CHAPTER — 4

REGULAR SEMIGROUPS IN SEMIRINGS
The concept of congruence on semigroup is introduced by Pondelicek. Congruence on regular semigroups have been explored extensively. The kernel-Trace approach is an effective tool for handling congruence on regular semigroups, which had been investigated in the previous literature, such as Crvenkovic and Dolinka, Feigenbaum, Gomes, Imaoka, Pastijn and Petrich and some other authors. In this chapter, we prove some results on regular semigroups by defining different congruence relations. We determine different additive structures of simple semiring which was introduced by Golan [16]. We also proved some results based on the papers of Fitore Abdullahu [1] and M.K.Sen.

4.1. Congruence on Regular Semigroups

This section deals with the properties of congruence on regular semigroups. The motivation to prove the theorems in this section is due to the results of J.M.Howie [19].

4.1.1. Definition: A semigroup $S$ is called medial if $xyzu = xzyu$, for every $x,y,z,u$ in $S$. 


4.1.2. Definition: A semigroup S is called left (right) semimedial if it satisfies the identity $x^2yz = xyxz$ ($zxy^2 = zxy$), where $x,y,z \in S$ and $x$, $y$ are idempotent elements.

4.1.3. Definition: A semigroup S is called a semimedial if it is both left and right semimedial.

Example: The semigroup S is given in the table is I-semimedial

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4.1.4. Definition: A semigroup S is called left(right) commutative if it satisfies the identity $xyz = yxz$ ($zxy = zyx$), where $x$, $y$ are idempotent elements.

4.1.5. Definition: A semigroup S is called I-commutative if it satisfies the identity $xy = yx$, where $x,y \in S$ and $x$, $y$ are idempotent elements.

Example: The semigroup S is given in the table is I-commutative.

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4.1.6. **Definition:** A semigroup $S$ is called I-left(right) distributive if it satisfies the identity $xyz = xyyz$ ($zyx = zxyx$), where $x, y, z \in S$ and $x, y$ are idempotent elements.

4.1.7. **Definition:** A semigroup $S$ is called I-distributive if it is both left and right distributive.

4.1.8. **Definition:** A semigroup $S$ is said to be cancellative for any $a, b, e \in S$, then $ac = bc \Rightarrow a = b$ and $ca = cb \Rightarrow a = b$.

4.1.9. **Definition:** A semigroup $S$ is called diagonal if it satisfies the identities $x^2 = x$ and $xyz = xz$.

4.1.10. **Definition:** A regular semigroup $S$ is said to be generalized inverse semigroup if all its idempotent elements form a normal band.

4.1.11. **Definition:** A regular semigroup $S$ is said to be locally inverse semigroup if $eSe$ is an inverse semigroup for any idempotent $e$ in $S$.

4.1.12. **Definition:** A regular semigroup $S$ is said to be orthodox semigroup if $E(S)$ is subsemigroup of $S$.

4.1.13. **Definition:** [16] A semiring $S$ is called yoked semiring if for every $a, b \in S$ there exist $c \in S$ such that $a = a + c$ and $b = b + c$.

4.1.14. **Definition:** An element "$a$" of $S$ is called $k$-regular if $a^k$ is regular element, for any positive integer $k$. If every element of $S$ is $k$-regular then $S$ is $k$-regular semigroup.
Examples:

(i) If $S$ is a zero semigroup on a set with zero, then $S$ is 2-regular but not regular, has a unique idempotent namely, zero but is not a group.

(ii) A left (right) zero semigroup is $k$-regular for every positive integer $k$ but not $k$-inverse unless it is trivial. It is to be noted that regular = 1-regular.

4.1.15. Definition: [16] A semiring $S$ is called simple if $a + 1 = 1 + a = 1$ for any $a \in S$.

4.1.16. Definition: A semiring is a non empty set $S$ on which operations of addition “$+$” and multiplication “$.$” have been defined such that the following conditions are satisfied:

(i) $(S, +)$ is a semigroup

(ii) $(S, .)$ is a semigroup

(iii) Multiplication distributes over addition from either side.

4.1.17. Definition: A semiring $(S, +, .)$ is called an additive inverse semiring if $(S, +)$ is an inverse semigroup, i.e., for each $a$ in $S$ there exists a unique element $a^1 \in S$ such that $a + a^1 + a = a$ and $a^1 + a + a^1 = a^1$

Example: Consider the set $S = \{0, a, b\}$ on $S$ we define addition and multiplication by the following cayley tables
It is easy to see that \((S, +, \cdot)\) is an additive inverse semiring.

4.1.18. **Definition:** A semiring \(S\) is called a regular semiring if for each \(a \in S\) there exist an element \(x \in S\) such that \(a = axa\).

