c_2, \ldots, c_n \in C$ and $d' \in D'$ and by 1.1.26.

And thus, $nc_1 c_2 \ldots c_n d' + b = 1$, where some $n \in N$ and $b \in B$. 179
Now \((C, \text{ being a left } N\text{–subset}), (nc_1c_2...c_n d' \in C^n D' (\subseteq Q)),\) we get \(1 \in B,\) as \(1 = n(1 - nc_1c_2...c_n d')\)
\[= nb\]
\(\in B,\) being left \(N\text{–subset of } N,\) a contradiction .

Hence \(C^n D' \subseteq J(N).\)

(ii) Suppose \(n > 1\) and \(C^{n-1} D' \not\subseteq J(N).\)

Now as above \(nc + b = 1,\) for some \(n \in N, b \in B\) and 
\(c (\notin B) \in C^{n-1} D'.\)

Again for any \(c_1c_2...c_{n-1}d' \in C^{n-1} D',\) where \(c_1, c_2, ..., c_{n-1} \in C, d' \in D\)
we get
\[c_1c_2...c_{n-1}d' = c_1c_2...c_{n-1}d'(nc + b) - c_1c_2...c_{n-1}d'nc + c_1c_2...c_{n-1}d'nc\]
\(\in B,\) as \(C,\) a left \(N\text{–subset giving thereby from above}\)
\(c_1c_2...c_{n-1}d'nc \in C^n D' \subseteq J(N)) \in B \text{ and } B, \text{ a left ideal.}\)

Hence \(C^{n-1} D' \subseteq B,\) a contradiction.

Therefore by induction, we get \(CD' \subseteq J(N)\) and by 3.1.10
we get \(CD \subseteq J(N).\)

Hence we get the following theorem establishing the link with the \(S_B\text{–nilpotent notion of a subset of } E\) and that of the radical of the \(N\text{–group as follows:}\)

4.3.9 Theorem [12]: If \(\bigoplus_{i=1}^n D_i \subseteq E\text{–nilpotent subset of } E,\) then \(\bigoplus_{i=1}^n C_i D_i \subseteq J(\bigoplus_{i=1}^n Ne_i),\) for some left \(N\text{–subset } C_1 \cup C_2 \cup ... \cup C_n\)
of \(N.\) ♦♦♦
4.4 Quasi-regular cyclic $S_{Ne}$-closed $N$-groups:

The ps-closeness of direct sum of the group sum of ideals related to quasi-regular left ideal of $N$ can be viewed when zero is the only element of $Ne_i$ that kills the $e_i$'s under the assumption that each of the $Q_{e_i}$ in $Ne_i$ with respect to the given $S_{Ne_i}$ is $S_{Ne_i}$-squeezed.

4.4.1 Theorem [[12]]: If $\text{Ann}(e_i) = 0$ and each $Q_{e_i}$ is $S_{Ne_i}$-closed in $Ne_i$ for each $i$, then $\bigoplus_{i=1}^{n} Q_{e_i}$ is a ps-closed ideal of $E$.

Proof: As above, the result for one component is sufficient. Now, by the proof of Theorem 4.3.3. $Se \subseteq Q_e$.

So $Se \subseteq Q_e$, as $Q_e$ is $S_{Ne}$-closed.

Again we have by the proof of the Theorem 4.3.3. $\overline{Se} \subseteq Se$.

Hence $Se$ is ps-closed. ♦♦♦

4.4.2 Theorem [[12]]: As in case of theorem 4.4.1, if $\bigoplus_{i=1}^{n} I_{e_i}$ is the unique maximal $N$-subgroup of $E$, then $\bigoplus_{i=1}^{n} I_{e_i}$ is ps-closed and each of $\bigoplus_{i=1}^{n} A_{e_i}$ and $J(E)$ is $\bigoplus_{i=1}^{n} I_{e_i}$.

