CHAPTER - 4

Antiflexible Derivation
Alternator Rings
Antiflexible algebras were introduced by Kosier [32], and a sub
class of antiflexible rings was studied by Kleinfeld[18]. A number of
properties of antiflexible rings are given by Kosier [32] and Rodabaugh [42].
In all these studies they have assumed the identity \((x,x,x) = 0\). But in derivation
alternator rings we have this identity. So in this chapter we present some
properties of primitive, prime and simple antiflexible derivation alternator
rings. In section 4.1, we show that a primitive antiflexible derivation alternator
ring is either associative or simple with an identity element. In section 4.2, we
prove that in an antiflexible derivation alternator ring, the nucleus and center
are equal. Also, we prove that a prime antiflexible derivation alternator ring is
associative. At the end of this section we give an example of an antiflexible
derivation alternator ring which is not alternative. In section 4.3, we prove that
a simple antiflexible derivation alternator ring is either associative or
commutative. Also, it is shown that antiflexible derivation alternator rings are
associative if and only if they satisfy the condition that \((x,y,x)^2 = 0\), implies
\((x,y,x) = 0\), for all elements \(x,y\) of the ring.
4.1 PRIMITIVE ANTIFLEXIBLE DERIVATION ALTERNATOR RINGS:

Celik [8] investigated the properties and the structure of primitive antiflexible rings. In [40] Paul and Saradha proved that a left primitive (-1,1) ring with commutators in the left nucleus is either associative or simple with right identity element. In this section we show that in an antiflexible derivation alternator ring, commutators are in the middle nucleus. Using this it is shown that a primitive antiflexible derivation alternator ring is either associative or simple with an identity element.

We know that a non-associative ring with characteristic \(\neq 2\) is called a derivation alternator ring if it satisfies the identities:

\[
(x,x,x) = 0, \quad 4.1.1
\]

\[
(yz,x,x) = y (z,x,x) + (y,x,x)z \quad 4.1.2
\]

and \(x,x,yz) = y(x,x,z) + (x,x,y) z. \quad 4.1.3
\]

In this section, we investigate the structure of non-associative, antiflexible rings that satisfy 4.1.1, 4.1.2 and 4.1.3. We note that antiflexible derivation alternator rings can be defined simply by 4.1.1, 4.1.2 and the identity

\[
D(x,y,z) = (x,y,z) - (z,y,x) = 0. \quad 4.1.4
\]

Throughout this section R denotes an antiflexible derivation alternator ring. A ring R is called primitive if it contains maximal ideal, which contains no two sided ideals of R other than \((0)\). A right ideal A of R is called regular if there exists an element \(g \in R\), such that \(x - gx \in A\) for all \(x \in R\).
From 4.1.1, 4.1.2 and 4.1.3, we obtain
\[(x, yz, x) = y(x, z, x) + (x, y, x) z.\] 4.1.5

By linearizing 4.1.5, we have
\[(x, yz, w) = y(x, z, w) + (x, y, w) z.\] 4.1.6

From 4.1.1 and 4.1.4, we obtain
\[J(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y).\] 4.1.7

The following identities hold in any arbitrary ring:
\[C(w, x, y, z) = (wx, y, z) - (w, xy, z) + (w, x, yz) - w(x, y, z) - (w, x, y)z = 0,\] 4.1.8
\[G(x, y, z) = (xy, z) - x(y, z) - (x, z)y - (x, y, z) - (z, x, y) + (x, z, y) = 0,\] 4.1.9

The following identities hold in antiflexible derivation alternator rings:
\[(xoy)oz - xo(yoz) = (y, (x, z)),\] 4.1.10
\[(xoy, z) + (yoz, x) + (zox, y) = 0,\] 4.1.11
\[E(x, y, z) = (xy, z) - x(y, z) - (x, z)y + 2 (x, z, y) = 0\] 4.1.12
and \[((x, y), z) + ((y, z), x) + ((z, x), y) = 0.\] 4.1.13

Using these identities first we prove the following lemmas.