**Examples:**
(i) Every regular ring is a regular semiring
(ii) Every distributive lattice is regular semiring.
(iii) The direct product of regular ring and distributive lattice is regular semiring.

4.1.19. **Definition:** An additive idempotent semiring \(S\) is \(k\)-regular if for all \(a \in S\) there is \(x \in S\) for which \(a + axa = axa\).

**Example:**
Let \(D\) be a distributive lattice. Consider \(S = M_2(D)\) the semiring of \(2 \times 2\) matrices on \(D\). Now consider \(A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\). Then for \(X = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\)

\(A + AXA = AXA\) and this shows that \(S\) is \(k\)-regular.

4.1.20. **Theorem:** Let \((S, \cdot)\) be a commutative left regular semigroup and let \(E(S)\) denotes the set of all idempotent elements in \(S\). If \(\eta\) be a...
relation defined on $S$ by $\eta = \{ (a, b) \in S, \ x(e)a = y(e)b, \text{ for every } e \in E(S) \}$ then $\eta$ is the maximum idempotent-seperative congruence on $S$.

**Proof:** Let $S$ be a commutative left regular semigroup and $E(S)$ denote the set of all idempotent elements in $S$.

Define the relation $\eta$ on $S$ by $\eta = \{ (a, b) \in S, \ x(e)a = y(e)b, \text{ for every } e \in E(S) \}$.

First we show that $\eta$ is an equivalence relation on $S$. For any $a$ in $S$ and $x$ in $S$, $x(e)a = x(e)a \Rightarrow a \eta a$. Hence $\eta$ is reflexive.

Let $a \eta b$ and $b \eta c \Leftrightarrow x(e)a = y(e)b$ and $y(e)b = z(e)c$ for some $x, y, z \in S$.

$\Leftrightarrow x(e)a = y(e)b = z(e)c \Leftrightarrow x(e)a = z(e)c \Leftrightarrow a \eta c$. Hence $\eta$ is transitive.

Let $a \eta b \Leftrightarrow x(e)a = y(e)b$

$\Leftrightarrow y(e)b = x(e)a$

$\Leftrightarrow b \eta a$. Hence $\eta$ is symmetric.

Therefore $\eta$ is an equivalence relation on $S$.

Left compatible: Let $a \eta b \Leftrightarrow x(e)a = y(e)b$

$\Leftrightarrow zx(e)ac = zy(e)bc$

$\Leftrightarrow xz(e)ca = yz(e)cb$

$\Leftrightarrow ca \eta cb$
\[ a \eta b \iff ca \eta cb. \]

\[ \therefore \eta \text{ is left compatible.} \]

Right compatible: Let \( a \eta b \iff x(e)a = y(e)b \)

\[ \iff zx(e)a = zy(e)b \]

\[ \iff zx(e)ac = zy(e)bc \]

\[ \iff ac \eta bc \]

\[ a \eta b \iff ac \eta bc. \text{ Hence } \eta \text{ is right compatible} \]

Therefore \( \eta \) is a congruence relation on \( S \).

Let \( e, f \in E(S) \) then \( e^2 = e, \ f^2 = f \)

Consider \( (e)^2 \eta (f)^2 \iff (e)^2(e)(e)^2 = (f)^2(e)(f)^2 \)

\[ \iff (e)(e)(e) = (f)(e)(f) \]

\[ \iff (e) \eta (f) \]

\[ (e)^2 \eta (f)^2 \iff (e) \eta (f). \]

Hence \( \eta \) is an idempotent congruence relation.

Let \( e \eta f \iff e(e)e = f(e)f \)

\[ e = e(f)(f) \]

\[ e = e(f)^2 \]

\[ e = ef \text{ and } e = fe \]

Again \( f \eta e \iff f(f)f = e(f)e \)

\[ f = e(ef) \]
= e^2 f

f = ef and f = fe

Therefore, e = ef, f = ef \Rightarrow e = f

Hence \( \eta \) is an idempotent separative congruence on S.

To prove \( \eta \) is maximum, let \( \mu \) be any idempotent separative congruence on S.

Let \((a, b) \in \mu \Rightarrow a \# b\). Since a and b are regular elements, there exist \(x, y \in S\) such that \(a = axa\) and \(b = byb\).

Hence \((a, b) \in \mu \Leftrightarrow (axa, byb) \in \mu \Leftrightarrow (x, y) \in \mu\). We know that for all \(e \in E(S)\), \((e, e) \in \mu\). \((a, b) \in \mu\), \((x, y) \in \mu\) and \((e, e) \in \mu \Rightarrow (xea, yeb) \in \mu\). Since \(a = axa\) and \(b = byb \Rightarrow (ax)^2 = ax\), \((by)^2 = by\) \((xa)^2 = xa\), \((yb)^2 = yb\) and also \(e^2 = e\). Hence, \(xea\) and \(yeb\) are idempotent elements. Therefore, \(xea = yeb \Leftrightarrow a \# b \Leftrightarrow (a, b) \in \eta\). So \((a, b) \in \mu \Rightarrow (a, b) \in \eta\). Thus \(\mu \subseteq \eta\)

Hence \(\eta\) is maximum idempotent separative congruence on S.

