CHAPTER - 5

Problems for further Study
This chapter is devoted to discussion on further possible developments of some results which we wish to study in our future work.

5.1 NILPOTENCY IN FLEXIBLE DERIVATION ALTERNATOR RINGS:

In [53] Zelmanov showed that a left or right nilpotent alternative ring must be nilpotent. Pokrass [41] studied the solvability and nilpotency in generalized alternative rings. Smith [49] extended the result of Pokrass and proved that for flexible derivation alternator rings either left or right nilpotence implies nilpotence. In this section we discuss the nilpotency in flexible derivation alternator rings.

We know that a non-associative ring with characteristic $\neq 2$ is called a derivation alternator ring if it satisfies the identities
(x,x,x) = 0, \hspace{1cm} 5.1.1
(yz,x,x) = y(z,x,x) + (y,x,x)z \hspace{1cm} 5.1.2
and \hspace{0.5cm} (x,x,yz) = y(x,x,z) + (x,x,y)z. \hspace{1cm} 5.1.3

A non-associative ring is called a flexible derivation alternator ring if it satisfies the identities: (x,y,x) = 0 \hspace{1cm} 5.1.4
and identity 5.1.2.

In linearized form, identities 5.1.4 and 5.1.2 are

(x,y,z) + (z,y,x) = 0. \hspace{1cm} 5.1.5
and \hspace{0.5cm} (yz,x,w)+(yz,w,x) = y \{(z,x,w)+(z,w,x)}+{(y,x,w)+(y,w,x)}z. \hspace{1cm} 5.1.6

Let R be a nonassociative ring. If for positive integer n every product of n elements from R is zero, no matter how the elements are associated, then R is called nilpotent. Let \( R^{[1]} = R \) and define inductively \( R^{[k]} = RR^{[k-1]} \). If \( R^{[n]} = 0 \) for some n, then R is said to be left nilpotent. Similarly, setting \( R_{[1]} = R \) and defining inductively \( R_{[k]} = R_{[k-1]} R \), then R is right nilpotent if \( R_{[n]} = 0 \) for some n. Throughout this section R denotes a flexible derivation alternator ring. For \( a \in R \), we denote by \( L_a \) and \( R_a \) the operators of left and right multiplication by \( a \). The notation \( S_a \) will be used when an operator \( s \) can be either \( L \) or \( R \). We note that, due to flexibility, the opposite ring of R is also a flexible derivation alternator ring. This means that when R satisfies some relation involving multiplication operators, it also satisfies the opposite relation where \( L \)'s and \( R \)'s are interchanged.

If we take as argument first \( x \) and then \( y \), 5.1.5 is equivalent in operator form to each of the following:
\[
R_yR_z - L_yL_z = R_{yz} - L_{zy}. \tag{5.1.7}
\]

\[
L_xR_z - R_yL_x + L_zR_x - R_xL_z = 0. \tag{5.1.8}
\]

Similarly, taking in turn \(y, z,\) and \(x\) as the argument, 5.1.6 can be written in operator form as

\[
R_zR_yR_w + R_xR_wR_y - R_xR_w = R_wR_x - R_yR_x = R_zR_wR_y - R_wR_z + R_{(z,y,w)} + (z,w,y). \tag{5.1.9}
\]

\[
L_yR_xR_w + L_yR_wR_x - R_yR_xL_y = L_yR_xL_w - R_yR_xL_y
\]

\[- R_wL_y + L(y,x,w) + (y,w,x). \tag{5.1.10}
\]

\[
L_zR_xL_y - R_yL_zL_y - L_wL_zL_y - L_yR_wR_x - R_wL_yR_x - L_wL_yR_x = - R_wL_y - L_yR_w + L_{(yw)}L_{(yz)} - L_{(yw)}L_{(yz)} + L_{(yz)}. \tag{5.1.11}
\]

Let \(B\) denote the ideal \(R^2\) of the ring \(R.\) For operators \(T\) and \(T^l\) on \(R,\) we write \(T \equiv T^l\) if \(T - T^l = \sum T_i,\) where each operator \(T_i\) has a factor of the form \(S_{bi}\) with \(b_i \in B.\)

**Lemma 5.1.1** If the operator \(T = S_1 \cdots S_{n+1},\) then \(T \equiv T^l,\) where \(T^l\) factors into \(m\) pairs of multiplication operators with each pair having at least one \(L.\)

**Proof:** By assumption \(T\) is itself the product of \(m\) pairs of multiplication operators. Each of these pairs either has an \(L\) or consists of two consecutive \(R's.\) But by 5.1.7, \(R_yR_z = L_yL_z + R_{yz} - L_{zy} = L_yL_z,\) since \(yz, zy \in R^2 = B.\) Thus
substituting for any double R pairs in $T$, we obtain $T = T^1$, where $T^1$ has the required form. □

**Lemma 5.7.2:** If the operator $T = S_{x_t} \ldots \ldots S_{x_j}$, where $t \geq 1$ of the $S$'s and $L$'s, then $T = \sum L_{y_t} \ldots \ldots L_{y_1} S \ldots \ldots S$.