**Lemma 4.1.1:** In an antiflexible derivation alternator ring, commutators are in the middle nucleus.

**Proof:** By forming \(0 = C(w, x, y, z) - C(x, y, z, w) + C(y, z, w, x) - C(z, w, x, y)\) and using 4.1.7, we obtain
\[H(w, x, y, z) = (w, (x, y, z)) - (x, (y, z, w)) + (y, (z, w, x)) - (z, (w, x, y)) = 0.\] 4.1.14
Expanding $0 = C(w,x,y,z) - C(z,y,x,w)$ and using 4.1.4 we get

$$0 = L(w,x,y,z) = ((w,x),y,z) - (w,(x,y),z) + (w,x,(y,z))$$

$$- (w,(x,y,z)) - ((w,x,y),z).$$ 4.1.15

Then we expand

$$0 = L(w,x,y,z) + L(x,y,z,w) + L(y,z,w,x) + L(z,w,x,y) - J((w,x),y,z)$$
$$- J((x,y), z,w) - J ((y,z),w,x) - J ((z,w),x,y)$$

to get

$$0 = K(w,x,y,z) = (w,(x,y),z) + (x,(y,z),w) + (y,(z,w),x) + (z,(w,x),y).$$ 4.1.16

From $0 = H(x,x,y,x) + (x,J(x,x,y))$ it follows that $2(x,(x,y,x)) = 0$, so that

$$(x,(x,y,x)) = 0.$$ 4.1.17

Hence from $0 = (x,J(x,y,x))$ and $0 = (x,D(x,x,y))$, we have

$$(x,(y,x,x)) = 0.$$ 4.1.18

and

$$(x,(x,x,y)) = 0.$$ 4.1.19

From 4.1.2, the identity 4.1.18 becomes

$$((y,x),x,x) = 0.$$ 4.1.20

Hence from 4.1.4, the identity 4.1.20 becomes

$$(x,x,(y,x)) = 0.$$ 4.1.21

From 4.1.20, 4.1.21 and 4.1.7, we get

$$(x,(y,x),x) = 0.$$ 4.1.22

Substituting $- z$ for $z$ in 4.1.22 and then adding to 4.1.22 yields

$$2(x,(y,z),z) + (z,(y,x),z) = 0.$$ 4.1.23
Next, linearize 4.1.23 and \( D ((y,x),z) = 0 \) to get

\[
I ((y,x),z) = (x,(y,z),w) + (x,(y,w),z) + (w,(y,x),z) = 0. \tag{4.1.24}
\]

Computing \( 0 = K(w,x,y,z) + I (w,x,y,z) + I (x,y,w,z) - D (z,(w,x),y) \),

we get \( M(w,x,y,z) = (x,(y,z),w) + (y,(w,x),z) = 0. \tag{4.1.25} \)

Expanding \( 0 = M(w,x,y,z) + M(x,w,y,z) - D(w,(y,z),x) \), we get

\[
(w,(x,y),z) = 0. \tag{4.1.26}
\]

Thus commutators are in the middle nucleus. This completes the proof of the lemma. \( \square \)

With the aid of 4.1.26, we can improve 4.1.15 to obtain

\[
0 = Q (w,x,y,z) = ((w,x),y,z) + (w,x,(y,z)) - (w,(x,y,z)) - ((w,x,y),z). \tag{4.1.27}
\]

Finally, applying 4.1.26 to \( 0 = J ((w,x),y,z) - D (y,z,(w,x)) \), we have

\[
0 = P(w,x,y,z) = ((w,x),y,z) + ((w,x),z,y). \tag{4.1.28}
\]

**Lemma 4.1.2:** In an antiflexible derivation alternator ring, \((x,x,(y,z)) = 0\).

**Proof:** From 4.1.5 and identity 4.1.26, we get

\[
(x,w,(y,z)) + (w,x,(y,z)) = 0 \tag{4.1.29}
\]

By taking \( w = x \) in 4.1.29 and \( t (x,x,(y,z)) = 0 \). \( \square \)
Lemma 4.1.4: Let $A$ be a right ideal of antiflexible derivation alternator ring $R$. Then,

(i) $S = \{ s \in A \mid Rs \subseteq A \}$ is a two-sided ideal of $R$,

(ii) $(R, A, R) \subseteq S$

Proof: (i) For any $s \in S$, $x \in R$, we consider $sx$ and $xs$.

Let $y \in R$. Then using 4.1.7, $y(sx) = -(y,s,x) + (ys)x = (s,x,y) + (s,y,x) + (ys)x$,
that is, $y(sx) \in A$, and hence $s x \in S$. Also, $y(xs) = -(y,x,s) + (yx)s = -(s,x,y) + (yx)s$ and hence $y(xs) \in A$, and $x s \in S$. It follows that $S$ is a two-sided ideal of $R$.