4.1.21. Theorem: Let S be a regular semigroup and let \(\eta\) be a congruence relation on S then \(S/\eta\) is a regular sub semigroup.

Proof: Let S be a regular semigroup and let \(\eta\) be a congruence relation on S. Therefore we can construct the congruence class \(S/\eta\) such that \(S/\eta = \{a\eta: a \in S\}\), where \(a\eta\) is a congruence class of a. Define \("o"\) on
S/ η in the following way. For any a η, b η ∈ S/ η such that (aη) o (bη) = (ab)η

Let a η = a^l η and b η = b^l η then (a η) o (b η) = (ab)η

(a^l η) o (b^l η) = (ab)η

Hence "o" is well defined and it is associative.

Hence, (S/ η, o) is a regular sub semigroup.

4.1.22. Theorem: Let η be a congruence relation on a regular semigroups S. Then η^n is also a congruence relation on S.

Proof: Let η be a congruence relation on regular semigroup S.

Let a η b then there exist t_1, t_2, t_3,.........,t_{n-1} ∈ S and by transitivity we have a η t_1, t_1 η t_2, t_2 η t_3,.........,t_{n-1} η b ⇒ a η^n b. It is easy to see that η^n is an equivalence relation.

Let c ∈ S then c η c (since η is compatible)

c η t_1, t_1 η t_2, t_2 η c,.........,t_{n-1} η c ⇒ c η^n c. Hence a η^n b ⇒ c η^n c.

Similarly a η^n b ⇒ a c η^n c ⇒ η^n is compatible. Therefore, η^n is congruence on S.

4.1.23. Theorem: If S is a commutative regular semigroup and E(S) is a set of all idempotent elements of S. Then the relation ρ defined on
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S by apb = {(a, b)eSxS, ea = eb, for every e∈ E(S)} is the congruence relation on S.

Proof: Let S be a commutative regular semigroup and E(S) be the set of all idempotent elements of S. Then the relation ρ on S is defined by apb ⇔ {(a, b)∈ SxS, ea = eb, for every e∈ E(S)}.

We know that for every a∈ S, e∈ S.

ea = ea ⇔ apa ⇔ ρ is reflexive

Let apb ⇔ ea = eb

⇔ eb = ea

⇒ apb ⇔ bpa

Hence ρ is symmetric

Let apb ⇔ ea = eb and bpc ⇔ eb = ec

⇒ apb ⇔ ea = eb

⇔ ea = eb = ec

⇒ ea = ec

⇒ apb ⇔ apc

Hence ρ is transitive

Therefore, ρ is an equivalence relation on S

To prove compatibility, consider apb ⇔ ea = eb
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\( \iff eac = ebc \quad \text{for some } c \in S. \)

\( \iff ac \rho bc \)

\( \iff \rho \text{ is right compatible} \)

Again \( apb \iff ea = eb \)

\( \iff cea = ceb \)

\( \iff eca = ecb \)

\( \iff capcb \)

\( \iff \rho \text{ is left compatible} \)

Hence \( \rho \) is congruence on \( S \).

4.2. Simple semirings

Section 4.2 reveals the additive structures of simple semirings by considering that the multiplicative semigroup is rectangular band.

4.2.1. Theorem: A simple semiring is additive idempotent semiring.

Proof: Let \( (S, +, \cdot) \) be a simple semiring. Since \( (S, +, \cdot) \) is simple, for any \( a \in S, a + 1 = 1 \). (Where 1 is the multiplicative identity element of \( S, S^1 = SU\{1\} \).)

Now \( a = a.1 = a(1 + 1) = a + a \Rightarrow a = a + a \Rightarrow S \) is additive idempotent semiring.
4.2.2. **Theorem:** Every singular semigroup \((S, +)\) is

(i) rectangular band

(ii) semilattice.

**Proof:** Let \((S, +)\) be a singular semigroup, then for all \(a, b \in S\), \(a + b = a \Rightarrow a + b + a = a + a \Rightarrow a + b + a = a\) (Since \(S\) is singular)

Similarly, \(b + a + b = b\). \((S, +)\) is a rectangular band

ii) Since \(S\) is singular, \(a + a = a\) for all \(a \in S\) \(\Rightarrow S\) is a band.