**Proof:** For $j = 1$ or 2 the lemma is immediately true. Thus we assume $j \geq 3$ and that the lemma holds for values less than $j$. Then for an operator $T$ of the form $LS \ldots \ldots S$, our induction assumption applied to the factor $S \ldots \ldots S$ completes the proof. Hence we are reduced to the case $t = RS \ldots \ldots S$, where we consider the initial factor $RSS$ of $T$. First using 5.1.7 to substitute for $RR$, we see $RRS = LLS$. Next using 5.1.7 to substitute for $LL$, we have $RLL = RRR = LLR$, since $RRS = LLS$. Then 5.1.11 gives $RLR = -RLL + LSS = LSS$, since $RLL = LLR$. This shows that in all cases we can make substitutions reducing $T = RS \ldots \ldots S$ to the form $T = \sum LS \ldots \ldots S$. But as already noted, the induction assumption now applied to the factors $S \ldots \ldots S$ completes the proof. □

**Lemma 5.1.3:** If the operator $T = SX_i \ldots \ldots S$ is such that $j$ of the $x_t \in B$ where $i < j < n$, then $T = \sum S_{y_t} \ldots \ldots S_{y_1} S \ldots \ldots S$ where in each term at least $j$ of the $y_t \in B$ and for each $1 \leq i \leq t$ either $y_t \in B$ or $y_{t+1} \in B$.

This is [49, Lemma 3]. This lemma is used to prove theorem 5.1.1.

**Lemma 5.1.4:** Let $R^{(2m)} = (0)$ where $m \geq 2$. If the operator $T = S_{x_t} \ldots \ldots S_{x_1}$, where for each $1 \leq i < 2m$ either $x_t \in B$ or $x_{t+1} \in B$, then $(B^k)T \subseteq (B^{k+1}) \sum S \ldots \ldots S$. 83
Proof: We first show that if $T = S_{x_1} \ldots S_{x_{n-1}} S_{x_n}$, where for each $1 \leq i < n - 2$ either $x_i \in B$ or $x_{i+1} \in B$ and where $x_{n-1}, x_n \in B$, then $(B^k) T \subseteq (B^{k+1}) \sum S \ldots S$.

We do this by induction on $n$, with $n = 2$ being immediate. Because of the induction assumption, we may assume $S_{x_{n-2}} S_{x_{n-1}} S_{x_n} = S_a S_b S_b$ where $b, b' \in B$.

Let us now set $x = a$, $w = b$, $z = b'$ in 5.1.9. Then $R_a R_b R_b = R_b R_a R_b + R_b R_b R_b - R_b R_b R_b - R_b R_b R_{b'}$. Similarly, setting $x = a$, $w = b$, $y = b'$ in 5.1.10 shows $R_a R_b R_b = \sum S_b \ldots S$; and setting $y = a$, $w = b$, $z = b'$ in 5.3.10 shows $L_a R_b R_b = \sum S_a \ldots S$. Since going to the opposite ring implies the remaining four cases, we see that $S_a S_b S_b = \sum S_{b''} \ldots S$. Now suppose $n = 3$. Then $(B^k) T = (B^k) S_a S_b S_b = (B^k) \sum S_a \ldots S$. On the other hand, if $n > 3$ then $x_n \in B$. Hence $T = S_{x_1} \ldots S_{x_{n-3}} S_b S_b S_b = \sum S_{b''} \ldots S_{b''} \ldots S$, and induction can be applied to each $S_{b''} \ldots S_{b''}$ to conclude $(B^k) T = (B^k) S_{x_1} \ldots S_{b''} \ldots S_{b''} \ldots S 
\subseteq (B^{k+1}) \sum S \ldots S$.

By the preceding, to prove the lemma we can now assume $T = S_{a_1} S_a \ldots S_{a_n},$ where each $b_i \in B$. Let us write $T \sim T'$ if $T - T' = \sum T_i$, where each $T_i$ has the form $S_b S_a S_b,$ $S_b S_b S_a,$ $S_b S_a S_b,$ $S_b S_a S_a,$ $S_b S_a S_b,$ or $S_a S_b S_b.$ We show that any $S_a S_b S_b S_b \sim L_a L_b L_b$. First, using 5.1.7 to substitute for $R_a R_b$, we see $R_a R_b L_a L_b = L_a L_b L_a L_b + R_{ab} L_a L_b - L_{ba} L_a L_b \sim L_a L_b L_a L_b$; and so similarly $R_a R_b R_b = R_a R_b L_a L_b + R_a R_b R_{ab} - R_a R_b L_{ba}$. 