Proof: Without loss of generality we see that, by hypothesis, $I_e$ is the unique maximal $N$-subgroup of $Ne$.

And $Q_e$ being $S_{Ne}$-closed, so $A_e = I_e$

$\subseteq \overline{A_e}$

$\subseteq Q_e$.

Again $Q_e$ is a proper subset of $Ne$, if not $Q = N$, as $\text{Ann}(e) = 0$, which is not true as $1 \notin Q$.  

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Hence by 4.1.28, we get \( J_e \) as ps-closed.

Again, for any left ideal \( B \) that is maximal as \( N \)-subgroup of \( N \), we have by 3.1.9, \( B e \), an ideal which is a maximal as \( N \)-subgroup.

Thus, by uniqueness of \( I_e \), we get \( B e = I_e \).

Hence \( J(Ne) = I_e \).

Thus \( \bigoplus_{i=1}^{n} I_i e_i \) is ps-closed and each of \( \bigoplus_{i=1}^{n} A e_i \) and \( J(E) \) is \( \bigoplus_{i=1}^{n} I_i e_i \).

**4.5 Pseudo nilpotent \( S_{Ne} \)-open \( N \)-groups:**

The notion of \( N_u \)-nilpotent element in \( N \)-group \( E \) gives the following results, some of which are analogous to those obtained above. The results obtained here are on the assumption that each \( Q_i \) is an \( S_{Ne} \)-open proper \( N \)-subset of \( Ne \) with \( \text{Ann} (e_i) = 0 \).

**4.5.1 Theorem \([12]\):** If \( \bigoplus_{i=1}^{n} B_i e_i \) is a maximal ideal of \( \bigoplus_{i=1}^{n} Ne \), then \( \bigoplus_{i=1}^{n} B_i e_i \) is ps-closed

**Proof:** At first we show the result for one component.

We have \( B e \) is a maximal ideal of \( Ne \).

Also by 4.1.28, \( \overline{Be} \) is an ideal of \( Ne \) such that \( Be \subseteq \overline{Be} \subseteq Ne \).

If possible let \( \overline{Be} = Ne \) \(...(i)\).

Since \( Q' \) is \( S_{Ne} \)-open, by 4.1.26, \( -Q' + e \) is ps-open.

Now, from (i), \( e \in \overline{Be} \), as \( 1 \in N \) and hence, by 4.1.19, \( (-Q' + e) \cap B e \neq \phi \).

Thus there is an element \( q' (= q e) \in Q' \) and \( b' (= b e) \in Be \)
such that \(- q' + e = b'\).
\[ \Rightarrow - qe + e = be. \]
\[ \Rightarrow - qe + e - be = 0. \]
\[ \Rightarrow (- q + 1 - b)e = 0, \text{ as } e = 1.e. \]
\[ \Rightarrow - q + 1 - b \in \text{Ann } (e). \]
\[ \Rightarrow - q + 1 - b = 0 \text{ [as } \text{Ann } (e) = 0]. \]
\[ \Rightarrow - q + 1 = b. \]
\[ \Rightarrow 1 = q + b \text{ \ldots (ii).} \]

Now as \( q' \in Q', \) we have \( N'_u q' = 0, \) but \( N'_u \neq 0, \) for some \( t \) (least) \( \in \mathbb{Z}^+. \)

Hence \( N'_u q = 0, \) as \( \text{Ann } (e) = 0. \)

Now for any \( n_1n_2...n_t = n_1n_2...n_t (q + b) - n_1n_2...n_t q \in B, \) as \( N'_u q = 0 \)
and \( B \) is a left ideal of \( N. \)
\[ \Rightarrow N'_u \subseteq B. \]

Again \( n_1n_2...n_{t-1} = n_1n_2...n_{t-1} (q + b) - n_1n_2...n_{t-1} q + n_1n_2...n_{t-1} q \)
\[ \in B, \text{ as } B \text{ being a left ideal of } N \text{ and } \]
\( n_1n_2...n_{t-1} q \in N'_u \subseteq B). \)