Also \(a + b = a = b + a \Rightarrow a + b = b + a \Rightarrow S\) is commutative. Hence \((S, +)\) is a semi lattice.

4.2.3. **Theorem:** Let \((S, +, \cdot)\) be a simple semiring. Then \(S\) is regular semiring if and only if it is \(k\)-regular semiring.

**Proof:** Let \((S, +, \cdot)\) be a simple semiring. Since \((S, +, \cdot)\) is a simple, for any \(a \in S\), \(a + 1 = 1\).

Consider \(a = a.1 = a(1 + b) = a + ab \Rightarrow a = a + ab\)

Similarly, \(a = a + ba\)

Assume that \(S\) be a regular semiring. Then for any \(a \in S\) there exist an element \(x\) in \(S\) such that \(axa = a\). Now consider \(axa = a = a.1 = a(1 + x) = a + ax = a + a(x + xa) = a + ax + axa = a + axa \Rightarrow axa = a + axa\).

Hence \((S, +, \cdot)\) is \(k\)-regular semiring.
Conversely, assume that \( (S, +, \cdot) \) be a \( k \)-regular semiring then for any \( a \in S \) there exist \( x \in S \) such that \( axa = a + axa \). Now consider \( a + axa = a(1 + xa) = a1 = a \Rightarrow axa = a \Rightarrow S \) is a regular semiring.

4.2.4. **Theorem**: Let \( (S, +, \cdot) \) be a simple semiring. If \( (S, +, \cdot) \) is \( k \)-regular semiring then \( S \) is additively regular semiring.

**Proof**: Let \( (S, +, \cdot) \) be a simple semiring. Since \( (S, +, \cdot) \) is simple, for any \( a \in S \), \( a + 1 = 1 \). Since \( S \) be a \( k \)-regular semiring, for any \( a \in S \) there exist an element \( x \) in \( S \) such that \( axa = a + axa \). To prove that \( S \) is additively regular semiring, consider \( axa = a + axa \Rightarrow a = a + a \) (by Theorem 4.2.1.) \( \Rightarrow (S, +) \) is a band.

Since \( (S, +) \) is a band then clearly \( (S, +) \) is regular. Therefore \( S \) is additively regular semiring.

4.2.5. **Theorem**: If the idempotent elements of a regular semigroup are commutes then \( S \) is generalized inverse semigroup.

**Proof**: Let \( S \) be a regular semigroup whose idempotent elements are commutes.

Let \( x, y, z \in S \) and are idempotent elements then \( xyz = xzy \Rightarrow xxyz = xxzy \Rightarrow xyxz = xzxy \Rightarrow xzyx \). Hence idempotent elements form a normal band. Therefore \( S \) is generalized inverse semigroup.
4.2.6. Theorem: Let $S$ be a regular semigroup and $E(S)$ is an $E$-inversive semigroup of $S$ then

\begin{itemize}
  \item[i)] $E(S)$ is subsemigroup
  \item[ii)] $S$ is an orthodox semigroup
  \item[iii)] $E(S)$ is locally inverse semigroup.
\end{itemize}

Proof: (i) Let $S$ be a regular semigroup and $E(S)$ is an $E$-inversive semigroup. If $a, b \in S$ then there exist some $x, y \in S$ such that

$$(ax)^2 = (ax) \text{ and } (by)^2 = (by), (xa)^2 = (xa) \text{ and } (yb)^2 = (yb)$$

Let $(axby)^2 = (ax)^2 (by)^2 = (ax)(by) \Rightarrow (axby)^2 = (ax)(by)$.

$(axby)$ is an idempotent of $E(S)$.

Hence $E(S)$ is subsemigroup of $S$ and its elements are idempotents.

(ii) Since $E(S)$ is a sub semigroup of $S \Rightarrow S$ is an orthodox semigroup.

(iii) To prove that $eE(S)e$ is regular for some $e \in S$. Let $f \in E(S)$ then

$$(efe)(efe)(efe) = efeefeefe = efeefe = efe \Rightarrow (efe)(efe)(efe) = efe$$

Similarly $(fef)(fef)(fef) = fefefefef = fefefef = fef \Rightarrow (fef)(fef)(fef) = fef \Rightarrow efe$ is an inverse element of $E(S) \Rightarrow E(S)$ is locally inverse semigroup.

4.2.7. Theorem: Let $(S, +, \cdot)$ be a semiring. Then the following statements are equivalent:

\begin{itemize}
  \item[i)] $a + 1 = 1$
  \item[ii)] $a^n + 1 = 1$
\end{itemize}
(iii) \((ab)^n + 1 = 1\) for all \(a, b \in S\).