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\[ \sim L_a L_b L_a L_b. \] In particular, this letter relation allows us to now make use of the opposite ring. Thus also \( L_a L_b R_a R_b \sim R_a R_b R_a R_b \sim L_a L_b L_a L_b. \)

We next use 5.1.7 to substitute for \( L_b L_a \) in \( R_a L_b L_a R_b \). This gives
\[
R_a L_b L_a R_b = R_a R_b L_a R_b - R_a R_b L_a R_b - R_a R_b L_a R_b + R_a L_a R_b R_a R_b \sim L_a L_b L_a L_b \quad \text{and from the opposite ring} \quad L_a \ R_b \ L_b \ L_a \ L_a \ R_b \sim L_a \ L_b \ L_a \ L_b. \quad \text{Then using 5.1.8 to substitute for} \\
R_a \ R_b \ \text{in} \ R_a L_b L_a R_b \sim R_a L_a R_b L_b, \ \text{we have} \ R_a \ L_b \ L_a \ R_b \sim R_a L_a R_b R_a L_b R_b L_a + R_a L_b L_a R_b - R_a L_b L_a R_b \sim L_a L_b L_a L_b. \ \text{From opposite ring, this manner also} \\
L_a \ R_b \ L_a \ R_b \sim L_a L_b L_a L_b. \ \text{We now set} \ z = a, w = b, y = a \ \text{in 5.1.11 to obtain} \\
L_a R_b L_a = R_b L_a R_a + L_b L_a R_b - L_a R_b R_b + R_b L_a R_a R_b + L_a L_b R_b - R_b L_a L_b + L_a L_b R_a - L_a L_b R_a R_b + L_a L_b R_a R_b + L_a L_b R_a R_b + L_a L_b R_b - L_a L_b R_b R_a \sim L_a R_b R_a R_b \sim L_a L_b L_a L_b. \ \text{From opposite ring, this manner also} \\
L_a R_b L_a R_b \sim L_a L_b L_a L_b. \ \text{Similarly, using 5.1.7 to substitute for} \\
L_a L_b \ \text{gives} \ L_a L_b L_a L_b \sim L_a L_b L_a L_b. \ \text{And from the opposite ring} \\
L_a R_b L_a L_b \sim L_a L_b L_a L_b. \ \text{These 15 cases establish our claim} \ S_a S_b S_a S_b \sim L_a L_b L_a L_b, \ \text{and we are now ready to consider} \ T = S_a S_b S_a S_b \ldots S_a S_b, \ \text{where} \ m \geq 2 \ \text{and each} \ b_i \in B. \ \text{By the preceding we can substitute for} \ S_a S_b S_a S_b \ldots S_a S_b \ \text{to obtain} \ T = \\
L_a L_b L_a L_b \ldots S_a S_b + \sum S_b S_a S_a S_b + \sum S_b S_a S_a S_b + \sum S_b S_a S_a S_b + \sum S_b S_a S_a S_b + \sum S_b S_a S_a S_b + \sum S_b S_a S_a S_b.
\[\sum S_k S_{k+1} S_{k+2} \ldots S_a + \sum S_k S_{k+1} S_{k+2} \ldots S_a + \sum S_k S_{k+1} S_{k+2} \ldots S_a.\]

Except in the first term, each summand in this expression either has initial factor \(S_b\) or a factor of the form \(S_b S_b\). If the first factor is \(S_b\), the power of \(B\) is clearly raised; and when there is a factor of the form \(S_b S_b\), the power of \(B\) is raised by our initial observation. We next substitute for \(L_a L_b S_a S_b\) to obtain

\[\sum \sum L_a L_b S_a S_b S_{a+1} \ldots S_a + \sum \sum L_a L_b S_a S_b S_{a+1} \ldots S_a + \sum \sum L_a L_b S_a S_b S_{a+1} \ldots S_a + \sum \sum L_a L_b S_a S_b S_{a+1} \ldots S_a.\]

Except for the first term, each summand has two consecutive operators of the form \(S_b\), and so by our initial observation must raise the power of \(B\).

