5.1 General Considerations

The attenuation of elastic waves (Acoustic) by different materials is a well known physical phenomenon. Experimental as well as theoretical investigations has been carried out extensively since the later part of the last century. These experimental works on attenuation of acoustic waves at different frequency ranges by different terrestrial materials performed in the laboratory as well as in the field using seismic waves have been reviewed by Howell (1963) and Knopoff (1964). These works indicate that the amplitude loss factor for seismic waves may be given by (White, 1965),

\[ \exp \left( -\frac{\omega t}{2Q} \right) \]  

(22)

where \(1/Q\) is a frequency independent loss parameter.

The parameter \(Q\) is related to the definition in electrical circuit theory.
where $\Delta E = \text{Energy dissipated per cycle of a harmonic excitation.} E$ being the peak energy in a certain volume.

The logarithmic decrement of a harmonic wave is equal to $\pi/\xi$. The attenuation factor for a wave function at a fixed time is of the form $\exp(-\alpha r)$ where $\alpha = \omega/2c\xi$ and $C = \text{phase velocity.}$ The wave function observed as a function of time at a fixed point in space gives the damping factor as $e^{-\gamma t}$. As a result $\xi$ is related to $\gamma$ by $\gamma = 2\omega/\xi$. This is however, true in homogeneous systems without dispersion.

The earth, which is suspected to be radially as well as horizontally inhomogeneous, may act as a dispersive medium for the seismic waves and the above relationships require modifications.

Table 7 shows some $Q$ values obtained by other investigators for earth using $P$ waves in the frequency range of our interest.

It is to be noted from the table that the values of $Q$ differ greatly even for the same frequency range depending on the method of evaluation.
Table 7

Q values on earth obtained by other investigators using p waves (Jackson et al., 1970)

<table>
<thead>
<tr>
<th>Investigator</th>
<th>Depth</th>
<th>Period of the waves</th>
<th>Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gutenberg (1945)</td>
<td>Whole earth</td>
<td>4</td>
<td>1300</td>
</tr>
<tr>
<td>Gutenberg (1958)</td>
<td>Whole earth</td>
<td>2</td>
<td>2500</td>
</tr>
<tr>
<td>Aceda and Takano (1963)</td>
<td>Whole earth</td>
<td>0.14-0.3</td>
<td>2000-4000</td>
</tr>
<tr>
<td>Carpenter (1964)</td>
<td>Whole earth</td>
<td>1</td>
<td>1000</td>
</tr>
<tr>
<td>Kanamori (1967)</td>
<td>Whole earth</td>
<td>0.8-2.5</td>
<td>410-630</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.5-2.0</td>
<td>435</td>
</tr>
<tr>
<td>Hirassawa and Takano (1966)</td>
<td>Whole earth</td>
<td>3-30</td>
<td>340</td>
</tr>
<tr>
<td>Kanamori (1967)</td>
<td>Upper 870</td>
<td>0.8-2.5</td>
<td>180</td>
</tr>
<tr>
<td></td>
<td>Below 870</td>
<td>0.8-2.5</td>
<td>1600-6000</td>
</tr>
</tbody>
</table>
5.2 Theoretical Considerations

The earliest attempt to explain the phenomenon of acoustic loss is due to Boltzmann (Knopoff, 1964). According to his theory release of strain developed in a body due to stress is delayed by some sort of memory behaviour. The stress is a complicated function of time which can be denoted as a convolution with a memory function $M(t)$ describing the delay. Thus,

$$\Pi(t) = \int_{-\infty}^{t} E(\tau) \ M(t-\tau) \ d\tau$$  \hspace{1cm} (24)

where $E(\tau)$ and $\Pi(t)$ are stress and strain functions respectively. $m(w)$, the fourier transform of $M(t)$ then stands out as a transfer function

$$p(w) = e(w) \ m(w)$$  \hspace{1cm} (25)

Now, if the nature of the loss mechanism is described in real space-time as an operator, the Fourier transform of this function may be found, which then can be compared with the experimental results. Maxwell, Mayer, Kelvin, Voigt and others suggested different models (Knopoff, 1964) which give the frequency independence of $Q$, while the viscoelastic model of Kelvin-Voigt indicate frequency dependence. Other models and based on linear mechanism like spring-dashpot systems based on hysteresis process have also been suggested (Knopoff, 1964,
The transform \( m(w) \) is of the form
\[
m(w) = \int_{-\infty}^{+\infty} e^{-i\omega t} M(t) \, dt
\] (26)

and since together with (25) it is given as a dynamic ratio between transform of stress and strain functions, it can be concluded that a scattering or absorptive medium may be characterised by complex modulii of elasticity, the real and imaginary part of the modulii should be related to the transform pair (Derajaguine, 1934) (after Knopoff, 1964)

\[
C(w) = \int_{0}^{\infty} M(t) \cos(\omega t) \, dt
\]
(27)
\[
S(w) = \int_{0}^{\infty} M(t) \sin(\omega t) \, dt
\]

