Chapter 2

A Class of Quasi Binomial Distributions of Order $k$. 
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2.1 Introduction

In this chapter, we have obtained a class of quasi binomial distributions of order $k$ by considering Abel's generalization of the binomial formula. A class of Generalized Poisson distributions of order $k$ is also defined as a limiting case of QBDs of order $k$. Also some recurrence relations have been studied.

2.2 Urn models with predetermined strategies

2.2.1 A two urn model

Let there be two urns marked I and II, urn I containing $a$ white and urn II containing $a$ white and $b$ black balls. Let $n$ and $z$ be two known
positive integers, for given $n$ and $z$, a strategy is determined by choosing an integer $k$ such that $0 \leq k \leq n$ before making two draws from urn I and $n$ draws from urn II under the following rules:

(i) $kz$ black balls will be added to urn I and $kz$ white, $(n-k)z$ black to urn II,

(ii) two balls are drawn from urn I with replacement, if both the balls are white, $n$ draws are made from urn II with replacement, otherwise no draws are made.

A success is achieved, if exactly $k$ out of $n$ draws is white balls.

Clearly, therefore the probability of success is equal to

$$
\Pr(X = k) = \binom{n}{k} \left( \frac{a}{a+kb} \right)^k \left( \frac{a+kb}{a+b+nz} \right) \left( \frac{b+(n-k)z}{a+b+nz} \right)^{n-k}
$$

(2.2.1)

with $p = \frac{a}{a+b+nz}$, $q = \frac{b}{a+b+nz}$ and $\phi = \frac{z}{a+b+nz}$

reparametrising (2.2.1), we get

$$
\Pr(X = k) = \binom{n}{k} p^2 (p+k\phi)^{k-1} (q+(n-k)\phi)^{n-k}
$$

(2.2.2)

In particular, for $\phi = 0$, the probability function (2.2.2) reduces to the classical binomial distribution.
2.2.2 An urn model with three urns

Let there be three urns marked I, II and III. Urn I contains ‘a’ white, urn II contains ‘b’ black and urn III contains ‘a’ white and ‘b’ black balls respectively. For given n and z, a strategy $0 \leq k \leq n$ is chosen before making two draws from urn I, one draw from urn II and $n$ draws from urn III under the following conditions:

(i) $kz$ black balls to urn I, $(n-k)z$ white balls to urn II and $kz$ white balls,

$(n-k)z$ black balls to urn III will be added,

(ii) Two balls are drawn with replacement from urn I, and, if both are white, then one is drawn from urn II, if black, then $n$ draws with replacement are made from urn III, otherwise the game is stopped as a failure. And a success is achieved, if out of $n$ draws from urn II exactly $k$ balls are white.

Then the probability of success is given by,

$$\Pr(X = k) = \binom{n}{k} p^k q^{n-k} (p + k\phi)^{k-2} (q + (n-k)\phi)^{n-k-1}$$ (2.2.3)
2.2.3 Another three urn model

Here the same setup as that in the last model has been considered only the drawing pattern is changed as follows:

Two balls are drawn from urn I if both white, then two draws are made from urn II, if both black \( n \) draws with replacement are made from urn III, a success is defined in the same manner as earlier. Hence, here

\[
\Pr(X = k) = \binom{n}{k} p^2 q^2 (p + k\phi)^{k-2} (q + (n-k)\phi)^{n-k-2}
\]  

(2.2.4)

None of these models are proper probability distribution. But using Abel’s generalization of the binomial identities Riordan [1968], it is possible to find norming constant \( C \) in such a way that \( \sum_x C \Pr(X = x) = 1 \)

All these and many more distributions are infact members of a class of quasi binomial distributions Das (1994) has defined using Abel’s formula as

\[
\Pr(X = x) = \frac{\binom{n}{x} (p + x\phi)^{x-1} (q + (n-x)\phi)^{n-x-1}}{B_n(p,q;\phi)}
\]  