We now substitute for \(L_a L_b S_a S_b\) in \(L_a L_b L_a L_b L_a L_b S_a \ldots \) and continuing in this fashion finally express \(T\) as \(L_a L_b \ldots L_a L_b\) plus terms which must raise the power of \(B\). But since by assumption \(R^{[2n]} = (0)\), then \(L_a L_b \ldots L_a L_b = 0\), and so \([B^k] T \subseteq [B^{k+1}] \sum s \ldots s\). This completes the Proof of the lemma. □

**Theorem 5.1.1:** If \(R\) is a left or right nilpotent flexible derivation alternator ring, then \(R\) is nilpotent.

**Proof:** Suppose first that \(R\) is left nilpotent. Let \(R^{(0)} = R\) and define inductively \(R^{(k)} = (R^{(k-1)})^2\). Then \(R\) is called solvable of index \(n\) if \(R^{(n)} = (0)\) and \(n\) is the least such integer. It is immediate that any left nilpotent ring is solvable. Thus \(R\) is solvable, and to prove \(R\) is nilpotent we induct on the index of solvability of \(R\).
To start, $R$ is clearly nilpotent when $R^2 = R^{(1)} = (0)$. Now by induction we can assume $B = R^2$ is nilpotent, since the ideal $B$ is a left nilpotent flexible derivation alternative ring with solvable index one less than that of $R$. Thus let $B^n = (0)$, and by the left nilpotence of $R$ let $R^{[2m]} = (0)$. Where $m \geq 2$.

We show that for $r = 2 (2m + 1) (2m(n-1)) + 1$ each operator $T = S_{\ldots}S_{\ldots}S_{\ldots}$ is zero, which by theorem 2.4 in [45] establishes that $R$ itself is nilpotent. We first note $(R) \subseteq (B) S_{\ldots}S_{\ldots}S_{\ldots}$. Now $S_{\ldots}S_{\ldots}S_{\ldots}$ is the product of $2m(n-1)$ factors each having length $2(2m + 1)$. By lemma 5.1.1 we can express each of these factors of length $2(2m + 1)$ as $T^1 + \sum T_i$, where $T^1$ has $2m + 1$ L’s and each $T_i$ has a factor of the form $S_b$. By lemma 5.1.2, $T^1$ can in turn be expressed as $\sum L_{\ldots}L_{\ldots}S_{\ldots}S_{\ldots}S + \sum T_i$. But since $R^{[2m]} = (0)$, any operator with $2m$ adjacent L’s is zero. Thus after making these substitutions and multiplying, we arrive at $S_{\ldots}S_{\ldots} = \sum T_i^1$ where each $T_i^1$ is the product of $2m(n-1)$ factors of the form $S_b$.

Now each $T_i^1$ is the product of $n-1$ factors each having $2m$ factors of the form $S_b$. In each $T_i^1$ we consider the first of these factors with $2mS_b$’s. Using lemma 5.1.3, we can substitute for these initial factor to obtain $T_i^1 = \sum T_1 T_2$, where $T_i = S_{\ldots}S_{\ldots}S_{\ldots}S_{\ldots}S_{\ldots}S$ with at least $2m$ of the $y_i \in R$ and for each $1 \leq i < t$ either $y_i \in B$ or $y_{i+1} \in B$, and where $T_2$ is the product of $n-2$ factor each having $2m$ factors of the form $S_b$. Thus by lemma 5.1.4, $(R) T \subseteq (B) S_{\ldots}S_{\ldots} = (B) \sum T_i^1 = (B) \sum T_1 T_2 \subseteq (B^2) \sum S_{\ldots}S_{\ldots}S T_2$. We now consider in each $T_2$ the first factor with $2mS_b$’s, and repeat the proceeding.
argument. Continuing in this fashion, we arrive at \((R) T \subseteq (B^n) \sum S \ldots S = (0)\), which shows \(R\) is nilpotent as claimed.

Finally, suppose \(R\) is right – nilpotent. Then the opposite ring of \(R\) is a left nilpotent flexible derivation alternator ring. Thus, by what has been already proved, the opposite ring is nilpotent, which of course means \(R\) itself is nilpotent as well.\(\square\)

We have seen that for a flexible derivation alternator ring either left or right nilpotency implies nilpotence. Using this we wish to try for the nilpotency of the derivation alternator rings without using flexible property.

5.2 ANTICOMMUTATIVE DERIVATION ALTERNATOR RINGS:

In [13] it is shown that derivation alternator Lie rings, which are anticommutative are solvable at most of index 2. Also it is proved that a simple flexible derivation alternator ring is either alternative or anticommutative. In [36] Nimmo developed the structure of simple anticommutative derivation alternator rings. In this section we present some properties of the multiplication in anticommutative derivation alternator rings.