**Proof:** Let \((S, +, \cdot)\) be a semiring

(i) \(\Rightarrow\) (ii)

Let \(a + 1 = 1\), for every \(a \in S\) then we have to prove that \(a^n + 1 = 1\) for some positive integer \(n\). We prove it by mathematical induction

If \(n = 1\) then \(a + 1 = 1\). If \(n = 1\) then the result is true.

If \(n = 2\) then \(a^2 + 1 = aa + 1 = aa + a + 1 = a(a + 1) + a + 1 = (a + 1)(a + 1) = 1.1 = 1\) \(\Rightarrow\) for \(n = 2\), the result is true.

Assume that the given statement is true for \(n = k\). That is,

\[a^k + 1 = 1.\]

To prove that the result is true for \(n = k + 1\), consider \(a^{k+1} + 1 = a^k a + 1 = a^k a + a^k + a + 1 = a^k(a + 1) + a + 1 = (a^{k+1})(a + 1) = 1\) \(\Rightarrow\) given statement is true for \(n = k + 1\). Therefore, \(a + 1 = 1 \Rightarrow a^n + 1 = 1\). This completes the induction.

(ii) \(\Rightarrow\) (iii)

Let \(a^n + 1 = 1\) and \(b^n + 1 = 1\) for every \(a, b \in S\). To prove that \((ab)^n + 1 = 1\).

Consider \((ab)^n + 1 = a^n b^n + 1 = a^n b^n + a^n + 1 = a^n b^n + a^n + b^n + 1 = a^n (b^n + 1) + b^n + 1 = (a^n + 1)(b^n + 1) = 1.1 = 1 \Rightarrow (ab)^n + 1 = 1.\)

(ii) \(\Rightarrow\) (i)
Let \( a^n + 1 = 1 \) for any \( a \in S \) and some positive integer \( n \).

If \( n = 1 \) then \( a^1 + 1 = 1 \Rightarrow a + 1 = 1 \).

**4.2.8. Theorem:** Let \((S, +, .)\) be simple semiring. Then for any \( a, b \in S \) the following holds:

(i) \( a + b + 1 = 1 \)

(ii) \( ab + 1 = 1 \).

**Proof:** Let \((S, +, .)\) be a simple semiring. Then \( a + 1 = 1 \) and \( b + 1 = 1 \) for every \( a, b \in S \).

(i) Consider \( a + b + 1 = a + (b + 1) = a + 1 = 1 \Rightarrow a + b + 1 = 1 \)

(ii) Consider \( ab + 1 = ab + (1) = ab + (a + 1) = ab + a + b + 1 \)

\[ = a(b + 1) + 1(b + 1) = (a + 1)(b + 1) = 1.1 \Rightarrow ab + 1 = 1 \]

**4.2.9. Theorem:** Let \((S, +, .)\) be a simple semiring in which \((S, .)\) is rectangular band then \((S, .)\) is singular.

**Proof:** Let \((S, +, .)\) be a simple semiring and \((S, .)\) be a rectangular band i.e, for any \( a, b \in S \) \( aba = a \). Since \( S \) is simple, \( 1 + a = a + 1 = 1 \), for all \( a \in S \). To prove that \((S, .)\) is singular, consider \((1 + a)b = 1.b \Rightarrow b + ab + b = a \Rightarrow (b + ab)a = ba \Rightarrow ba + aba = ba \Rightarrow ba + a = ba \Rightarrow (b + 1)a = ba \Rightarrow 1.a = ba \Rightarrow a = ba \Rightarrow ba = a \Rightarrow (S, .) \) is a right singular.
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Again \( b(1 + a) = b \cdot 1 \Rightarrow b + ba = b \Rightarrow a(b + ba) = ab \Rightarrow ab + aba = ab \)

\( \Rightarrow ab + a = ab \Rightarrow a(b + 1) = ab \Rightarrow a \cdot 1 = ab \Rightarrow a = ab \Rightarrow ab = a \Rightarrow (S, .) \)

is left a singular. Therefore \((S, .)\) is singular.

4.2.10. Theorem: Let \((S, +, .)\) be a simple semiring in which \((S, .)\) is rectangular band then \((S, +)\) is one of the following:

(i) I-medial

(ii) I-semimedial

(iii) I-distributive

(iv) L-commutative

(v) R-commutative

(vi) I-commutative

(vii) digonal

(viii) external commutative

(ix) conditional commutative.

Proof: Let \((S, +, .)\) be a semiring in which \((S, .)\) is a rectangular band.

Assume that \( S \) satisfies the identity \( 1 + a = 1 \) for any \( a \in S \). Now for any \( a, b, c, d \in S \).

(i) Consider \( a + b + c + d = a + (b + c) + b \) (by Theorem 4.2.2.(ii))

\[ a + c + b + d \]

\((S, +)\) is I-medial.
(ii) Consider $a + a + b + c = a + (a + b) + c = a + b + a + c \implies (S, +)$ is I-left semi medial.

