Chapter 4

A Family of Multivariate Abel series Distributions of Order k
Chapter 4

A family of multivariate Abel series distributions of order $k$

4.1 Introduction

In this chapter, considering the multivariate Abel series of order $k$, we have defined multivariate Abel series distributions of order $k$. From the MASDs of order $k$, a new distribution called Quasi Multivariate logarithmic series distribution of order $k$ is obtained. Also a variant of quasi negative multinomial distribution of order $k$ is studied. Moreover, on using a new method of derivation some well known distributions, viz. quasi multinomial distribution of type-I of order $k$ (QMD-I ($k$)), quasi multinomial distribution of type-II of order $k$ (QMD-II ($k$)), multiple generalized Poisson distribution of order $k$ (MGPD ($k$)) etc. are obtained. A property, i.e. the limiting distribution of the QNMD of order $k$ has been found out.
4.2 Some Multivariate distributions of order $k$

On passing from univariate to multivariate distributions, some essentially new features require attention. These are connected with relationships among sets of random variables and include regression, correlation, and, more generally, conditional distributions.

The relatively recently discovered and investigated univariate distributions of order $k$ are multivariate geometric distributions of order $k$, multivariate negative binomial distributions of order $k$, multivariate Poisson distributions of order $k$, multivariate logarithmic series distributions of order $k$, multinomial distributions of order $k$, multivariate Polya and inverse Polya distributions of order $k$. Most of the generalizations are quite straightforward, and it is indeed remarkable that the class of distributions of order $k$ is rather easily extended to the multivariate case. Nevertheless, there are some novel results which shed additional light on the structure and flexibility of the classical multivariate discrete distributions, and they certainly provide an avenue for further generalizations.

Philippou et. al (1988) obtained three multivariate distributions of order $k$ by extending respective results of Philippou (1987) to the
Proposition 4.1. An urn contains balls bearing the letters $F_{i_1}, \ldots, F_{i_k}$ and $S$ ($=S_{\infty}$) with respective proportions $q_{i_1}, \ldots, q_{i_k}$ and $p$ $(0 < q_{i} < 1$ for $1 \leq i \leq m$ and $1 \leq j \leq k$, $q_{i_1} + \ldots + q_{i_k} < 1$ and $q_{i_1} + \ldots + q_{i_k} + p = 1$). Balls are drawn from the urn with replacement until $r$ balls $(r \geq 1)$ bearing the letter $S$ appear. Let $X_i (1 \leq i \leq m)$ be a random variable denoting the sum of the second indices of the letters on the balls drawn whose first index is $i$. Then

$$
\Pr(X_1 = x_1, \ldots, X_m = x_m) = p^r \sum_{i_1, \ldots, i_k = x} \left( \sum_{i_l = 1}^{x_i} x_{i_l} + \ldots + x_{i_k} + r - 1 \right) q_{i_1}^{x_{i_1}} \ldots q_{i_k}^{x_{i_k}},
$$

$$
x_i = 0, 1, \ldots, x_m = 0, 1, \ldots
$$

Proof. For any fixed non-negative integers $x_1, \ldots, x_m$, a typical element of the event $(X_1 = x_1, \ldots, X_m = x_m)$ is an arrangement $a_1 a_2 \ldots a_{x_i} \ldots x_{i_k} + r - 1 S$ of the letters $F_{i_1}, \ldots, F_{i_k}$ and $S$, such that $r - 1$ of the $a$'s are $S$, $x_{i_l}$ of the $a$'s are $F_{i_l}$ $(1 \leq i \leq m$ and $1 \leq j \leq k$), and

$$
\sum_j jx_{i_l} = x_i, i = 1, 2, \ldots, m
$$

(4.2.1)

Fix $x_{i_1}, \ldots, x_{i_k}$ ($r$ is fixed). Then the number of the above arrangements is
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\[
\left( x_{11} + \cdots + x_{m k} + r - 1 \right)
\left( x_{11}, \ldots, x_{m k}, r - 1 \right),
\]

and each of them has probability \( p^r q_{11}^{x_{11}} \cdots q_{m k}^{x_{m k}} \).