(ii) For any $x,y \in R$, $a \in A$, $(x,a,y) = -(a,x,y) - (a,y,x)$ by 4.1.7.

Hence $(R,A,R) \subseteq A$.

Let $z \in R$ be arbitrary. Then using 4.1.8, and 4.1.26, we get that

$$(z,xa,y) = (z,ax,y)$$

and by 4.1.7, $$(zx,a,y) = -(a,zx,y) - (a,y,zx).$$

Thus $z(x,a,y) = -(a,zx,y) - (a,y,zx) - (z,ax,y) + (ay,x,z) - (a,x,z)y$ and hence $z(x,z,y) \in A$. This implies that $(R,A,R) \subseteq S$. □

Theorem 4.1.1: If an antiflexible derivation alternator ring $R$ has a maximal right ideal $A \neq (0)$, which contains no two-sided ideal of $R$ other than $(0)$, then $R$ is associative.
Proof: By lemma 4.1.4, $S$ is a two-sided ideal of $R$ contained in $A$. Therefore $S = (0)$, and hence $(R,A,R) = (0)$. On the other hand, it is easy to verify that $A + RA$ is a two-sided ideal of $R$. Since $A \subseteq A + RA$, we must have $A + RA = R$. Thus, considering $(R,R,R) = (R,A + RA,R) = (R,RA,R)$. But $(R,RA,R) = (R,AR,R) \subseteq (R,A,R) = (0)$. Therefore, $(R,R,R) = (0)$, that is, $R$ is an associative ring. □

Theorem 4.1.2: If $R$ is a primitive antiflexible derivation alternator ring, then either $R$ is associative or it is simple with an identity element.

Proof: Let $A$ be a regular maximal right ideal of $R$, with a modular element $g$. Either $A = (0)$ or $A \neq (0)$. If $A \neq (0)$, by theorem 4.1.1, $R$ is associative. Thus, we assume that every regular maximal right ideal of $R$ is $(0)$. In particular, there exists $g \in R$, such that $x - gx = 0$ for all $x \in R$. Therefore every right ideal of $R$ is regular. By Zorn's lemma any regular right ideal of $R$ can be imbedded in a regular maximal right ideal of $R$. Therefore $R$ has no proper right ideals, and hence $R$ is simple. The proof will be completed if we show that $g$ is the identity element of $R$. By assumption $g$ is a left identity element.

We consider the set $T = \{x \in R / \exists g (xg = x)\}$.

Let $y \in R$, and $x \in T$. Then, $0 = (g,x,y) = (y,x,g) = (yx)g - yx$.

Therefore, $yx \in T$, which implies that $T$ is a left ideal of $R$. Since $g \in T$, $T \neq (0)$.
Since $R$ is simple, we have that $T = R$, and thus $g$ is a right identity element of $R$, therefore it is the identity element of $R$. □
4.2 PRIME ANTIFLEXIBLE DERIVATION ALTERNATOR RINGS:

Some properties of prime antiflexible rings are given by Celik [8]. In [39] Paul considered prime ring $R$ satisfying $(x,y,z) = (x,z,y)$ with nucleus $N$ and center $C$. He proved that if $R$ has commutators in the middle nucleus then either $R$ is associative or $N = C$. In this section, we prove that in a prime antiflexible derivation alternator ring, the nucleus and center are equal. Also, we prove that a prime antiflexible derivation alternator ring with idempotent $e \neq 1$ is alternative. At the end of this section we give an example of an antiflexible derivation alternator ring which is not alternative.

We know that the middle nucleus $N_m = \{ n \in R / (R,n,R) = (0) \}$, the nucleus $N = \{ n \in R / (n,R,R) = (R,n,R) = (R,R,n) = (0) \}$ and the center $C = \{ c \in N / (c,R) = 0 \}$. A ring $R$ is called purely antiflexible if the nucleus $N$ of $R$ contains no ideal of $R$.

Throughout this section $R$ denotes an antiflexible derivation alternator ring.

We derive some properties of the nucleus of $R$ in the following lemmas.