In this section, we investigate the structure of anticommutative rings that satisfy 5.1.1, 5.1.2 and 5.1.3. We note that anticommutative derivation alternator rings can be defined simply by identity 5.1.2 and identity

\[ xy = -yx. \]  

5.2.1

Using 5.2.1, we can simplify the linearized version of 5.1.2 to

\[ (yz.x)w + (yz.w)x = y (zx.w) + (yx.w)z + y (zw.x) + (yw.x)z. \]
Let $A$ be an algebra satisfying 5.1.2 and 5.2.1. We know that an eigenvector of a square matrix $M$ is a nonzero vector $x$ such that $Mx = \lambda x$ for some scalar $\lambda$. A scalar $\lambda$ is called an eigenvalue of $M$ if there is a nontrivial solution $x$ of $Mx = \lambda x$. The set of all solutions of $(M-\lambda I)x = 0$ is called the eigenspace of $M$ corresponding to $\lambda$. For $a \in A$, let the operator of right multiplication by $a$ be denoted by $Ra$. Via this operator, we decompose $A$ into generalized eigenspaces. We assume that $A$ is finite dimensional and we adopt the notation that $x_i^{(j)}$ is a generalized eigenvector of order $j$ with the associated eigenvalue $r_i$, $i, j \in \mathbb{N} \cup \{0\}$. In this notation, $x_i^{(0)}$, a generalized eigenvector of order 0, is a pure eigenvector.

We let the set $\{x_i^{(j)} / i = 0, 1, 2...n, j = 0,1,2,3 ...., N_i\}$ be a basis for $A$.

Theorem 5.2.1: Let $A$ be an algebra that satisfying 5.1.2 and 5.2.1. Also let $y_p^{(k)}$ and $z_q^{(l)}$ be generalized eigenvectors with associated eigenvalues $r_p$ and $r_q$, respectively, and let $\{ r_i / i = 1,............n\}$ be the eigenvalues for $Ra$. If we define $I = \{ i / r_i^2 = r_p^2 + r_q^2 \}$ then $y_p^{(k)} z_q^{(l)} = \sum_{i,j} t_{i,j} x_i^{(j)}$, where $i \in I$, $r_p, t_{i,j} \in \mathbb{C}$.

Proof: We intend to show that $y_p^{(k)} z_q^{(l)} = 0$ unless there is some $r_i$ such that $r_p^2 + r_q^2 = r_i^2$, and if such an $r_i$ exists, then $y_p^{(k)} z_q^{(l)} = \sum_{i,j} t_{i,j} x_i^{(j)}$, where $i \in I$.

Since $y_p^{(0)} Z_q^{(0)} = = \sum_{i,j} t_{i,j} x_i^{(j)}$, then by 5.1.2

\[
(y_p^{(0)} Z_q^{(0)}) a.a = \left( \sum_{i,j} t_{i,j} x_i^{(j)} \right) a.a
\]

\[
= \sum_{i,j} t_{i,j} (x_i^{(j)} a.a)
\]

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where we use the notation that the term $x^{(i)}$ is zero if $j < 0$.

But also

\[
(y_p^{(0)} z_q^{(0)})_{a.a} = (y_p^{(0)} a.a) z_q^{(0)} + y_p^{(0)} (z_q^{(0)} a.a)
\]

\[
= \left(r_p^2 + r_q^2\right) z_q^{(0)}
\]

\[
= \left(r_p^2 + r_q^2\right) \sum_{i,j} t_{i,j} x_i^{(j)}.
\]

Equating the coefficients of $x_i^{(j)}$ yields

\[
\left(r_p^2 + r_q^2\right) t_{i,j} = r_i^2 t_{i,j} + 2r_{i,j+1} + t_{i,j+2},
\]

which implies $t_{i,j+2} = \left(r_p^2 + r_q^2 - r_i^2\right) t_{i,j} - 2r_{i,j+1} + t_{i,j+1}$.

If we now set $j = N_i$, then $t_{i,j+1} = t_{i,j+2} = 0$ and so either $r_p^2 + r_q^2 - r_i^2 = 0$ or $t_{i,j} = 0$. If $r_p^2 + r_q^2 - r_i^2 \neq 0$ then $t_{i,N_i} = 0$ and by finite induction, $t_{i,j} = 0$ for all $j$.

Therefore $y_p^{(0)} z_q^{(0)} = \sum_{i,j} t_{i,j} x_i^{(j)}$ where $i \in I$.

Now consider $y_p^{(1)} z_q^{(0)} = \sum_{i,j} t_{i,j} x_i^{(j)}$. Proceeding as we did above

\[
(y_p^{(1)} z_q^{(0)})_{a.a} = \sum_{i,j} t_{i,j} \left(r_p^2 x_i^{(j)} + 2r_i x_i^{(j+1)} + x_i^{(j+2)}\right)
\]

and also

\[
(y_p^{(1)} z_q^{(0)})_{a.a} = y_p^{(1)} (z_q^{(0)} a.a) + (y_p^{(1)} a.a) z_q^{(0)}
\]

\[
= \left(r_p^2 + r_q^2\right) y_p^{(1)} z_q^{(0)} + 2r_p y_p^{(0)} z_q^{(0)}
\]

\[
= \left(r_p^2 + r_q^2\right) \sum_{i,j} t_{i,j} x_i^{(j)} + 2r_p y_p^{(0)} z_q^{(0)}.
\]
Equating the coefficients of $x_i^{(0)}$ and using the previous case for $y_p^{(0)}z_q^{(0)}$, we obtain
\[ (r_p^2 + r_q^2)_{i,j} + 2r_p \left( \sum_{i \in I, j} S_{i,j} \right) = r_p^2 t_{i,j} + 2r_p t_{i,j} + t_{i,j+2}. \]
Solving for $t_{i,j+2} = (r_p^2 + r_q^2 - r_p^2) t_{i,j} - 2r_p t_{i,j+1} + 2r_p \left( \sum_{i \in I, j} S_{i,j} \right)$.