The proposition then follows, since the non-negative integers \( x_{n_1}, \ldots, x_{n_k} (1 \leq i \leq m) \) may vary subject to (4.2.1).

It may be seen by means of the transformations \( x_y = n_y \) and \( x_i = n_i + \sum_j (j - 1) n_j \) \( (1 \leq i \leq m \) and \( 1 \leq j \leq k \) that the above derived probability function is a proper probability distribution.

**Definition 4.1** A random vector \( X = (X_1, \ldots, X_m) \) is said to have the multivariate negative binomial distribution of order \( k \) with parameters \( r, q_{11}, \ldots, q_{m k} \) \((r > 0, 0 < q_j < 1 \text{ for } 1 \leq i \leq m \text{ and } 1 \leq j \leq k \text{ and } q_{11} + \cdots + q_{m k} < 1 \)) to be denoted by \( MNB_k (r; q_{11}, \ldots, q_{m k}) \) if

\[
\Pr(X_1 = x_1, \ldots, X_m = x_m) = p^r \sum_{\sum_j x_j = r} \frac{\Gamma \left( r + \sum_j x_j \right)}{\prod_j \Gamma (r)} \prod_j q_j^{x_j}, \quad x_i = 0, 1, \ldots, 1 \leq i \leq m
\]

For \( k = 1 \), this distribution reduces to the usual multivariate negative binomial distribution and for \( m = 1 \), it reduces to the multiparameter negative binomial distribution of order \( k \) of Philippou (1987). For \( r = 1 \), it
reduces to a multivariate distribution of order \( k \), which we call multivariate geometric distribution of order \( k \) with parameters \( q_{11}, \ldots, q_{mk} \) and denoted by \( MG_k(q_{11}, \ldots, q_{mk}) \). We write

\[
MNB_K(r; q_{11}, \ldots, q_{mk}) = MG_K(q_{11}, \ldots, q_{mk})
\]  

\text{(4.2.2)}

Proposition 4.2 specialized to the case \( r = 1 \), provides a genesis scheme for \( MG_k(q_{11}, \ldots, q_{mk}) \). Furthermore, if \( q_i = P^{r+1}Q_i \) (\( Q_i = 1 - P_i \), \( 1 \leq i \leq m \) and \( 1 \leq j \leq k \)) so that \( p = 1 - \sum_i (1 - P_i^k) = P \), we observe that \( MNB_K(r; q_{11}, \ldots, q_{mk}) \) reduces to the following multivariate negative binomial distribution of order \( k \):

\[
\Pr(X_1 = x_1, \ldots, X_m = x_m) = P^r \sum_{x_1, \ldots, x_m} \frac{\Gamma \left( r + \sum_i x_i \right)}{\prod_i \left( \Gamma(r) \right) \left( \prod_i x_i! \right)} \left( \frac{Q_i}{P_i} \right)^{x_i} \left( P_i \right)^{x_i},
\]

\[x_i = 0, 1, \ldots, 1 \leq i \leq m \]  

\text{(4.2.3)}

which is the multivariate analogue of the (shifted) negative binomial distribution of order \( k \) of Philippou et al. (1983). We call it multivariate negative binomial distribution of order \( k \), type I, with parameters \( r, Q_1, \ldots, Q_m \), and denote it by \( \text{MNB}_{\text{K}}(r; Q_1, \ldots, Q_m) \). For \( r = 1 \), (4.2.3) reduces to a multivariate distribution of order \( k \), which we call (shifted) multivariate
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geometric distribution of order $k$, type I, with parameters $Q_1,\ldots,Q_m$ and
denote by $MG_{k,I}(Q_1,\ldots,Q_m)$. It is the multivariate analogue of the (shifted)
geometric distribution of order $k$. We write