**Lemma 4.2.1:** Let $R$ be an antiflexible derivation alternator ring with nucleus $N$. Then

(i) $N (R,R,R) = (NR,R,R)$,

(ii) $(R,R,R) N = (R,R,RN)$,

(iii) $N(R,R,R) = (R,NR,R)$,

(iv) $(R,R,R) N = (R,RN,R)$,

(v) $(N,(R,R,R)) = (0)$.
**Proof:** Applying 4.1.8 to $n \in \mathbb{N}$, $x, y, z \in R$, we get

$$(nx, y, z) = n(x, y, z) \text{ and } (x, y, zn) = (x, y, z)n$$

which imply (i) and (ii).

(iii) follows from the following:

$$(x, ny, z) = n(x, y, z) \text{ by 4.1.6.}$$

(iv) follows from the following:

$$(x, yn, z) = (x, y, z)n \text{ by 4.1.7. }$$

For (v), we subtract (iv) from (iii), then $(n,(x,y,z)) = (0)$.

This implies that $(N,(R,R,R)) = (0)$. □

**Corollary 4.2.1:** $(R,N) \subseteq N$.

**Proof:** For any $n \in \mathbb{N}$, $x, y, z \in R$,

$$(nx, y, z) = n(x, y, z) = (x, y, z)n = (z, y, xn) = (xn, y, z) \text{ or}$$

$$(n, x, y, z) = (0), \text{ which implies } ((R,N), R, R) = (0).$$

Therefore $(R,N) \subseteq N$. □

**Lemma 4.2.2:** Let $R$ be an antiflexible derivation alternator ring. If $I$ is an ideal generated by the set $(R,N)$, then

1. $I = (R,N) + R (R,N)$
2. $I \subseteq N$.

**Proof:** (i) For any $w, x, y, z \in R$, and $n_1, n_2 \in N$,

$$(x, n_1)w = ((x, n_1), w) + w (x, n_1),$$

hence, $(x,n_1)w \in (R,N) + R (R,N)$
Also, \((y(z,n^2))_w = y \left( ((z,n^2)_w + w(z,n^2)) \right) = y \left( (z,n^2)_w + (yw)(z,n^2) \right)\), hence, \((y(z,n^2))_w \in (R,N) + R(R,N)\).

Thus \((R,N) + R(R,N)\) is an ideal of \(R\) and it contains \((R,N)\).

Therefore \(I \subseteq (R,N) + R(R,N)\).

The converse inclusion is clear. Thus
\[ I = (R,N) + R(R,N). \]

(ii) Since \(I = (R,N) + R(R,N)\), and \((R,N) \subseteq N\), using 4.1.7 and lemma 4.2.1, it is immediate that the following are all equivalent:

(a) \(I \subseteq N\);
(b) \((I,R,R) = (0)\);
(c) \((R(R,N),R,R) = (0)\);
(d) \((R,N) (R,R,R) = (0)\).

Now, let \(x,y \in R\) and \(n \in N\). Then by 4.1.9
\[
(xn,y) - x(n,y) - (x,y)n = 0
\]
or \(x(y,n) = - (xn,y) + (x,y)n\).

So \(x(y,n) \in (R,R) + (R,R)N\).

But \((R,R) + (R,R) N \subseteq N_m\), hence, \(x(y,n) \in N_m\), which implies that \(R(R,N) \subseteq N_m\).

By the definition of the middle nucleus \(N_m\) of the ring \(R\),
\[ (R,R(R,N),R) = (0). \]

But lemma 4.2.1 and corollary 4.2.1 imply that
\[ (R,R(R,N),R) = (R,N) (R,R,R). \]
Therefore \((R,N) (R,R,R) = (0)\).

Thus by (d), we have \(I \subseteq N\) and the proof of the lemma is completed. □

**Theorem 4.2.1:** If \(R\) is a prime antiflexible derivation alternator ring, then \(N=C\).

**Proof:** Suppose that \(R\) is not associative. Then there exists \(x,y,z,t \in R\) such that \((x,y,z) \neq 0\). Let \(A\) be the ideal generated by \((x,y,z)\). \(A\) is a non zero ideal of \(R\).

Suppose \(p \in I\) and \(t \in R\). Since \(I\) is an ideal and \(I \subseteq N\), \((x,y,z)p = (x,y,zp) = 0\).

Then this identity together with \((x,y,z)t = -x(y,z,t) + (xy,z,t) - (x,yz,t) + (x,y,zt)\) implies that \(AI = (0)\).

Since \(A \neq (0)\) and \(R\) is prime, we must have \(I = (0)\).