Again $b + c + a + a = b + (c + a) + a = b + a + c + a \implies (S, +)$ is I-right semi medial.

Therefore, $(S, +)$ is I-semi-medial.

(iii) consider $a + b + c = (a) + b + c = a + a + b + c = a + (a + b) + c = a + b + a + c \implies (S, +)$ is I-left distributive.

Consider $b + c + a = b + c + (a) = b + c + a + a = b + (c + a) + a = b + (a + c) + a \implies b + c + a = b + a + c + a \implies (S, +)$ is I-right distributive.

Hence $(S, +)$ is I-distributive.

Similarly we can prove the remaining.

4.2.11. Theorem: Let $(S, +, \cdot)$ be a simple semiring and $(S, \cdot)$ is rectangular band then $(S, +)$ is

(i) quasi-seprative
(ii) weakly-seperative
(iii) seperative.

Proof: Let $(S, +, \cdot)$ be a simple semiring and $(S, \cdot)$ is a rectangular band i.e, for any $a, b \in S$, $aba = a$. Since $S$ is simple, $1 + a = a + 1 = 1$, for all $a \in S$. 

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Let \( a + a = a + b \Rightarrow a + a + 1 = a + b + 1 \Rightarrow a + 1 = b + 1 \Rightarrow a = b \)

Again, \( a + b = b + b \Rightarrow a + b + 1 = b + b + 1 \Rightarrow a + 1 = b + 1 \Rightarrow a = b \)

Hence \( a + a = a + b = b + b \Rightarrow a = b \Rightarrow (S, +) \) is quasi-separative.

(ii) Let \( a + b = (a) + b = ba + b = b + ab = b + a \Rightarrow a + b = b + a \Rightarrow (1) \)

From (i) and (ii) \( a + a = a + b = b + a = b + b \Rightarrow a = b \Rightarrow (S, +) \) is weakly separative

(iii) Let \( a + a = a + b \)

\( b + b = b + a \)

From (1) \( a + b = b + a \) and from theorem 4.2.1 \((S, +)\) is a band

Therefore, \( a = a + a = a + b = b + b = b \Rightarrow a = b \).

Hence \((S, +)\) is separative.

4.2.12. Theorem: Let \((S, +, .)\) be a simple semiring in which \((S, .)\) is rectangular band then \((S, +)\) is cancellative in which case \(|S| = 1\).

Proof: Let \((S, +, .)\) be a simple semiring in which \((S, .)\) is rectangular band. Since \(S\) is simple then for any \(a \in S, 1 + a = a + 1 = 1\).

Let \(a, b, c, \in S\). To prove that \((S, +)\) is cancellative, for any \(a, b, c \in S,\) consider \(a + c = b + c\). Then \(a + c.1 = b + c.1 \Rightarrow a + c(a + 1) = b + c(b + 1) \Rightarrow a + ca + c = b + cb + c \Rightarrow a + ca + cac = b + cb + cbc \) (since \((S, .)\) is rectangular) \(\Rightarrow a + ca(1 + c) = b + cb(1 + c) \Rightarrow a + ca.1 = b + cb.1 \Rightarrow a + ca = b + cb \)

\(\boxed{75}\)
\[ (1 + c)a = (1 + c)b \Rightarrow 1.a = 1.b \Rightarrow a = b \Rightarrow a + c = b + c \]

\[ a = b. \Rightarrow (S, +) \text{ is right cancellative} \]

Again \( c + a = c + b \Rightarrow c.1 + a = c.1 + b \Rightarrow c(1 + a) + a = c(1 + b) + b \)

\[ \Rightarrow c + ca + a = c + cb + b \Rightarrow cac + c + a = cbc + cb + b \Rightarrow cac + ca + a \]

\[ = abc + cb + b \Rightarrow ca(c + 1) + a = cb(c + 1) + b \Rightarrow ca.1 + a = cb.1 + b \]

\[ \Rightarrow a + a = cb + b \Rightarrow (c + 1)a = (c + 1)b \Rightarrow 1.a = 1.b \Rightarrow a = b \Rightarrow c + a \]

\[ = c + b \Rightarrow a = b \Rightarrow (S, +) \text{ is left cancellative}. \]

Therefore, \((S, +)\) is cancellative semigroup. Since \(S\) is simple semiring we have \(1 + a = 1 \Rightarrow 1 + a = 1 + 1\). But \((S, +)\) is cancellative \(\Rightarrow a = 1\) for all \(a \in S\). Therefore \(|S| = 1\).

4.3. Semirings satisfying the identity \(a + 1 = 1 + a = a\)

In this section we consider some class of semirings satisfying the identity \(a + 1 = 1 + a = a\) for all \(a\) in \(S\).