$$
MG_{k,I}(Q_1,\ldots,Q_m) = MG_{k,(1;0,\ldots,0)} = MG_{k,(1;0,\ldots,0)}
$$

If $q_i = \frac{Q_i}{k} \quad (1 \leq i \leq m \text{ and } 1 \leq j \leq k)$ so that $p = 1 - Q_1 - \cdots - Q_m = P$, we
note that $MNB_k(r, q_1, \ldots, q_m)$ reduces to the following multivariate negative
binomial distribution of order $k$:

$$
\Pr(X_i = x_1, \ldots, X_m = x_m) = p^r \sum_{x_1,\ldots,x_m} \frac{\Gamma(r + \sum_{j} x_j)}{\Gamma(r) \prod_j \Gamma(x_j)} \prod_j \left( \frac{Q_j}{P} \right)^{x_j},
$$

$$
x_i = 0,1,\ldots,1 \leq i \leq m
$$

which is the multivariate analogue of the compound Poisson (or negative
binomial) distribution of order $k$ of Philippou (1983). We call it multivariate
negative binomial distribution of order $k$, type II, with parameters
$r,Q_1,\ldots,Q_m$, and denote it by $MNB_{k,II}(r;Q_1,\ldots,Q_m)$. For $r = 1$, (4.2.5) reduces
to a multivariate distribution of order $k$, which we call multivariate
geometric distribution of order $k$, type II, and denote by $MG_{k,II}(Q_1,\ldots,Q_m)$. 

We write

\[ MNB_{k,n}(r; Q_1, \ldots, Q_m) = MG_{k,n}(Q_1, \ldots, Q_m) \]  

(4.2.6)

It is obvious from the above deliberations that proposition 4.2 provides a genesis scheme for each one of the type I and type II multivariate negative binomial distributions of order \( k \) (and hence for \( MG_{k,l}(Q_1, \ldots, Q_m) \) and \( MG_{k,n}(Q_1, \ldots, Q_m) \), as well)

We proceed now to derive a multivariate Poisson distribution of order \( k \) as a limiting case of the multivariate negative binomial distribution of the same order.

**Theorem 4.1.** Let \( X_r, r > 0 \), be \( m \times 1 \) random vectors distributed as \( MNB_{k}(r; q_{i1}, \ldots, q_{im}) \), and assume that \( q_y \to 0 \) and \( rq_y \to \lambda_y \) \((0 < \lambda_y < \infty \) for \( 1 \leq i \leq m \) and \( 1 \leq j \leq k \)) as \( r \to \infty \). Then for \( x_i = 0,1, \ldots, 1 \leq i \leq m \), we have

\[
\Pr(X_1 = x_1, \ldots, X_m = x_m) \to \sum_{r_y \to \infty} \exp(-\sum_j \sum_i \lambda_y) \frac{\prod \lambda_y^y}{\prod \prod x_y!}
\]

**Proof.** For \( x_i = 0,1, \ldots, 1 \leq i \leq m \), we have
Pr\( (X_1 = x_1, \ldots, X_m = x_m) = p^r \sum_{\sum_j x_j = r} \prod_{i=1}^{m} x_i! \left( \prod_{j} q_j \right)^{x_j} \}

= \left( 1 - \frac{r \sum_j \sum x_j}{r} \right) \sum_{\sum_j x_j = r} \left( \prod_{i} \lambda_i \right)^{x_i} 

\prod_{i=1}^{m} x_i! 

\exp\left( -\sum_{i} \sum_j \lambda_i \right) \prod_{i=1}^{m} \lambda_i^{x_i} \prod_{i=1}^{m} x_i! 

which establishes the proposition.