In particular \((R,N) = (0)\) or \(N = C\).

This completes the proof of the theorem. □

Using the definition of purely antiflexible, we state the above theorem 4.2.1 in a slightly more general form in the following corollary:

**Corollary 4.2.2:** A prime antiflexible derivation alternator ring \(R\) is either associative or purely antiflexible.

We now prove the following theorem:

**Theorem 4.2.2:** Let \(R\) be a prime antiflexible derivation alternator ring with idempotent \(e\), then \(e\) is the identity element of \(R\).

**Proof:** By theorem 3.2.1, in derivation alternator rings, we have \(e \in N_t(R)\).

i.e., \((x,e,x) = 0\) or \((e,x,e) = 0\). □
By linerizing 4.2.1, we get $(e,x,y) + (y,x,e) = 0$. Thus $2(e,x,y) = 0$,
since $D (e,x,y) = (e,x,y) - (y,x,e) = 0$. Since $R$ is of characteristic $\neq 2$, we have $(e,x,y) = 0$. Hence $e \in N$. So $e \in C$ by theorem 4.2.1.

We consider the Pierce decomposition $R = R_{11} + R_{10} + R_{01} + R_{00}$ of $R$ with respect to $e$, where $R_{ij} = \{x \in R: ex = ix, xe = jx, i,j = 0,1\}$.

Since $R_{10} = e R_{10} = R_{10} e = (0)$,

\[ R_{01} = R_{01} e = e R_{01} = (0), \]

\[ R = R_{11} + R_{00}. \]

Also $e \in N$ implies that $R_{11}$ and $R_{00}$ are ideals of $R$. Also we have $R_{11} R_{00} = (0)$.

$R$ is prime, $e \in R_{11}$ imply that $R_{00} = (0)$. Thus $R = R_{11}$ and $e$ is the identity element of $R$.

**Theorem 4.2.3:** Let $R$ be a prime antiflexible derivation alternator ring with idempotent $e \neq 1$. Then $R$ is alternative.

**Proof:** From lemma 4.1.2, we have $(x,x,(x,y)) = 0$.

In particular, for any idempotent $e$ we have $(e,e,(e,y)) = 0$. 4.2.2

Thus using Albert decomposition, 4.2.2, 4.1.3 and 4.1.1 imply $(e,e,x) = (e,e,ex) = 2 (e,e,x)e$. Iteration then gives $2 (e,e,x)e = 4 [(e,e,ex)e]e = 4 (e,e,x)e$, so that $2 (e,e,x)e = 0$. This in turn means $(e,e,x) = 2 (e,e,x)e = 0$ for any idempotent $e$. At this point the argument given in section 3 of [28] shows that $R$ is alternative, which completes the proof of the theorem. □

**Corollary 4.2.3:** Let $R$ be a prime antiflexible derivation alternator ring with idempotent $e \neq 1$. Then $R$ is associative.
Now we give an example of an antiflexible derivation alternator ring which is not alternative.

**Example 4.2.1:** Suppose that the ring $R$ is defined by the following multiplication table together with all finite sums of $e,a,b,c,d,h$ such that $x + x = 2x \neq 0$.

$$
\begin{array}{ccccccc}
  & e & a & b & c & d & h \\
 e & e & b & 0 & 0 & 0 & 0 \\
a & h & c & 0 & 0 & 0 & 0 \\
b & 0 & 0 & 0 & 0 & 0 & 0 \\
c & 0 & 0 & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 & 0 & 0 \\
h & h+b & b & 0 & 0 & 0 & 0 \\
\end{array}
$$

We observe that $(a,a,a) = a^2 a - aa^2 = ca - ac = 0$.

Therefore, $R$ satisfies 4.1.1.

This is enough to check the identity 4.1.4, $(e,e,a) = (a,e,e)$,

$$(e,e,a) = ea - e(ce) = b$$

and $(a,e,e) = (ae)e - ae = he - h = h + b - h = b$.

Also, to check 4.1.2,

$$(ab,e,e) - a(b,e,e) - (a,e,e)b = (ab) (e,e) - (ab)e - a \{(be)e - b(e,e)\} - \{(ae)e - ae\}b = 0.0 - 0.e - a \{0.e - b.e\} - \{he - h\}b = 0 - 0 - a \{0 - 0\} - \{h + b - h\}b = b.b = 0.$$

Hence, $R$ is an antiflexible derivation alternator ring but not alternative.