Setting $j = N$, we get
\[ 0 = (r_p^2 + r_q^2 - r_p^2) t_{i,j} + 2r_p \left( \sum_{i \in I, j} S_{i,j} \right). \]

Thus if $i \notin I$, then $t_i, N = 0$, and by induction we see that $t_i, j = 0$ for all $j$.

Hence, $y_p^{(0)} z_q^{(0)} \sum t_{i,j} x_i^{(j)}$ where $i \in I$.

Now we proceed by induction on $m = k + l$. We assume that
\[ y_p^{(k)} z_q^{(l)} = \sum_{i,j} s_{i,j} x_i^{(k)} \quad \text{where } i \in I \text{ for } k + l < m. \]

Consider the product $y_p^{(k)} z_q^{(l)} = \sum_{i,j} t_{i,j} x_i^{(k)}$, where $k + l = m$.

Then by 5.1.2,
\[
(y_p^{(k)} z_q^{(l)})_{a.a} = \left( \sum_{i,j} t_{i,j} x_i^{(k)} \right)_{a.a}
= \sum_{i,j} t_{i,j} (x_i^{(k)})_{a.a}
= \sum_{i,j} t_{i,j} \left( r_i x_i^{(k)} + 2r_i x_i^{(j-1)} + x_i^{(j-2)} \right).
\]

Also
\[
(y_p^{(k)} z_q^{(l)})_{a.a} = y_p^{(k)} (z_q^{(l)})_{a.a} + (y_p^{(k)})_{a.a} z_q^{(l)}
= (r_p^2 + r_q^2) y_p^{(k)} z_q^{(l)} + 2(r_p y_p^{(k-1)} z_q^{(l-1)} + r_q y_p^{(k)} z_q^{(l-1)}) + y_p^{(k-2)} z_q^{(l)} + y_p^{(k)} z_q^{(l-2)}
= (r_p^2 + r_q^2) \sum_{i,j} t_{i,j} x_i^{(k)} + \left( \sum_{i \in I, j} S_{i,j} x_i^{(k)} \right).
\]
by the induction assumption. Equating coefficients of \( x_i^{(0)} \) and solving for \( t_{ij+2} \), we obtain

\[
t_{ij+2} = (r_p^2 + r_q^2 - r_i^2)t_{ij} - 2r_i t_{ij+1} + \left( \sum_{(u,i,j)} S_{i,j} \right).
\]

Setting \( j = N_1 \) yields

\[
0 = (r_p^2 + r_q^2 - r_i^2)t_{ij} + \left( \sum_{(u,i,j)} S_{i,j} \right).
\]

Thus if \( i \not\in I \) then \( t_i N_1 = 0 \) and so by induction \( t_{ij} = 0 \) for all \( j \).

Therefore \( y_p^{(x)} z_q^{(x)} = \sum_{i,j} t_{ij} x_i^{(j)} \) where \( i \in I \). □

**Theorem 5.2.2:** Let \( A \) be an algebra that satisfies 5.1.2 and 5.2.1, and let \( a \in A \), if \( b \) and \( c \) are pure eigenvectors for \( R_a \) with associated eigenvalues \( \lambda \) and \( \mu \), respectively, then \( bc \) is a sum of two pure eigenvectors or \( \lambda^2 + \mu^2 = 0 \).

**Proof:** Suppose \( \lambda^2 + \mu^2 = t^2 \neq 0 \) and that

\[
bc = k_0 x_t^{(0)} + \ldots + k_0 x_t^{(0)}.
\]

Invoking the same technique we used in theorem 5.2.1,

\[
(bc.a) a = b(ca.a) + (ba.a) c
\]

\[
= t^2 bc
\]

\[
= t^2 \left( k_j x_t^{(0)} + \ldots + k_0 x_t^{(0)} \right)
\]

and also

\[
(bc.a) a = (k_j x_t^{(0)} + k_{j-1} x_t^{(0)}) + \ldots + k_0 x_t^{(0)} a.a
\]

\[
= k_j t^2 x_t^{(0)} + (2k_j t + k_{j-1} t^2) x_t^{(0-1)} + \ldots
\]