**Definition 4.2** A random vector \( X = (X_1, \ldots, X_m) \) is said to have the multivariate Poisson distribution of order \( k \) with parameters \( \lambda_{ij}, \ldots, \lambda_{mk} \) (\( 0 \leq \lambda_i < \infty \) for \( 1 \leq i \leq m \) and \( 1 \leq j \leq k \)), to be denoted by \( MP_k(\lambda_{ij}, \ldots, \lambda_{mk}) \), if

\[
Pr(X_1 = x_1, \ldots, X_m = x_m) = \sum_{\sum_j x_j = r} \exp\left( -\sum_{i} \sum_j \lambda_i \right) \prod_{i=1}^{m} \lambda_i^{x_i} \prod_{i=1}^{m} x_i! 

\]

For \( k = 1 \), this distribution reduces to the usual multivariate or multiple Poisson distribution. For \( m = 1 \), it reduces to the multiparameter Poisson distribution of order \( k \) of Philippou (1987). The latter was called by
Aki (1985) extended Poisson distribution of order $k$, since it extends the Poisson distribution of order $k$, and therefore, $MP_\kappa(\lambda_{11}, \ldots, \lambda_{mk})$ may also be called multivariate extended Poisson distribution of order $k$.

**Definition 4.3** A random vector $X = (X_1, \ldots, X_m)$ is said to have the multivariate logarithmic series distribution of order $k$ with parameters $q_{11}, \ldots, q_{mk}$ ($0 \leq q_y < \infty$ for $1 \leq i \leq m$ and $1 \leq j \leq k$ and $q_{11} + \cdots + q_{mk} < 1$), to be denoted by $MLS_k(q_{11}, \ldots, q_{mk})$, if

$$
Pr(X = x) = \alpha \sum_{\sum_j x_j = x} \frac{\left(\sum_j x_j - 1\right)!}{\left(\prod_j x_j \right)!} \prod_j q_j^{x_j},
$$

$$x_i = 0, 1, \ldots, 1 \leq i \leq m,$n

where $\alpha = -(\log p)^{-1}$ and $p = 1 - q_{11} - \cdots - q_{mk}$.

For $k = 1$, this distribution reduces to the usual multivariate logarithmic series distribution. For $m = 1$, it reduces to the multiparameter logarithmic series distribution of order $k$ of Philippou (1987).
4.3 Multivariate Abel series distribution of order \( k \) and its special cases

Let us consider a finite and positive function \( f(a) \) of \( a = (a_{i1}, \ldots, a_{mk}) \), where each \( a_y(i = 1 \ (1) \ m), j = 1 \ (1) \ k \) is a non-negative integer. For any real \( z \), we have the multinomial Abel series expansion of order \( k \) as,

\[
f(a) = f(a, z) = \sum_{x_1, x_2, x_3, \ldots, x_k} \left[ \prod_{i=1}^{m} \prod_{y=1}^{k} a_y (a_y - x_y z)^{y_0 - 1} / x_{y}^! \right] \left[ d(y) f(a) \right] \tag{4.3.1}
\]

where the summation is over \( x_1, x_2, x_3, \ldots, x_k \) such that \( \sum_j jx_y = x_i \) and each \( x_i(i = 1 \ (1) \ m) \) being a non-negative integer and the factor \( \left[ d(y) f(a) \right] \left( \prod_j x_y^! \right) \) is denoted by \( \beta(x, z) \) is independent of ‘\( a \)’, which is always greater than zero. The domain of \( a = (a_{i1}, \ldots, a_{mk}) \) is a subspace of an \( mk \)-dimensional parameter space subject to restrictions \( a_y \geq 0 \), if \( z \leq 0 \) and \( (a_y - x_y z) \geq 0 \), if \( z \geq 0 \), \( z \) belonging to a suitable subject of real numbers. Thus, (4.3.1) can be written,

\[
f(a) = f(a, z) = \sum_{x_1, x_2, x_3, \ldots, x_k} \beta(x, z) \prod_i \prod_j a_{yj} (a_y - x_y z)^{y_0 - 1}
\tag{4.3.2}
\]
using the formal series expansion (4.3.1), we suggest the following definition for the multivariate abel series distribution of order $k$ (MASD ($k$))