**4.3 SIMPLE ANTIFLEXIBLE DERIVATION ALTERNATOR RINGS:**

Anderson and Outcalt [3] studied the simple antiflexible rings.

The result of Rodabaugh that a simple, antiflexible, power - associative, finite - dimensional, algebra of characteristic $\neq 2$ has a unit element provided it is
not nil [42] is extended by showing that such an algebra with the additional assumption of characteristic \( \neq 3 \) can not be nil. In this section, we prove that in a simple antiflexible derivation alternator ring, \(((x,y),z) = 0\) and also we show that in a simple antiflexible derivation alternator ring, \(((x,y,z),R) = 0\). That is, commutator and associator are in the center. Using this we prove that a simple antiflexible derivation alternator ring is either associative or commutative. Also we prove that antiflexible derivation alternator rings are associative if and only if they satisfy the condition that \((x,y,x)^2 = 0\) implies \((x,y,x) = 0\), for all elements \(x,y\) of \(R\). We know that the middle nucleus \(N_m = \{n \in R/ (R,n,R) = 0\}\).

Throughout this section \(R\) denotes an antiflexible derivation alternator ring.

**Lemma 4.3.1:** If \(R\) is a simple antiflexible derivation alternator ring, then \(R\) has no proper one-sided ideals.

**Proof:** Suppose that \(I\) is a non zero right ideal of \(R\).

Then \((I,R,R) \subseteq I\); hence from \(0 = D (I,R,R)\), we have \((R,R,I) \subseteq I\), and then \((R,I,R) \subseteq I\) because of \(0 = J (I,R,R)\). Now we can show that \(I + RI\) is a two-sided ideal of \(R\)

\[
(I + RI) R \subseteq IR + (RI) R \subseteq I + R (IR) + (R,I,R) \subseteq I + RI
\]

and \(R(I + RI) \subseteq RI + R (RI) \subseteq RI + (RR) I + (R,R,I) \subseteq RI + I\). Since \(I \neq 0\) and \(R\) is simple, \(R = I + RI\). Then from the identity 4.1.26, we have

\[
(R,R,R) \subseteq (R,I + RI,R) \subseteq (R,I,R) + (R,RI,R) \subseteq (R,I,R) + (R,IR,R) \subseteq (R,I,R) \subseteq I.
\]

Since \(IR \subseteq I\), \((R,R,R)+(R,R,R)R \subseteq I\). But it is shown [48] that \((R,R,R) + (R,R,R)R\)

is a two-sided ideal of \(R\), hence it is equal to \(R\) since \(R\) is not associative.
Therefore \( I = R \). A similar argument shows \( R \) has no proper left ideals as well. \( \square \)

**Lemma 4.3.2:** If \( R \) is a simple antiflexible derivation alternator ring, and if \( T \) is a subset of \( N_m \) such that \( (T,R,R) \subseteq T \), then \( (T,R) = 0 \).

**Proof:** We first show that \((T,R) + (T,R)R\) is a right ideal of \( R \). Evidently, it is sufficient to show that \(((T,R),R,R) \subseteq (T,R)\), since \((T,R) R R \subseteq (T,R) (RR) + ((T,R), R,R) \subseteq (T,R)R + ((T,R),R,R)\).

Thus we consider \(((t,x),y,z)\), where \( t \in T, x,y,z \in R \). We have
\[
0 = Q(t,x,y,z) = ((t,x),y,z) + (t,x,(y,z)) - (t,(x,y,z)) - ((t,x,y),z),
\]
and since \( t,(y,z) \in N_m \), it follows from \( 0 = J(t,x,(y,z)) \) that \( 0 = (t,x,(y,z)) \), so that
\[
((t,x),y,z) = (t,(x,y,z)) + ((t,x,y),z) \in (T,R).
\]

According to lemma 4.3.1, either \((T,R) = 0\) or \( R = (T,R) + (T,R)R \).

We show that in the next case, \( R \) would be associative. Indeed, let \( t \in T \) and \( y,z \in R \). By expanding \( 0 = E(t,y,z) \), we get \( (t,z)y = (ty,z) - (y,z) + 2(t,z,y) \).