4.3.1. Theorem: Let \((S, +, \cdot)\) be a semiring satisfying the identity \(a + 1 = 1 + a = a\) for all \(a \in S\). If \((S, \cdot)\) be a rectangular band then \((S, \cdot)\) is singular.

Proof: Let \((S, +, \cdot)\) be a semiring in which \((S, \cdot)\) be a rectangular band. Assume that \(S\) satisfies the identity \(1 + a = a + 1 = a\) for every \(a \in S\).

Now \(1 + a = a \Rightarrow b(1 + a) = ba\) for every \(b \in S\)

\[ a(b + ba) = aba \]
ab + aba = a (since (S, .) is rectangular band)
ab + a = a
ab = a

Hence (S, .) is left singular

Again 1 + a = a \Rightarrow (1 + a)b = ab for every b \in S
b + ab = ab
(b + ab)a = aba
ab + aba = aba
ba + a = a
(b + 1)a = a
ba = a

Hence (S, .) is right singular.

Therefore, (S, .) is singular

4.3.2. Theorem: Let (S, +, .) be a semiring and (S, .) be a rectangular band. If S satisfies the identity \(1 + a = a + 1 = a\) for all \(a \in S\), then
(S, +) is one of the following:

(i) I- medial
(ii) I- L commutative
(iii) I- R commutative
(iv) I- commutative
(v) I- semi medial

(vi) I- distributive

(vii) diagonal.

Proof: Let \((S, +, .)\) be a semiring in which \((S, .)\) be a rectangular band.

Assume that \(S\) satisfies the identity \(1 + a = a + 1 = a\) for any \(a \in S\).

Now for any \(a, b, c, d \in S\),

(i) Consider \(a + b + c + d = a + (b) + e + d = a + cb + c + d = a + c(b + 1) + d = a + cb + d = a + (c + 1)b + d = a + c + b + d\)

\(\Rightarrow a + b + c + d = a + c + b + d \Rightarrow (S, +)\) is I- medial.

(ii) Consider \(a + b + c = (a) + b + c = ab + b + c = (a + 1)b + c\)

\(= ab + c = (1 + a)b + c = b + ab + c = b + a + c \Rightarrow a + b + c = b + a + c\). Therefore \((S, +)\) is I- L commutative

(iii) Consider \(c + a + b = c + a + (b) = c + a + ba = c + (1 + b)a\)

\(= c + ba = c + (b + 1)a = c + ba + a = c + b + a \Rightarrow a + b + c = c + b + a\)

\(\Rightarrow (S, +)\) is I- R commutative

(iv) Consider \(a + b = a + (b) = a + ba = (1 + b)a = ba = b(1 + a) = b + ba = b + a \Rightarrow ab + b = b + a\).

Therefore, \((S, +)\) is I- commutative

(v) Consider \(a + a + b + c = a + (a + b) + c = a + b + a + c\) \((\text{from(i)})\)

\(\Rightarrow (S, +)\) is I- L semimedial.
A gain \( c + b + a + a = c + (b + a) + a = c + a + b + a \) (from (ii))

\( \Rightarrow (S, +) \) is I -R semimedial. Therefore, \( (S, +) \) is I-semimedial.

(vi) Consider \( a + b + c = (a) + b + c = a + a + b + c \Rightarrow a + b + c = a + b + a + c \)

Similarly, \( c + a + b + a = c + a + b + a \). Therefore, \( (S, +) \) is I-distributive.

(vii) Let \( a + a = a \Rightarrow a + b + c = (a) + b + c = ab + b + c = (a + 1)b + c = ab + c \Rightarrow a + b + c = a + c \). Therefore \( (S, +) \) is diagonal.

4.3.3. Theorem: Let \( (S, +, .) \) be a semiring and \( (S, .) \) be a rectangular band and \( S \) satisfies the identity \( 1 + a = a + 1 = a \) then \( (S, +) \) is

(i) quasi-seperative

(ii) weakly-seperative

(iii) seperative.

Proof: Let \( (S, +, .) \) be a semiring in which \( (S, .) \) be a rectangular band.