**Definition 4.4** A multivariate discrete distribution of order $k$, $p_k(x)$ is said to be a MASD ($k$) family, if it has the following probability function (p.f.),

$$p_k(x) = p_k(x;a,z) = \sum_{\sum_i x_i = x} \left[ \prod_j a_j \left( a_j - x_j z \right)^{y_j} \right] ^{z-1} / \left( k_y \right) ! \left[ d^{k_y} f(a) \right] _{a=x} f(a) \quad (4.3.3)$$

where $\sum_j jx_j = x$, each $x_j$ being a non-negative integer, the $a_j (i = l(1)m; j = l(1)k)$ and $z$ are parameters and $f(a)$ are stated in (4.3.1)

For $z = 0$, the probability function (4.3.3) becomes multivariate power series distribution of order $k$.

For $z = 0$ and $k = 1$, the probability function (4.3.3) becomes multivariate Abel series distribution (Nandi and Das, 1996) and the $z = 0$, it becomes usual multivariate power series distribution (Patil, 1965).

### 4.3.1 Derivation of some distributions

**4.3.1.1 Multivariate logarithmic series distribution of order $k$**

Here we derive a new distribution from MASD ($k$), called the multivariate logarithmic series distribution of order $k$. 


Let us consider Abel series expansion of order \( k \) of the logarithmic series function \( f(a) \) given by,

\[
- \log \left( 1 - \sum_i \sum_j a_y \right) = \sum_{y=x_i} \frac{\left( \sum_j x_j - 1 \right)!}{\prod_i \prod_j (x_y)!} \prod_i \prod_j a_y^y ;
\]

\( x_i = 0, 1, \ldots; \text{for} \ 1 \leq i \leq m, \sum_i x_i > 0, 0 < a_y < 1 (\text{for} \ 1 \leq i \leq m \ \text{and} \ 1 \leq j \leq k); \sum_i \sum_j a_y < 1 \)

(4.3.4)

Thus the associated MASD-family of order \( k \) has the following probability function,

\[
p_k(x) = \frac{1}{\log \left( 1 - \sum_i \sum_j a_y \right)} \sum_{y=x_i} \frac{\left( \sum_j x_j - 1 \right)!}{\prod_i \prod_j (x_y)!} \prod_i \prod_j a_y^y ; \quad (4.3.5)
\]

\( x_i = 0, 1, \ldots; \text{for} \ 1 \leq i \leq m, \sum_i x_i > 0, 0 < a_y < 1 (\text{for} \ 1 \leq i \leq m \ \text{and} \ 1 \leq j \leq k); \sum_i \sum_j a_y < 1 \)

\sim MLSD_k (a_1, \ldots, a_m) \quad (\text{Aki, Kuboki and Hirano (1984)})

For \( k = 1 \), the probability function (4.3.5) becomes usual MLSD with parameters \( a_1, a_2, a_3, \ldots, a_m \), i.e.,

\[
p(x_1, \ldots, x_m) = \frac{\left( \sum x_i - 1 \right)!}{\left( \prod x_i \right)! \left( -\log \left( 1 - \sum a_i \right) \right)} \prod_i a_i^y ; x \geq 0; \sum_i x_i \geq 0 \quad (4.3.6)
\]
For \( m = 1 \), the pf (4.3.5) becomes the multiparameter logarithmic series distribution of order \( k \) (Philippou, 1988).

### 4.3.1.2 Quasi-multivariate logarithmic series distribution of order \( k \)

Here we derive a new distribution from MASD \((k)\), called the quasi-multivariate logarithmic series distribution of order \( k \) (QMLSD \((k)\)) as follows.