Since \((R,R) \subseteq N_m\) and since \( N_m \) is a subring, it follows from our assumption on \( T \) that \( (t,z)y \in N_m \). Therefore \( R = (T,R) + (T,R)R \subseteq N_m \), and \( R \) would be associative. Hence \((T,R) = 0 \). \( \square \)

**Lemma 4.3.3:** If \( R \) is an antiflexible derivation alternator ring with characteristic \( \neq 3 \), then \(((R,R),R,R) \subseteq ((R,R),R)\).

**Proof:** Since \( 0 = J((a,b), c, (x,y)) \), it follows from 4.1.26 that \(((a,b),c,(x,y))=0\).

Hence from \( 0 = Q((a,b),c,x,y) \) we have
\[((a,b),c),x,y\) = \(((a,b),c,x),y\) + ((a,b), ((c,x,y))), from which it follows that
\[((a,b),c),x,y\) \equiv \(((a,b),c,x),y\) (mod T),  
4.3.1

where T = ((R,R),R). We obtain
\[((a,b),c),x,y\) \equiv - ((a,b),x),c,y (mod T)  
4.3.2

from 0 = (P (a,b,c,x),y) and 4.3.1. Also,
\[((a,b),c),x,y\) \equiv - (((c,x),b),a,y) (mod T).  
4.3.3

Indeed, expanding 0 = Q (a,b,c,x), we see that
\[((a,b),c),x,y\) \equiv - ((a,b,(c,x)),y) (mod T),
and since 0 = (D(a,b,(c,x)),y), we get
\[((a,b),c),x,y\) \equiv - (((c,x),b),a,y) (mod T),

from which 4.3.3 follows immediately, using 4.3.1.

Now we can derive
\[((a,b),c),x,y\) \equiv ((c,a),b),x,y (mod T).  
4.3.4

Computing, mod T, using 4.3.2, 4.3.3, and the properties of the commutator,
\[((a,b),c),x,y\) \equiv - (((c,x),b),a,y) + P((c,x),b,a,y)
\equiv (c,x),b),y,a)
\equiv - ((b,y),x),c,a)
\equiv (((b,y),c),x,a) - P((b,y),c,x,a)
\equiv - (((b,y),c),a,x)
\equiv ((y,b),c),a,x)
\equiv - (((c,a),b),y,x) + P((c,a),b,y,x)
\equiv ((c,a),b),x,y)

76
Finally, \( 3(((a,b),c),x,y) = 0 \text{ (mod } T) \),

for according to 4.3.4, \( (((a,b),c),x,y) \) remains unchanged mod \( T \) when \( a,b,c \) are
cyclically permuted. Hence

\[ 3(((a,b),c),x,y) = (((a,b),c),x,y) + (((c,a),b),x,y) + (((b,c),a),x,y) \text{ (mod } T) \].

But because of the jacobi identity 4.1.13,

\[ (((a,b),c),x,y) + (((c,a),b),x,y) + (((b,c),a),x,y) = 0. \quad 4.3.5 \]

Let \( a',b,c,x,y \) be arbitrary elements of \( R \). Since \( a^1 = 3a \) for some \( a \in R \),

\[ (((a',b),c),x,y) = 3 (((a,b),c),x,y) = 0 \text{ (mod } T) \],

which completes the proof of the Lemma. □

**Lemma 4.3.4:** If \( R \) is a simple antiflexible derivation alternator ring with
characteristic \( \neq 3 \), then 0 = \( (((R,R),R),R) = ((R,R), (R,R)) \).

**Proof:** Since \( 3R \) is an ideal of \( R \), it follows from our assumption that
characteristic \( \neq 3 \). Let \( T = ((R,R)R) \); from lemma 4.3.3 we have that \( (T,R,R) \subseteq T \). But \( T \) is a subset of the middle nucleus because of Lemma 4.1.1, hence from
lemma 4.3.2, we conclude that \( (((R,R),R),R) = 0 \), and since 4.1.13, 0 = \( ((R,R), (R,R)) \) as well. □

**Theorem 4.3.1:** If \( R \) is a simple antiflexible derivation alternator ring with
characteristic \( \neq 3 \), then \( (x,y),z) = 0. \)

