CHAPTER III

FUZZY INTEGRABLE FUNCTIONS

3.0 INTRODUCTION

Here we define fuzzy integrable function. Then we prove that a function \( f \) is integrable on \( X_p \) iff the fuzzy total variation is fuzzy integrable. Also we prove that \( f \) is fuzzy integrable then \( f, A(x) \) and \( T, f \) are fuzzy integrable where \( A(x) \) is the membership function and \( T \) is a bounded operator on a Banach space \( Y \).

3.1 FUZZY INTEGRABLE FUNCTION

Definition 3.1.1

A function \( f \) from \( X_p \) to \( Y \) is fuzzy integrable on \( X_p \) if there is a sequence \((f_n)\) of integrable fuzzy simple functions satisfying the conditions.

\[
\text{Lim } \begin{array}{l}
1) \quad m, n \rightarrow \infty \int_{X_p} \| f_m(x) - f_n(x) \| d\mu = 0 \\
2) \quad f_n \text{ converges to } f \text{ in fuzzy measure} \\
\end{array}
\]

For each \( E \in C \), integral over \( E \) of fuzzy integrable function \( f \) defined in terms of such a sequence \((f_n)\) of integrable fuzzy simple functions by equation \( \int f(x) \, d\mu = \lim_{n} \int_{E} f_n(x) \, d\mu \).
Remark 3.1.2

Suppose f is fuzzy integrable, then there exists a sequence \((f_n)\) of integrable fuzzy simple functions satisfying the above conditions (1) and (2). For each \(f_n\) we can find a finite valued fuzzy simple function \(h_n\) such that \(f_n - h_n\) is a fuzzy \(\mu\)-null function. Then the sequence \((h_n)\) converges to \(f\).

Since \(|h_n - f| \leq |h_n - f_n| + |f_n - f| \rightarrow 0\)

\[
\lim_{m,n \to \infty} \int_{X_p} \|h_n(x_\lambda) - h_m(x_\lambda)\| \, d\mu = \lim_{m,n \to \infty} \int_{X_p} \|f_n(x_\lambda) - f_m(x_\lambda)\| \, d\mu = 0 \text{ by definition}
\]

Thus \(\{f(x_\lambda)\} \, d\mu = \lim_{n} \int_{X_p} h_n(x_\lambda) \, d\mu\)

Therefore if \(f\) is fuzzy integrable then there exists a sequence of finite valued fuzzy simple functions satisfying the conditions of definition 3.1.1 and

\[
\{f(x_\lambda)\} \, d\mu = \lim_{n} \int_{X_p} h_n(x_\lambda) \, d\mu
\]

Notation 3.1.3

The set of all fuzzy integrable functions \(f\) from \(X_p\) to \(Y\) be denoted by \(K(X_p)\).
Proposition 3.1.4

A function \( f \) on \( X_p \) is fuzzy integrable iff \( \nu(\mu,.) \) is fuzzy integrable. If \( f \) is fuzzy integrable so is \( \| f(.) \| \). If \( (f_n) \) is a sequence of integrable fuzzy simple functions determining \( f \) in accordance with definition 3.1.1, the sequence

\[
(\| f_n(.) \|) \text{ determines } \| f(.) \| \text{ and further } \lim_{n \to \infty} \int_{X_p} \| f_n(x) - f(x) \| \, dv = 0
\]

Proof

If \( f \) is fuzzy integrable on \( X_p \), there exists a sequence \( (f_n) \) of integrable fuzzy simple functions converging to \( f \) in fuzzy measure and satisfying.

\[
\lim_{m, n \to \infty} \int_{X_p} \| f_m(x) - f_n(x) \| \, dv = 0
\]

Since \( (f_n) \) converges in fuzzy measure \( n \to \infty \) \( f_n \to f \)

ie. \( \lim_{n \to \infty} \nu^*(\mu, X_p (\| f_n(x) - f(x) \| > \varepsilon)) = 0 \) \( \forall \varepsilon > 0 \), by Proposition 1.5.2.