Consider the logarithmic series function \( f(a) = -\log(1-a) \), where \( a = a_{11} + \cdots + a_{nk} \) and \( a = (a_{11}, \ldots, a_{nk}) \). Then the multivariate Abel series expansion of \(-\log(1-a)\) of order \( k \) is,

\[
-\log(1-a) = \sum_{\sum_{j} x_{ij} = 1} \left( \sum_{j} x_{ij} \right)! \prod_{j} (x_{ij})! \prod_{j} \prod_{i} a_{ij} \left( a_{ij} - x_{ij} z \right)^{v_{ij}} \left(1 - \sum_{j} x_{ij} z\right)^{-v_{ij}}
\]

(4.3.7)

where \( \sum_{\sum_{j} x_{ij} = 1} \) is defined in (4.3.1) and \( 0 < a = a_{11} + \cdots + a_{nk} < 1 \).

Thus the associated MASD family of order \( k \) has the following p. f.
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The pf (4.3.8) is called the QMLSD ($k$).

If $z \to 0$, then the pf (4.3.8) becomes the multinomial logarithmic
distribution of order $k$.

For $k = 1$, the pf (4.3.8) becomes QMLSD (Nandi & Das, 1996) and
then as $z \to 0$, it becomes the common multinomial logarithmic series
distribution (Johnson & Kotz, 1969, p.303).

### 4.3.1.3 Quasi-multinomial distribution of type I of order $k$

Next we obtain quasi multinomial distribution of type-I of order $k$
(QMD-I ($k$)).

Let us consider the simple series function $f(a) = (a+b)^n$, where
$a = (a_{i1}, \ldots, a_{nk})$ and $a = a_{i1} + \cdots + a_{nk}$. Then the multivariate Abel series
expansion of $(a+b)^n$ of order $k$ is,
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\[(a+b)^n = \sum_{x \in \mathbb{N}} \binom{n}{x} \prod_i a_i \left( a_i - x_i z \right)^{v_i - 1} (b + xz)^{v_i - x} \]

Hence the corresponding MASD family finds the pf of QMD-I ($k$),

\[\Pr(X = x) = p_k(x) = \binom{n}{x} \prod_{i} a_i \left( a_i - x_i z \right)^{v_i - 1} (b + xz)^{v_i - x} / (a+b)^n, \quad (4.3.9)\]

where, $0 \leq \sum_j jx_j = x \leq n$, each $x_y$ being a non-negative integer and

\[\binom{n}{x} = \frac{n!}{(n-x)! \prod_{i} x_i !} \]

Suppose, $p_y = \frac{a_y}{(a+b)}; i = l(l)m \& j = l(l)k$;

\[p_0 = \frac{a}{(a+b)} \text{ and } \phi = \frac{z}{(a+b)} \quad (4.3.10)\]

Using (4.3.10) in (4.3.9), we get

\[\Pr(X = x) = p_k(x) = \binom{n}{x} \left(p_0 + x\phi\right)^{v_i - 1} \prod_{i=1}^{m} \prod_{j=1}^{k} p_y \left(p_y - x_y \phi\right)^{v_i - 1} \quad (4.3.11)\]

where, $0 \leq x \leq \sum_{i=1}^{m} \sum_{j=1}^{k} p_y = 1$ and

For $k = 1$, we get the pf (4.3.9) as QMD-I (Janardan, 1975).
4.3.1.4 Multiple generalized Poisson distribution of order $k$

Now, we derive the multiple generalized Poisson distribution of order $k$.

Let us consider the exponential series function $f(a) = e^a$, where $a = (a_1, \ldots, a_m)$ and $a = a_1 + \cdots + a_m$. Then the multivariate Abel series expansion of $f(a) = e^a$ is,

$$e^a = \sum_{\sum_{y=1}^m r_y = n} \left[ \prod_{y=1}^m a_y (a_y - x_y)^r_y / (x_y)! \right] e^a$$

(4.3.12)

where $\sum_{y=1}^m r_y = n$, is stated in (4.3.1)

Thus the corresponding pf of the MASD family of order $k$ is,

$$Pr(X = x) = p_k(x) = \prod_{y=1}^m \prod_{j=1}^k [a_y (a_y - x_y)^r_y / (x_y)!]$$

(4.3.13)

where $\sum_{j=1}^k x_{ij} = x$, and each $x_{ij}$ being a non-negative integer.