Equating the coefficients of \( x_t^{(0-1)} \) we obtain

\[
k_{j-1} t^2 = 2k_j t + k_{j-1} t^2
\]

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which implies that $2k_jt = 0$. Thus $k_j = 0$ or $t = 0$, but both of these possibilities lead to a contradiction. Hence, $bc$ is a pure eigenvector. In the case that $bc$ lives in the sum of the generalized eigenspaces corresponding to $t$ and $-t$,

$$bc = k_j x_t^{(0)} + \ldots + k_0 x_t^{(0)} + l_1 y_t^{(i)} + \ldots + l_0 y_t^{(0)},$$

then an analogous argument shows that $bc$ must be a sum of two pure eigenvectors. □

Now we give an example of an algebra that satisfy equations 5.1.2 and 5.2.1.

**Example 5.2.1**: Let $A$ consist of the elements $a, b, ab, a.ab, a(a.ab), \ldots$.

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
<th>$ab$</th>
<th>$a.ab$</th>
<th>$a(a.ab)$</th>
<th>$a.a(a.ab)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0</td>
<td>$ab$</td>
<td>$a.ab$</td>
<td>$a(a.ab)$</td>
<td>$a.a(a.ab)$</td>
<td></td>
</tr>
<tr>
<td>$b$</td>
<td>-$ab$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$ab$</td>
<td>-$a.ab$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$a.ab$</td>
<td>-$a(a.ab)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$a(a.ab)$</td>
<td>-$a.a(a.ab)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Here we verify 5.1.3. Since 5.2.1 holds, 5.1.3 simplifies to

$$x(x.yz) = y(x.xz) + (x.xy)z.$$  \hspace{1cm} 5.2.2

By assumption, any product containing more than one $b$ is zero. Hence, 5.2.2 holds trivially. For the same reason, only one of the two elements $y$ and $z$ can contain a $b$. In view of 5.2.1, we may assume that $y = a$. If we use the notation that $a(a\ldots b)$ stands for stacking $a$'s onto $b$ from the left, we need only show that

$$a(a.a(a(a\ldots b))) = a(a.a(a(a\ldots b))) + (a.aa)(a(a\ldots b)).$$

But $aa = 0$ by 5.2.1 and this equation holds. This shows the existence of an anticommutative derivation alternator ring.
We have presented some properties of the multiplication in anticommutative derivation alternator rings. Using these we wish to study the structure of general anticommutative derivation alternator rings under certain conditions.

5.3 ON THE COMMUTATIVITY OF PRIME RINGS WITH DERIVATION

In many results on commutativity of rings, we have seen that if a certain subset S of the ring R is central, then R is commutative. There are also some results which state that if a non zero derivation maps a suitably chosen subset onto {0}, then R is commutative. Bell [4] proved that if d is a non-zero derivation and d(s) is central for appropriately – chosen subsets S of R, then R must be commutative. In this section we discuss the commutativity of prime rings with derivation.

Throughout the section R denotes ring, and C its center. For arbitrary subsets H of R, \(|H|\) denotes the cardinal number of H. We gather together some elementary facts which will be used repeatedly.

In any ring R,
\[
[xy,z] = x[y,z] + [x,z] y \quad \text{and} \quad [x,yz] = y[x,z] + [x,y] z,
\]
for all \(x,y,z \in R\).

If R is any ring and d any derivation, \(d(C) \subseteq C\).

and \(d(x^n) = x^{n-1} d(x) + x^{n-2} d(x)x + \ldots + d(x) x^{n-1}\) for all \(x \in R\). In particular, if \((x, d(x)) = 0\), then
\[
d(x^n) = nx^{n-1} d(x).
\]
If R is any ring and d any derivation, then
\[ d((x,y)) = (x, d(y)) + d(x),y), \text{ for all } x,y \in R. \] 5.3.4

The center of a prime ring \( R \) contains no non zero elements which are zero divisors in \( R \) 5.3.5

If \( c \in C \setminus \{0\} \) and \( r \) is an element of \( R \) such that \( rc \in C \), then \( r \in C \).

We also require several lemmas.

**Lemma 5.3.1** A finite field admits no non zero derivation.

**Proof:** Let \( K = \text{GF}(p^n) \). If \( x \in K \), then \( x = x^{p^n} \) and it follows from 5.3.3 that \( d(x) = d(x^{p^n}) = p^n x^{p^n-1} d(x) = 0 \).

**Lemma 5.3.2:** Let \( R \) be a prime ring satisfying an identity \( q(X) = 0 \), where \( q(X) \) is a polynomial in a finite number of noncommuting indeterminates, its coefficients being integers with highest common factor 1. If there exists no prime \( p \) for which the ring of \( 2 \times 2 \) matrices over \( \text{GF}(p) \) satisfies \( q(X) = 0 \), then \( R \) is commutative.