(2.2.5)

\[
B_n(p,q;\phi) = \sum_{x=0}^{n} \frac{n! \binom{n}{x} (p + x\phi)^{x-1} (q + (n-x)\phi)^{n-x-1}, \quad \text{(Riordan, 1968)}}
\]  

(2.2.6)

where \( p \) and \( q \) being the non-negative fractions, \( p + q + n\phi = 1; \)
- \frac{n \phi}{n} < \phi < \frac{(1-p)}{n} \quad \text{and} \quad s, t \quad \text{are integers.} \quad \text{Alternatively, (2.2.5) can also be written as}

\begin{equation}
p_k = \binom{n}{k} \frac{(p + k\phi)^s (q - k\phi)^n}{B_n(p, q - n\phi; s; t; \phi)}
\end{equation}

\textbf{Some special cases of (2.2.5)}

Substituting,

i) \( s = -1, \ t = 0, \) QBD type I (Consul, 1974).

ii) \( s = -1, \ t = -1, \) QBD type II (Consul and Mittal, 1975).

iii) \( s = -2, \ t = 0, \) QBD type III (Das, 1994).

iv) \( s = -2, \ t = -1, \) QBD type IV (Das, 1994).

v) \( s = 0, \ t = 0, \) QBD type VII (Das, 1994), etc.

Let \( k \) be a positive integer. Suppose we are given independent trials with success probability \( p. \) The distribution of the number of occurrences of consecutive \( k \)-successes until the \( n^{th} \) trial is called the binomial distribution of order \( k \) and is denoted by \( \text{B}_k(n, p). \) Consul (1974) considering the above mentioned simple urn model with a predetermined strategy and derived the quasi binomial distribution (QBD).
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Here we have defined some QBDs of order $k$.

### 2.3 QBD I of order $k$

**Definition 1.** The probability function of QBD I of order $k$ has the following form

$$\Pr(X = x) = \sum_{m=0}^{k-1} \sum_{\sum_{i=1}^{k} x_i = n - m - kx} \left( \sum_{i=1}^{k} x_i \right)^{\frac{k}{2}} \phi^{\frac{k}{2}} \left( 1 - p + \phi \sum_{i=1}^{k} x_i \right)^{n-k-x} \left( 1 - p - \phi \sum_{i=1}^{k} x_i \right)^{x}$$

(2.3.1)

for $x = 0, 1, 2, \ldots, \left[ \frac{n}{k} \right]$; $[a]$ means the largest integer not exceeding $a$, and the inner summation is over all the non-negative integers $x_1, x_2, \ldots, x_k$ such that $x_1 + 2x_2 + \cdots + kx_k = n - m - kx$.

Putting $k = 1$, the pf (2.3.1) reduces to the pf of QBD I (Consul, 1974), as

$$\Pr(X = x) = \binom{n}{x} p^{x} \left( 1 - p \right)^{n-x} \phi^{x}$$

where $x = 0(1)n$, $0 < p < 1$ and $-\frac{P}{n} < \phi < \frac{1-p}{n}$.

And substituting $\phi = 0$, it reduces to the classical binomial distribution.
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2.4 QBD II of order \( k \)

**Definition 2.** The pf of QBD II of order \( k \) has the following form

\[
\Pr(X = x) = \sum_{a=0}^{k} \sum_{\sum_{i=1}^{k} x_i = a} \left( \frac{x!}{x_1!x_2!\cdots x_k!} \right) p^{x_1}(q-n\phi)^{x_2} \left( p + \phi \sum_{i=1}^{k} x_i \right)^{x_3} \left( 1 - p - \phi \sum_{i=1}^{k} x_i \right)^{x_4} 
\]

(2.4.1)

and the conditions imposed are same as in (2.3.1)

Substituting \( k = 1 \), the probability function (2.4.1) reduces to the pf of QBD II (Consul and Mittal, 1975) as follows

\[
\Pr(X = x) = \binom{n}{x} p(q-n\phi)(p+x\phi)^{x-1}(q+x\phi)^{n-x} / (1-n\phi) 
\]

where \( x = 0(1)n, \ 0 < p < 1 \) and \( -\frac{p}{n} < \phi < \frac{1-p}{n} \)

and for \( \phi = 0 \), it reduces to the classical binomial distribution.