**Proof:** First we prove that \( ((R,R),R)(R,R) \subseteq ((R,R),R) \).

Indeed if \( w \in ((R,R),R)(R,R) \), then \( w = \sum t_i u_i \), where \( t_i \in ((R,R),R) \) and
\( u_i \in (R,R) \). Hence it would be sufficient to prove that \( ((a,b),c) (r,s) \in ((R,R),R) \)
for all \(a, b, c, r, s \in R\). Since \(0 = E ((c,(r,s),(a,b)) = (c(r,s),(a,b)) - c ((r,s), (a,b)) - (c,(a,b)) (r,s) + 2 (c,(a,b), (r,s))\), we have from 4.1.26 and lemma 4.3.4 that 
\[
((a,b),c) (r,s) = (c(r,s), (a,b)) \in ((R,R),R).
\]

Next, we note that \(W = ((R,R),R) + ((R,R),R)R\) is a right ideal of \(R\):
\[
WR \subseteq W + ((R,R),R)R \subseteq W + ((R,R),R) (RR) + (((R,R),R),R,R) \subseteq W
\]
because of lemma 4.3.3. According lemma 4.3.1, \(W = 0\) or \(W = R\).

Suppose \(W = R\). Then \((R,R) \subseteq (W,R) \subseteq (((R,R),R),R)R\)
because of lemma 4.3.4. Now we prove that \(((R,R),R)R,R) \subseteq ((R,R),R)\).

Evidently, it is sufficient to show that if \(x,y,z,b,c\) are arbitrary elements in \(R\),
then \(((x,y),z)b,c) \in ((R,R),R). Thus, set \(a = ((x,y),z)\).

Now \(0 = E (a,b,c) = (ab,c) - a(b,c) = (a,c)b + 2 (a,c,b)\).

But \((a,c)b = 0\) because of lemma 4.3.3, and \(2(a,c,b) \in ((R,R),R)\) because of lemma 4.3.1. Therefore \((ab,c) \in ((R,R),R) + ((R,R),R)(R,R) \subseteq ((R,R),R)\).

Altogether we have \((R,R) \subseteq ((R,R),R)\).

But then \(((R,R),R) \subseteq (((R,R),R),R) = 0\) because of lemma 4.3.4.

Therefore \(W = 0\), and this completes the proof of the theorem. □

**Theorem 4.3.2:** If \(R\) is an antiflexible derivation alternator ring, then \(R\) is either associative or commutative.

**Proof:** By taking the commutator \(v\) for \(z\) in 4.1.9 and then using 4.1.7 and theorem 4.3.1, we get \((yv,x) + (vx,y) = 0\), so that \((yv,x) = - (vx,y) = (xy,v) = 0\). 4.3.6

Therefore \((y(R,R),x) = 0\).

4.3.6

By taking \(z = x\) in 4.1.9 and then commuting with \(r\), we get
\[
((xy,x),r) + (x(x,y)r) = ((x,y,x),r)
\]
4.3.7
By theorem 4.3.1 and identity 4.3.6, the identity 4.3.7 becomes

\[(x,y,x),r = 0.\]  \hspace{1cm} \text{(4.3.8)}

By Linearizing (replacing \(x\) by \(x + z\)) 4.3.8, we obtain

\[2((x,y,z),r) = 0, \text{ so that } ((x,y,z),r) = 0.\]

Therefore \((x,y,z),R = 0.\)  \hspace{1cm} \text{(4.3.9)}

We know that simple ring \(R\) satisfying \((x,y,z),R = 0\) and \((x,x,x) = 0\) is either associative or commutative \([52]\). This completes the proof of the theorem. \(\Box\)

We know that \((x,y,x)^2 = 0\) holds in any derivation alternator ring. Using this we now prove the following theorem.

**Theorem 4.3.3:** An antiflexible derivation alternator ring \(R\) is associative if and only if \((x,y,x)^2 = 0\), implies \((x,y,x) = 0\), for all elements \(x,y\) of \(R\).

**Proof:** Clearly the only if part is automatically satisfied.

Conversely, if the above condition is satisfied then \((x,y,x) = 0\), a result of lemma 3.1.1, A linearization of this identity leads to \((x,y,z) + (z,y,x) = 0\). But \(0 = D(x,y,z) = (x,y,z) - (z,y,x)\), so that \(2(x,y,z) = 0\). Since \(R\) is of characteristic \(\neq 2\), \((x,y,z) = 0\), for all elements \(x,y,z\) of \(R\). The ring is therefore associative. This completes the proof of the Theorem. \(\Box\)

An immediate application of this theorem is the following:

**Corollary 4.3.1:** An antiflexible derivation alternator ring without nilpotent elements is associative.