**Lemma 5.3.3:** If \( R \) is a prime ring and \( x \) is an element of \( R \) such that \( (x,(y,w)) = 0 \), for all \( y, w \in R \), then \( x \in C \).

**Lemma 5.3.4:** Let \( R \) be a prime ring with characteristic \( \neq 2 \). If \( R \) admits a non zero derivation \( d \) such that \( (d(y),d(y)) = 0 \), for all \( x, y \in R \), then \( R \) is commutative.

**Lemma 5.3.5:** Let \( R \) be a prime ring \( L \) a non zero left ideal of \( R \). If \( R \) admits a non zero derivation \( d \) such that \( (d(x),x) = 0 \) for all \( x \in L \), then \( R \) is commutative.
Lemma 5.3.6: Let $D$ be a division algebra over a finite field $F$, and let $S$ be a subalgebra. If for each $x \in D$, there exists a nonconstant polynomial $f(t) \in F(t)$ for which $f(x) \in S$, then $D$ is commutative.

Lemma 5.3.7: Let $n > 1$ be a positive integer and $R$ a prime ring. If $R$ admits a non zero derivation $d$ such that $d(x^n) = 0$ for all $x \in R$, then $R$ is commutative and infinite. Moreover, characteristic $= p > 0$ and $p/n$.

Theorem 5.3.1: Let $n > 1$, and let $R$ be a prime ring. If $R$ admits a non zero derivation $d$ such that $d(x^n) \in C$ for all $x \in R$, then $R$ is infinite and commutative.

Proof: $R$ is infinite is immediate from lemma 5.3.1 once we establish commutativity. If characteristic $= p$, then by 5.3.3 we have $d((x^n)^p) = p(x^n)^{p-1}d(x^n) = 0$, for all $x \in R$, so commutativity follows from theorem 5.3.1. Thus we assume that characteristic $= 0$.

Now for $x \in R$ and $c \in C$, we have $d((xc)^n) = d(c^n x^n) = c^n d(x^n)$ + $d(c^n) x^n \in C$; and if $d(c^n) \neq 0$, this yields $x^n \in C'$ for all $x \in R$, so that $R$ is commutative by lemma 5.3.2. Therefore we assume $d(c^n) = 0 = nc^{n-1}d(c)$ for all $c \in C$, hence $d(C) = \{0\}$. Thus $d^2(x^n) = 0$ for all $x \in R$, so that $d^2(x^n) = d^2(x^n) x^n + 2d(x^n)^2 + x^n d^2(x^n)^2 = 0 = 2d(x^n)^2$, for all $x \in R$. Since characteristic $\neq 2$, it follows form 5.3.4 that $d(x^n) = 0$ for all $x \in R$, so we need only invoke lemma 5.3.7. □
It is proved in [5] that if $R$ is a prime ring admitting a nonzero derivation $d$ such that $d(xy - yx) = 0$ for all $x, y \in R$, then $R$ is commutative. Our next theorem is a natural extension.

**Theorem 5.3.3**: Let $R$ be a prime ring with characteristic $\neq 2$. If $R$ admits a nonzero derivation $d$ such that $d((x,y)) \in C$ for all $x, y \in R$, then $R$ is commutative.

**Proof:** By 5.3.4, we have $(x,d(y)) + (d(x),y) \in C$ for all $x,y \in R$. 5.3.6

Replacing $y$ by $cy$, $c \in C$, we get $c(x,d(y)) + d(c)(x,y) + c(d(x),y) \in C$.

That is, $c((x,d(y)) + (d(x),y)) + d(C)(x,y) \in C$.

Recalling 5.3.6, we see that

$$d(c)(x,y) \in C \text{ for all } x, y \in R \text{ and } c \in C.$$  

If $d(C) \neq \{0\}$, this implies $(x,y) \in C$ for $x, y \in R$, so that $R$ is commutative by lemma 5.3.2.

We now suppose that $d(C) = \{0\}$, in which case

$$d^2(xy - yx) = 0 \text{ for all } x, y \in R.$$  

5.3.7

Expanding this condition, we get

$$(x, d^2(y)) + (d^2(x),y) + 2(d(x), d(y)) = 0 \text{ for all } x, y \in R.$$  

5.3.8
Letting $y$ be a commutator $v$ and applying 5.3.7, we get $(d^2(x),v) = 0$ for all $x \in R$ and all commutators $v$. It follows by lemma 5.3.3 that $d^2(R) \subseteq C$, so by 5.3.8 we obtain $(d(x),d(y)) = 0$ for all $x,y \in R$. Therefore, $R$ is commutative by lemma 5.3.4. □

We have studied the commutativity of prime rings with derivation on commutator. Using this we wish to study the alternativity of prime rings with derivation on alternators.