2.5 Limiting distributions

The limiting form of both the QBDs of order \( k \), as \( n \to \infty \) and \( \phi \to 0 \), such that \( np = \lambda \) and \( n\phi = \nu \), tend to generalized Poisson distribution of order \( k \) with the probability function
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$$\Pr(X = x) = \sum_{\mathclap{\sum_{i=1}^{k} x_i = x, \sum_{i=1}^{k} \lambda_i = \lambda}} \frac{e^{-\sum_{i=1}^{k} \lambda_i} \left(\lambda + \sum_{i=1}^{k} x_i \psi \right)^{\sum_{i=1}^{k} x_i - 1}}{\prod_{i=1}^{k} x_i !} ; \left(\lambda + \sum_{i=1}^{k} x_i \psi \right) > 0 \quad (2.5.1)$$

substituting $k = 1$, the probability function (2.5.1) reduces to the pf of GPD I (Consul and Jain. 1973) as follows

$$\Pr(X = x) = p(x) = \frac{e^{-(\lambda + x\psi)} (\lambda + x\psi)^{x-1}}{x !} ; \ x = 0, 1, 2, \ldots, |\psi| < 1$$

and for $\psi = 0$ we get the simple Poisson distribution.

2.6 A class of Quasi Binomial distributions of order $k$

Here we have used Abel’s generalization of the binomial formula (Riordon, 1968, p. 18). By extending Abel’s generalization of the binomial formula to order $k$, we have defined a class of QBDs of order $k$ with parameters $n, p$ and $\phi$, and integers $s$ and $t$, whose probability function is given by:

$$P_k(x; s, t; \phi) = \left[ \sum_{m=0}^{l-1} \sum_{x_1, x_2, \ldots, x_k} \left( \frac{x_1 + x_2 + \cdots + x_k + x}{x_1, x_2, \ldots, x_k, x} \right) \left( p + \phi \sum_{i=1}^{k} x_i \right)^{\sum_{i=1}^{k} x_i} \left( 1 - p - \phi \sum_{i=1}^{k} x_i \right)^{n-\sum_{i=1}^{k} x_i} \right]$$

$$/ B_k(n, p, q; s, t; \phi) , \quad (2.6.1)$$
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where \( x=0(1)n, \; p+q+n\phi = 1 \) and \( \frac{P}{n} < \phi < \frac{1-P}{n} \)

and

\[
B_k(n; p,q,s,t;\phi) = \sum_{x=0}^{\infty} \sum_{i=0}^{\infty} \left( \sum_{x_1, x_2, \ldots, x_i} \binom{x_1 + x_2 + \cdots + x_i + x}{x} \left( p + \phi \sum_{x_i} \right)^{x_i} \left( q - \phi \sum_{x_i} \right)^{x - \sum x_i} \right)
\]

(2.6.2)

for \( k = 1 \), equations (2.6.1) and (2.6.2) reduces to the probability function of a class of QBD (Das, 1993).

2.6.1 Some particular cases

i) If \( s = -1 \) and \( t = 0 \) then the pf (2.6.1) reduces to the pf (2.3.1), i.e. QBD I of order \( k \)

ii) If \( s = -1 \) and \( t = -1 \) then the pf (2.6.1) reduces to the pf (2.4.1), i.e. QBD II of order \( k \)

iii) If \( k = 1, s = 0, t = 0 \) and \( \phi = 0 \) then the pf (2.6.1) transforms to the common binomial distribution (Johnson and Kotz, 1969, p. 50).