Assume that \( S \) satisfies the identity \( 1 + a = a + 1 = a \) for any \( a \in S \). Now for any \( a, b, \in S \),

(i) consider \( a + a = a + b \Rightarrow a(1 + 1) = ab + b \Rightarrow a.1 = b + b \Rightarrow a = b(1 + 1) \Rightarrow a = b \cdot 1 \Rightarrow a = b \)

Similarly, we prove that \( a + b = b + b \Rightarrow a = b \)

Hence \( a + a = a + b = b + b \Rightarrow a = b \Rightarrow (S, +) \) is quasi-seperative
(ii) Let $a + b = (a) + b = ba + b = b + ab = b + a \Rightarrow a + b = b + a \rightarrow (1)$

From (i) and (ii) $a + a = a + b = b + a = b + b = a = b \Rightarrow (S, +)$ is weakly separative

(iii) From (i), we have

\[ a + a = a + b \]

\[ \Rightarrow a = b \rightarrow (2) \]

\[ b + b = a + b \]

From (1) $a + b = b + a$

\[ a + a = b + a \]

\[ \Rightarrow a = b \rightarrow (3) \]

\[ b + b = b + a \]

From (2) and (3), $(S, +)$ is separative semigroup.

4.3.4. Theorem: If $(S, +)$ is I-commutative then the following conditions are equivalent:

(i) I-medial (ii) I-semimedial (iii) I-distributive.

Proof: Proof is similar to Theorem 4.3.2.

4.3.5. Theorem: Let $(S, +, \cdot)$ be a semiring in which $(S, +)$ satisfies the identity $a + 1 = 1 + a = a$ for all $a \in S$. Then for any $a, b \in S$, the following conditions hold

(i) $a + b + 1 = ab$
(ii) \(ab + 1 = ab\).

**Proof:** Let \((S, +, \cdot)\) be a semiring in which \((S, +)\) satisfies the identity 
\[1 + a = a + 1 = a, \text{ for all } a \in S.\] Let \(a, b \in S\) then

(i) consider 
\[a + b + 1 = a + (b + 1) = a + b \Rightarrow a + b + 1 = a + b\]

(ii) Again 
\[ab + 1 = a(b) + 1 = a(b + 1) + 1 = ab + a + 1\]
\[= (a + 1)b + a + 1 = (a + 1)(b + 1) \Rightarrow ab + 1 = a \cdot b\]

4.3.6. **Theorem:** Let \((S, +, \cdot)\) be a semiring then the following statements are equivalent:

i) \(a + 1 = a\)

ii) \(a^n + 1 = a^n\)

iii) \((ab)^n + 1 = (ab)^n\) for all \(a, b \in S\).

**Proof:** Let \((S, +, \cdot)\) be a semiring.

Let \(a + 1 = 1 + a = a\) for all \(a \in S\) then we have to prove that \(a^n + 1 = a^n\) for some positive integer \(n\). We prove it by mathematical induction

Case (i) If \(n = 1\) then \(a + 1 = a \Rightarrow a + 1 = a \Rightarrow\) Given statement is true for \(n = 1\)

Case (ii) Assume that given result is true for \(n = k\), i.e., \(a^k + 1 = a^k\) for some positive integer \(k\). To prove that the result is true for \(n = k + 1\)
\[a^{k+1} + 1 = a^k a + 1 = (a^k + 1)a + 1 = a^k a + a + 1 = a^k(a + 1) + a + 1\]
= (a^k + 1)(a + 1) = a^k a = a^{k+1} \Rightarrow \text{Hence the result is true for } n = k + 1 \Rightarrow 
\text{Therefore, } a^1 + 1 = a \Rightarrow a^n + 1 = a^n.

(ii) \Rightarrow (iii)

Let a^n + 1 = a^n and b^n + 1 = b^n for all a, b \in S.

Consider, (ab)^n + 1 = a^n b^n + 1 = a^n (b^n + 1) + 1 = a^n b^n + a^n + 1 = (a^n + 1)b^n + a^n + 1 = (a^n + 1)(b^n + 1) = a^n b^n \Rightarrow (ab)^n + 1 = (ab)^n.

(ii) \Rightarrow (i)

Let a^n + 1 = an for any a \in S and some +ve integer n. If n = 1 then

a^1 + 1 = a^1 \Rightarrow a + 1 = a \Rightarrow a^n + 1 = a^n \Rightarrow a + 1 = a.

4.3.7. Theorem: Let (S, +, \cdot) be a b-lattice semiring. Then

i) if (S, \cdot) is commutative semigroup then for any a, b, c, d \in S \quad ac = b

and bd = a then a = b

ii) if a + c = b and b + d = a then a = b.

Proof: Let (S, +, \cdot) be a b-lattice semiring and let (S, \cdot) be commutative semigroup.

If ac = b and bd = a for any a, b, c, d \in S. Since (S, \cdot) is a band, a = aa = a(bd) = a(ac)d = a^2 cd = acd = bdcd = bddc = bd^2 c = bdc = ac = b \Rightarrow a = b
Similarly, since \((S, +)\) is a band, for every \(a \in S\),
\[
 a = a + a \\
= a + (b + d) = a + (a + c) + d = a + c + d = b + d + c + d = b + d + d \\
+ c = b + d + c = a + c = b \\Rightarrow a = b.
\]