Then \( \lim_{n \to \infty} \| f_n(x) \| - \| f(x) \| \leq \lim_{n \to \infty} \| f_n(x) - f(x) \| = 0 \)

ie. \( \lim_{n \to \infty} \nu^*(\mu, X_p (\| f_n(x) \| - \| f(x) \| > \varepsilon)) = 0 \) \( \forall \varepsilon > 0 \)

Hence \( (\| f_n(.) \|) \) converges in fuzzy measure to \( \| f(.) \| \) as \( n \to \infty \)
Also \[
\lim_{m,n \to \infty} \int_{X_p} (\|f_m(x_\lambda)\| - \|f_n(x_\lambda)\|) \, dv
\]
\[
\leq \lim_{m,n \to \infty} \int_{X_p} \|f_m(x_\lambda) - f_n(x_\lambda)\| \, dv = 0
\]
using (1)

Thus \( \|f(.)\| \) is fuzzy integrable and \( (\|f_n(.)\|) \) determines \( \|f(.)\| \)

For a fixed \( m \), \( \|f_n(x_\lambda) - f_m(x_\lambda)\| \to 0 \) as \( n \to \infty \)

Thus \[
\lim_{n \to \infty} \int_{X_p} \|f_n(x_\lambda) - f(x_\lambda)\| \, dv = 0
\]

We have \( v^*(v(\mu, .), E) = \inf_{F \supseteq E} v(v(\mu, .), F) \)
\[
= \inf_{F \supseteq E} v(\mu, F) = v^*(\mu, E)
\]

Hence \[
\lim_{n \to \infty} v^*(v(\mu, .), X_p (|f_n-f| > \varepsilon)) = \lim_{n \to \infty} v^*(\mu, X_p (|f_n-f| > \varepsilon)) = 0
\]

Thus the convergence in fuzzy measure is the same as the convergence in fuzzy total variation \( v(\mu, .) \). Hence a function \( f \) is fuzzy integrable on \( X_p \) iff the fuzzy total variation is fuzzy integrable.
Remark 3.1.5

If $f$ is fuzzy integrable then $||f(\cdot)||$ is also fuzzy integrable and

$$\int_{x_p} ||f_n (x_\lambda)||dv$$

has a definite limit i.e.

$$\int_{x_p} ||f_n (x_\lambda)||dv < \infty$$

Proposition 3.1.6

Let $\mu$ be a finitely additive fuzzy measure defined on a $\sigma$- algebra of fuzzy sets $C$ then $K (X_p)$ is a set of fuzzy integrable function on $X_p$ which is a fuzzy linear space and for each $E \in C$, the fuzzy integral $\int_E f \mu$ is linear on $K (X_p)$.

Proof

Let $f, g \in K (X_p)$. Then there exists sequences of integrable fuzzy simple functions $(f_n)$ and $(g_n)$ converging to $f$ and $g$ respectively in fuzzy measure and satisfying the conditions,

$$\lim_{m, n \to \infty} \int_{X_p} ||f_m (x_\lambda) - f_n (x_\lambda)||dv = 0$$

and

$$\lim_{m, n \to \infty} \int_{X_p} ||g_m (x_\lambda) - g_n (x_\lambda)||dv = 0$$

since $f_n \to f$ and $g_n \to g$.

Therefore

$$\lim_{n \to \infty} v^* (\mu, X_p (|af_n + b g_n| - |af + bg| > \epsilon)) = 0$$

for all $\epsilon > 0$ by 1.5.2.
i.e. \( af_n + bg_n \to af + bg \) in fuzzy measure. Let \( h_n = \langle \gamma \rangle_{\mathbb{R}} + \delta_n \) Then \( (h_n) \) is a sequence of integrable fuzzy simple functions such that \( h_n \to \langle af_n + bg_n \rangle \)

\[
\lim_{m,n \to \infty} \int_X \| h_n(x) - h_m(x) \| \, d\nu
\]

\[
= \lim_{m,n \to \infty} \int_X \| af_n(x) + bg_n(x) - (af_m(x) + bg_m(x)) \| \, d\nu
\]

\[
\leq \lim_{m,n \to \infty} |a| \int_X \| f_n(x) - f_m(x) \| \, d\nu + \lim_{m,n \to \infty} |b| \int_X \| g_n(x) - g_m(x) \| \, d\nu = 0 \quad \text{by definition 3.1.1}
\]

This means that \( af + bg \) is fuzzy integrable, i.e. \( af + bg \in \mathcal{K}(X_p) \). Hence \( \mathcal{K}(X_p) \) is fuzzy linear.