Similarly, setting different combinations of \( s \) and \( t \), we may get different QBDs of order \( k \), using different recurrence relations of the expression \( B_k(n; p,q;s,t;\phi) \).
2.6.2 Recurrence relations

Here we have found the following recurrence relations:

\[
B_k(n; p, q, s, t; \phi) = B_k(n-1; p+\phi, q; s+1, t; \phi) + B_k(n-1; p, q+\phi; s, t+1; \phi)
\]

(2.6.3)

Separating the factor \( p + \phi \sum_{i=1}^{k} x_i \) from the equation (2.6.2) we have the following recurrence relation

\[
B_k(n; p, q; s-1, t; \phi) = \left( \frac{1}{p} \right) \left[ B_k(n; p, q; s, t; \phi) - n\phi B_k(n-1; p+\phi, q; s, t; \phi) \right]
\]

(2.6.4)

and with the help of the identity:

\[
\left( p + \phi \sum_{i=1}^{k} x_i \right) + \left( q - \phi \sum_{i=1}^{k} x_i \right) = 1
\]

we can get the following recurrence relation,

\[
B_k(n; p, q; s, t; \phi) = B_k(n; p, q; s+1, t; \phi) + B_k(n; p, q; s, t+1; \phi)
\]

(2.6.5)

2.6.3 Limiting distributions of QBDs of order \( k \)

By setting \( n \to \infty \) and \( \phi \to 0 \), such that \( np = \lambda \) and \( n\phi = \psi \) (finite), in equation (2.6.1), then the limiting distribution of QBDs of order \( k \) is the
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probability function of a class of GPDs of order $k$ for $\lambda > 0$, and $|\psi| < 1$ is
given by,

$$g_k(x;\lambda,s;\psi) = \sum_{\mathclap{\sum_{i=1}^{\infty} x_i = x, x_i \geq 0}} \frac{e^{\left(\lambda + \psi \sum_{i=1}^{\infty} x_i\right)} \left(\lambda + \psi \sum_{i=1}^{\infty} x_i\right)^{\lambda + \psi x}}{\prod_{i=1}^{\infty} x_i ! K_k(\lambda,s;\psi)}$$

(2.6.6)

where $K_k(\lambda,s;\psi)$ is defined in (2.6.7).

Substituting various values of $s$, one may obtain different GPDs of order $k$.

**Definition 3.** A random variable ‘$X$’ is said to follow a class of GPDs of
order $k$, when it assumes only non-negative values having parameters $\lambda$ and
$\psi$, such that $\lambda > 0$, and $|\psi| < 1$, having the probability function:

$$g_k(x;\lambda,s;\psi) = \sum_{\mathclap{\sum_{i=1}^{\infty} x_i = x, x_i \geq 0}} \frac{e^{\left(\lambda + \psi \sum_{i=1}^{\infty} x_i\right)} \left(\lambda + \psi \sum_{i=1}^{\infty} x_i\right)^{\lambda + \psi x}}{\prod_{i=1}^{\infty} x_i ! K_k(\lambda,s;\psi)}$$

(2.6.8)

where $K_k(\lambda,s;\psi)$ is defined in (2.6.7).

Substituting various values of $s$, one may obtain different GPDs of order $k$. 
2.6.4 Particular cases

(i) If $s = -1$, the probability function (2.6.6) reduces to the probability function of (2.5.1)

(ii) If $s = 0$, and $\psi = 0$, then the probability function (2.6.6) becomes common Poisson distribution (Johnson and Kotz, 1969).

2.6.5 Recurrence Relation

By separating the factor $\left( \lambda + \psi \sum_{i=1}^{k} x_i \right)$ in $K_k(\lambda, s; \psi)$, we get the recurrence relation

$$K_k(\lambda, s-1; \psi) = \left( \frac{1}{\lambda} \right) \left[ K_k(\lambda, s; \psi) - \psi K_k(\lambda + \psi, s; \psi) \right]$$  \hspace{1cm} (2.6.9)