The pf (4.3.13) is known as the MGPD of order $k$ and for $k = 1$, the pf (4.3.13) reduces to MGPD (Janardan, 1975).

4.3.1.5 Quasi negative multinomomial distribution of order $k$

Finally, we derive the quasi negative multinomial distribution of order
k.

Let us consider the series function, \( f(a) = (b-a)^{-\alpha} \), where \( a = (a_1, \ldots, a_m) \) and \( a = a_1 + \cdots + a_m \). Then the multivariate Abel series expansion of \( (b-a)^{-\alpha} \) of order \( k \) is,

\[
(b-a)^{-\alpha} = \sum \frac{\Gamma\left(n + \sum_i \sum_j x_{ij}\right)}{\prod_j x_j! \Gamma(n)} \prod_i a_i \left(a_i - x_{ij} \right)^{-\alpha-1} \left(b - \sum_i \sum_j x_{ij}z\right)^{-\alpha} \sum_j x_j z
\]

where \( \sum_j x_j \) are given in (4.3.1).

Then the associated family of the MASD of order \( k \) has the p.f.

\[
p_h(x) = \frac{\Gamma\left(n + \sum_i \sum_j x_{ij}\right)}{\prod_j x_j! \Gamma(n)} \prod_i a_i \left(a_i - x_{ij} \right)^{-\alpha-1} \left(b - \sum_i \sum_j x_{ij}z\right)^{-\alpha} \sum_j x_j z (b-a)^{-\alpha},
\]

\( x_{ij} \geq 0; i = l(1)m \) and \( j = l(1)k \) \hspace{1cm} (4.3.14)

Suppose, \( p_g = \frac{a_g}{(b-a)^{\alpha}} \); \( i = l(1)m \) & \( j = l(1)k \);

\[
Q = \frac{b}{(b-a)} \text{ and } \phi = \frac{z}{(b-a)} \hspace{1cm} (4.3.15)
\]

Applying (4.3.15) to the p.f. (4.3.14), we get
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\[
\Pr(X = x) = p_{k}(x) = \frac{\Gamma\left(n + \sum_{j} x_{j}\right)}{\prod_{j} x_{j}! \Gamma(n)} \prod_{j} \prod_{i} p_{i} P_{j} (p_{j} - x_{j} \phi)^{(j-1)} \left(O - \sum_{j} x_{j} \phi\right)^{-\sum_{j}}
\]

(4.3.16)

where \( p_{j} - x_{j} \phi \geq 0 \) and \( O - \sum_{j} P_{j} = 1 \)

The probability functions (4.3.14) and (4.3.16) are called QNMD of order \( k \).

If \( z = 0 \), then (4.3.14) and (4.3.16) reduces to negative multinomial distribution of order \( k \).

If \( k = 1 \), then (4.3.14) and (4.3.16) reduces to QNMD and then for \( z = 0 \), it reduces to common negative multinomial distribution (Johnson and Kotz, 1969, p. 292).

4.4 Properties of quasi negative multinomial distribution of order \( k \)

The QNMD of order \( k \), (4.3.16) with \( P_{i} (i = 1 \ 1 \ m, j = 1 \ 1 \ k), Q, \phi \) and \( n \) tends to multiple generalized Poisson distribution with parameters \( \tilde{\lambda}_{i} (i = 1 \ 1 \ m, j = 1 \ 1 \ k) \) and \( \phi \), as \( n \to \infty \), \( P_{i} \to 0 \) and \( \phi \to 0 \), such that \( nP_{i} = \tilde{\lambda}_{i} \).
and $n\phi = \phi$. The probability function of this limiting distribution is given in (4.3.13).