Assuming density to remain unchanged the attenuation is associated with the complex modulus, the complex velocity can be written as
\[
\nu = \left( \frac{\omega}{\rho} \right)^{1/2} = \left( \frac{C + i S}{\rho} \right)^{1/2}
\] (28)

Since the wave function is of the form \( \exp \left[ i \omega t (x/\nu - t) \right] \) the phase velocity \( C \) and attenuation factor \( \alpha \) which may be function of frequency, better denoted by \( C(f) \) and \( \alpha(f) \), are given by
Then the specific attenuation factor \( \frac{1}{Q} \), becomes

\[
\frac{1}{Q} = \frac{2 \times c}{\omega}.
\]

For \( \frac{Q}{\omega} \gg 1 \), \( C(\omega) \gg S(\omega) \), so it can be approximated as

\[
\frac{1}{Q} \approx \frac{S}{C}.
\]

A function \( M(t) \) representing a real mechanism for attenuation having the transform pair should corroborate with the observed frequency independence of \( Q \). The mechanism can be denoted as an operator. Such operator should satisfy causality condition in the Kramer-König (K-K) relation or others. The problem is then reduced to deriving two characteristic parameters, namely, \( C(f) \), the velocity parameter and \( \alpha(f) \) the attenuation parameter, associated with the wave propagation through the medium which are related to the transform pairs linearly.

Futtermann (1962) investigated a number of models, and
has shown that in order that the causality condition should hold dispersion becomes a necessity. Consequently $\phi$ needs also to be frequency dependant. Taking all these facts into consideration, he finally derived a model with a low cut-off frequency for the dispersion of body waves. The transform pair given by him is

$$c = c_0 \left[1 - \frac{1}{\pi \phi_0} \log_e \left(\frac{\gamma \omega}{\omega_0}\right)\right]^{-1}$$  \hspace{1cm} (32)

$$\phi = \phi_0 \left[1 - \frac{1}{\pi \phi_0} \log_e \left(\frac{\gamma \omega}{\omega_0}\right)\right]^{-1} \hspace{1cm} \omega \gg \omega_0$$  \hspace{1cm} (33)

where, $c =$ phase velocity,

$c_0$, $\phi_0$, $\omega_0$ are characteristic constants

$\gamma =$ Euler's constant $= 0.57724$.

Relation (32) indicate the dispersion condition while (33) show that $\phi$ may be independant over a wide range of frequency provided the cut-off frequency $\omega_0$ is small. These seem to agree with experimental results of frequency independence of $\phi$.

Consideration of wave propagation in a medium with complex elastic coefficient lead to this result in the following way. The loss of energy during each cycle of stress, is assumed to be linear function of maximum energy stored (White, (1965)).

The complex elastic coefficients characterizing absorptive medium may be taken as complex Lamé's constants.
\[ \lambda_c = \lambda + i \text{sgn}(\omega) \lambda^* \]
\[ \mu_c = \mu + i \text{sgn}(\omega) \mu^* \]

The factor \( \text{sgn}(\omega) \) required to satisfy the equation
\[ y(\omega) = y(-\omega) \]
relating real function to its complex conjugate for negative \( \omega \).

For a plane compressional wave the wave equation in the spectral form can be written as

\[ \left( M + i \text{sgn}(\omega) M^* \right) \frac{2}{\varepsilon \rho} [A(\omega, r)] = -i\omega A(\omega, r) \quad (34) \]

where \( M = \lambda + 2\mu \)
\( M^* = \lambda^* + 2\mu^* \)

A solution written as
\[ A(\omega, r) = A(\omega) \exp\left( \frac{W r}{\rho} \right) \]

will satisfy the equation when

\[ W = \pm i \omega \left( \frac{1}{M} \right)^{1/2} \left[ M + i \text{sgn}(\omega) M^* \right]^{-1/2} \]

or
\[ W = \pm i \omega \left( \frac{1}{M} \right)^{1/2} \left[ 1 - i \text{sgn}(\omega) \frac{M^*}{2M} + \cdots \right] \]

For \( M^* \ll M \)
where,

\[ \left( \frac{M}{F} \right)^{\frac{1}{2}} = C = \text{phase velocity} \]

Consequently an approximate solution to the wave equation is

\[ A (w, r) = A (w) \exp \left[ - i \frac{\omega r}{c} - \frac{i \omega |r|}{2c \alpha} \right] \quad (36) \]

where,

\[ Q = \frac{M^*}{M} \]