If \( n_1n_2...n_{t-1} q \notin N'_u, \) then \( q \notin N_u. \)

So, for some \( n \in N, \) we get \( nq = 1. \)

Now \( e = 1.e \)
\[ = nqe \]
\[ \in Q', \text{ being } N\text{-subset of } Ne. \]

So, \( N'_u e = 0, \) but \( N'_u \neq 0, \) for some \( l \) (least) \( \in \mathbb{Z}^+. \)

Hence \( N'_u = 0, \) as \( \text{Ann } (e) = 0, \) which contradicts \( N'_u \neq 0. \)

Thus \( N'_u^{-1} \subseteq B. \)
So, by induction, $N_u \subseteq B$ which is again a contradiction, as $q \in B$ and $b \in B$ gives $1 \in B$, [by (ii)].
Hence by 4.3.1, $B_e$ is closed, as $B_e$ being maximal.

Thus it follows from 4.1.15 that $\bigoplus_{i=1}^n B_e_i$ is ps–closed. ♦♦♦

4.5.2 Corollary: $J(\bigoplus_{i=1}^n N_e_i)$ is ps–closed.

**Proof:** As the ideal of $\bigoplus_{i=1}^n N_e_i$ maximal as $N$–subgroup is also maximal as ideal of $\bigoplus_{i=1}^n N_e_i$, so, by 4.5.1, $J(\bigoplus_{i=1}^n N_e_i)$ is the intersection of ps–closed subset of $\bigoplus_{i=1}^n N_e_i$ and giving thereby it as ps–closed. ♦♦♦

4.5.3 Theorem [[12]]: If $N$ is $\mathcal{F}$–bounded where $E = \bigoplus_{i=1}^n N_e_i$ then

$J(\bigoplus_{i=1}^n N_e_i)$ is ps–open.

**Proof:** Since $Q'$ is an ps–open subset of $N$ containing 0 (zero of $N$) and $N$ is $\mathcal{F}$–bounded, so there exists a ps–open subset $V$ of $N$ such that $NV \subseteq Q'$.
Now for each $x \in V$, by 1.1.26, $N_x$ is an $N_u$–nil $N$–subgroup of $N$. Hence, by 3.4.3, $N_x \subseteq J(N_e)$, for each $x \in V$.
Thus $V \subseteq J(N_e)$.
Now for $y \in V$ and $x \in J(N_e)$, we get, by 4.1.26, $x + (-y) + V$ as a ps–open set containing $x$ and $x + (-y) + V \subseteq J(N_e)$.
Hence, by 4.1.20, $J(N_e)$ is ps–open.
Consequently, $J(\bigoplus_{i=1}^n N_{e_i})$ is ps–open. ♦♦♦

In other words, $J(\bigoplus_{i=1}^n N_{e_i})$ is a proper subset of $\bigoplus_{i=1}^n N_{e_i}$ which is both ps–open and ps–closed and hence we get the following:

4.5.4 Corollary: If $N$ is $(\subseteq)$-bounded where $E = \bigoplus_{i=1}^n N_{e_i}$, then $\bigoplus_{i=1}^n N_{e_i}$ is ps–disconnected.

4.5.5 Corollary: If $N$ is $(\subseteq)$-bounded where $E = \bigoplus_{i=1}^n N_{e_i}$ and $J(\bigoplus_{i=1}^n N_{e_i})$ is the zero ideal, then $\bigoplus_{i=1}^n N_{e_i}$ is ps–discrete.

Proof: For $x \in \bigoplus_{i=1}^n N_{e_i}$, we have, by $0 = J(\bigoplus_{i=1}^n N_{e_i})$ is ps–open and so is $\{x\} = 0 + x$.

Hence $\bigoplus_{i=1}^n N_{e_i}$ is ps–discrete. ♦♦♦

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