If C and Q are assumed frequency independent, the second part of the exponent in equation (36) corresponds to the observed form (22). Frequency independence, however, do not satisfy causality condition. Puttermann's high frequency solution taking into account of the causality condition is given by

\[ A (\omega, r) = A (\omega) \exp \left[ - \frac{i \omega}{2c} \alpha \left( 1 - \exp \left( \frac{-i \omega |r|}{\omega_c} \right) \right) + i \phi \right] \quad (37) \]

with \[ \phi = \frac{i \omega r}{c} \quad ; \quad \alpha = \alpha_0 \left[ 1 - \frac{1}{\pi \alpha_0} \log \left( \frac{\gamma \omega}{\omega_c} \right) \right]^{-1} \]

\[ \gamma = \text{Euler's constant} \]

Choosing \( \omega_c \) significantly lower than the frequency ranges considered, equation (37) will also yield the observed form (22), with Q independent of frequency.
Strick (1967) has raised pertinent questions regarding validity of Futtermann's result. Firstly his result of linear frequency dependence of as given in (59), satisfies K-K-relation for causality but it cannot be extended to infinite frequency as would violate Paley-Wiener principle of causality. Similarly it cannot be extended to zero frequency, which would give rise to unboundedness in K-K relation. Also the values for the low frequency \( \omega_c \) and \( C_0 \) are arbitrary and therefore vague. Strick assumed a power relation for \( \alpha(f) \) as

\[
\chi(f) = k_0 f^{-\delta}, \quad 0 < \delta < 1
\]

which satisfies Paley-Wiener relation. Hilbert transform of (38) gives

\[
\hat{\chi}(f) = \tan \left( \frac{\pi f}{2} \right) \sin \theta(f)
\]

(39)

\( \sin \theta(f) \) is +1 for \( f > 0 \), \(-1 \) for \( f < 0 \)

The phase lag function \( \theta(f) \) is considered identical with \( \hat{\chi}(f) \).

Defining the spatial \( \hat{Q}_R \) (in contrast with the temporal \( \hat{Q}_T \) which is given by the negative of the tangent of the hose angle)

\[
\hat{Q}_R = \frac{\theta(f)}{2 \hat{\chi}(f)}
\]

(40)

one gets

\[
\hat{Q}_R = \frac{1}{2} \tan \left( \frac{\pi f}{2} \right)
\]

from (39)
This yields $Q_R$ as independent of frequency. But to explain some of the experimental results of Letherisich (1950), Wmenehel (1965) and Jordan (1966), he added a third parameter as

$$\theta(f) = \frac{2\pi f}{\sqrt{f}} + \omega(f) + 2\pi f \tau$$

which leads to $Q_R$ as a slowly increasing function of frequency.

$$Q_R = \frac{1}{2} \tan \left( \frac{\theta}{2} \right) + \frac{\pi \tau}{K} \left( 1 - \varepsilon \right)$$

where $\tau, K =$ travel time of waves of finite frequency.

Thus it seems that theoretically the quality factor $\sqrt{Q}$ is frequency dependent. Puttermann's theory imposing a limit $(\omega_0)$, which, from all accounts is arbitrary beyond which it is independent of frequency also imply the variation of $Q$ with frequency in general. While the constant-$Q$ hypothesis find most of the support from the different investigators including Knopoff (1964) who has reviewed the works of many other workers. The recent works of Raitian and Khalturin (1978), Aki (1980) and Thouvenot (1983) apart from Strick (1967) seem to point towards a relationship like

$$Q = Q_0 f^s$$

where $s < 1$

The scope of this work being very limited as mentioned
in the introduction it is not possible to discuss the dependency of on frequency. In this study, we shall be assuming to be independent of frequency.

5.3 Body-wave Attenuation, Spectral Ratio Method

In case of attenuation of seismic body waves by the earth another factor to be considered is the geometrical spreading factor. Both Teng (1968) and Long (1969) assumed the geometrical factor to be independent of frequency. This follows from the work of Kishimoto (1964), which means that the normalised source spectrum is radiated regardless of direction in time. Teng and Menahem (1965) have shown that the source function can be separated into a spatial part and temporal part as,

\[ A_\circ (t, \theta, \phi) = A_s (\theta, \phi) A_T (f) \] (41)

The wave from observed at an epicentral distance can then be put as

\[ A (f, \Delta) = A_\circ (f) F(p) A_T (f) B(t) \] (42)

where \( A_T \) = Attenuation function

\[ = \exp \left( - \frac{\pi f}{c \tilde{Q}} \right) \]

\[ = \exp \left( - \frac{\nu f}{2 c \tilde{Q}} \right) \]
\[ ds = \text{element of the ray path} \]
\[ B(f) = \text{function comprised of instrumental transfer function and crustal transfer function} \]
\[ F(r) = \text{Geometrical spreading function.} \]

The attenuation function is cumulative being different at different point of the interior of the earth. So the integration of the attenuation function is to be taken along the ray path.

The wave forms observed at two stations \((A_i, A_j)\) are then

\[ A_i(t, A_i) = A_0(t) F(r_i) B(f) e^{-\frac{t^* t_i^*}{t}} \]
\[ A_j(t, A_j) = A_0(t) F(r_j) B(f) e^{-\frac{t^* t_j^*}{t}} \]

Putting \( t^* = \int_{\text{ray}} \frac{ds}{c_\beta} \)

Taking ratio of these two and then logarithm of the ratio, one gets,

\[ \log_e \left[ \frac{A_i(t, A_i)}{A_j(t, A_j)} \right] = L + f(t_j^* - t_i^*) \] \( (44) \)

where \( L = \log_e \left[ \frac{F(r_i)}{F(r_j)} \right] \)

\( B(f) \) is assumed same for all the stations. The left hand side of (44) is termed as the spectral ratio between the two stations, is a linear function of frequency. The slope of
this function, \( \mathcal{R} A_j^i \), which is termed as 'differential attenuation' is related to the Q structure as (Teng, 1968)

\[
\frac{\partial}{\partial \mathcal{R}} \left[ \log \left( \frac{A_j}{K_j} \right) \right] = \mathcal{R} A_j^i = \Pi \left( \int_j \frac{dS}{c \mathcal{R}} - \int_i \frac{dS}{c \mathcal{R}} \right) \tag{45}
\]

\( L \) is frequency independent constant in (44) which is not important in the calculation of Q.

The calculation of differential attenuation with respect to the source function would have given the absolute value of the attenuation suffered by the rays. But since the source function is not known directly, spectral ratio method gives the value of Q with respect to a certain station only. But with the following procedure of Teng it can lead to Q structure of the earth.

5.4 Inversion of Travel Time Matrix

Equation (45) can be written as,

\[
\frac{1}{\Pi} \mathcal{R} A_j^i = \int_j \frac{dT}{Q_j} - \int_i \frac{dT}{Q_i} \tag{46}
\]

with \( j \) fixed and \( i = 1, 2, \ldots, N \).

Now if the mantle is subdivided into \( M \) layers with characteristic constant values \( Q_k \) (\( k = 1, 2, \ldots, M \)), the integral can be approximated by a sum as,
\[ \sum_{K=1}^{M} \left( \frac{\Delta t_{k}^{i}}{\delta t_{k}} \right) - \sum_{K=1}^{M} \left( \frac{\Delta t_{k}^{j}}{\delta t_{k}} \right) = \frac{1}{N} \gamma A_{j} \]  

(47)

\[ \Delta t_{k}^{i} \text{ (or } \Delta t_{k}^{j} \text{)} \text{ are the transit time of the } i\text{th ray (or } j\text{th ray) through the } k\text{th layer, which may be termed as 'differential travel time'. It is zero if the ray does not traverse the layer.} \]

A travel time matrix \( \mathcal{T} \) can be defined as

\[
\mathcal{T} = \begin{bmatrix}
\Delta t_{1}^{j} - \Delta t_{1}^{1} & \Delta t_{2}^{j} - \Delta t_{2}^{1} & \cdots & \Delta t_{M}^{j} - \Delta t_{M}^{1} \\
\Delta t_{1}^{2} - \Delta t_{1}^{2} & \Delta t_{2}^{2} - \Delta t_{2}^{2} & \cdots & \Delta t_{M}^{2} - \Delta t_{M}^{2} \\
\cdots & \cdots & \cdots & \cdots \\
\Delta t_{1}^{N-1} & \Delta t_{2}^{N-1} & \cdots & \Delta t_{M}^{N-1}
\end{bmatrix}
\]

and defining two column vectors \( \mathcal{Q} \) and \( \mathbf{A} \) as,

\[
\mathcal{Q} = \left[ \frac{1}{\mathcal{A}_1}, \frac{1}{\mathcal{A}_2}, \cdots, \frac{1}{\mathcal{A}_M} \right]
\]

\[
\mathbf{A} = \left[ -\frac{1}{N} \delta \mathbf{A}_1, \hat{\mathbf{A}}_2, \cdots, \hat{\mathbf{A}}_{N-1} \right]
\]

Equation (42) can be written as

\[ \mathcal{T} \mathcal{Q} = \mathbf{A} \]  

(48)

If \( N = N - 1 \), the solution of (45) can be written as,
Given the matrices, $Q$ values for different layers can be obtained and $Q$-depth profile of the earth can be constructed.