To show that \( \int f(x) \, d\mu \) is linear on \( \mathcal{K}(X_p) \)
Let $h = af + bg$

$$\int \limits_{E} (af(x_{\lambda}) + bg(x_{\lambda})) \, d\mu = \int \limits_{E} h(x_{\lambda}) \, d\mu.$$ 

$$= n \to \infty \int \limits_{E} \eta_n(x_{\lambda}) \, d\mu.$$ 

$$= \lim_{n \to \infty} \int \limits_{E} (af_n(x_{\lambda}) + bg_n(x_{\lambda})) \, d\mu$$ 

$$= \lim_{n \to \infty} a \int \limits_{E} f_n(x_{\lambda}) \, d\mu + \lim_{n \to \infty} b \int \limits_{E} g_n(x_{\lambda}) \, d\mu$$

$$= a \int \limits_{E} f(x_{\lambda}) \, d\mu + b \int \limits_{E} g(x_{\lambda}) \, d\mu$$

Since $(f_n)$ and $(g_n)$ are integrable fuzzy simple functions. Hence $\int \limits_{E} f \, d\mu$ is linear on $K(X_{\rho})$

**Proposition 3.1.7**

Let $\mu$ be a finitely additive fuzzy measure defined on a $\sigma$ algebra of fuzzy sets $C$. If $A(x)$ is the membership function of a set $A \in C$ and $f$ is fuzzy integrable, then $f \cdot A$ is fuzzy integrable and $\int \limits_{E} f \cdot A \, d\mu = \int \limits_{E \cdot A} f \cdot d\mu \quad E \in C.$
Proof

Since \( f \) is fuzzy integrable, then there exists a sequence \((f_n)\) of integrable fuzzy simple functions converging to \( f \) in fuzzy measure on \( X_p \) and satisfying

\[
\lim_{m,n \to \infty} \int_{X_p} \|f_n(x_a) - f_m(x_a)\| dv = 0.
\]

Since \( f_n \)'s are integrable fuzzy simple functions, therefore \( f_n \) differs from a fuzzy \( \mu \)-null functions of the form

\[
h_n = \sum_{i=1}^{n} y_i A_i \quad (A) \in C
\]

We have \( \nu^* (\mu, X_p (|A(x)f_n - A(x)f| > \alpha)) \leq \nu^* (\mu, X_p (|f_n - f| > \alpha)) \quad \forall \alpha > 0 \)

ie.

\[
\inf_{\alpha > 0} \tan^{-1} \left\{ \alpha + \nu^* (\mu, X_p (|A(x)f_n - A(x)f| > \alpha)) \right\}
\]

\[
\leq \inf_{\alpha > 0} \tan^{-1} \left\{ \alpha + \nu^* (\mu, X_p (|f_n - f| > \alpha)) \right\}
\]

Hence \( |Af_n - Af| \leq |f_n - f| \).

Thus \( f_n \to f \) implies \( A f_n \to A f \) in fuzzy measure

\( f_n \)'s are integrable fuzzy simple functions which differ from \( h_n \)'s by a fuzzy \( \mu \)-null function.
In the following proposition we will show that if \( f \) is fuzzy integrable and if \( T \) is a bounded operator on the Banach space, then \( T \ f \) is fuzzy integrable

**Proposition 3.1.8**

Let \( \mu \) be a finitely additive fuzzy measure defined on a \( \sigma \)-algebra of fuzzy sets \( C \). If \( T \) is a bounded operator on a Banach space \( Y \) to another Banach space \( Z \) and if \( f \) is fuzzy integrable then \( T \ f \) is fuzzy integrable and

\[
\int_{E} T( f(\lambda)) \, d\mu = T \left( \int_{E} f(\lambda) \, d\mu \right) \quad E \in C
\]
**Proof**

Let $Y$ and $Z$ be two Banach spaces and $X_p$ be the (crisp) collection of fuzzy points in $X$

We have $T : Y \rightarrow Z$

$f : X_p \rightarrow Y$

$Tf : X_p \rightarrow Z$

Let $(f_n)$ be a sequence of integrable fuzzy simple functions on $X_p$ converging to $f$ and $\lim_{m,n \rightarrow \infty} \int_{X_p} \|f_n (x_\lambda) - f_n (x_\lambda)\| d\nu = 0$. $(Tf_n)$ is a sequence of integrable fuzzy simple functions. Now we can show that $Tf_n \rightarrow Tf$ in fuzzy measure.

Since $T$ is bounded

$v^* (\mu, X_p (|Tf - Tf_n| > \varepsilon))$

$\leq v^* (\mu, X_p (|f - f_n| > \varepsilon / |T|)) \quad \forall \varepsilon > 0 \text{ Since } (f_n) \text{ converges to } f \text{ in fuzzy measure}$

Hence $v^* (\mu, X_p (|f - f_n| > \varepsilon / |T|)) \neq 0$

ie. $v^* (\mu, X_p (|Tf - Tf_n| > \varepsilon)) = 0 \quad \forall \varepsilon > 0, n \rightarrow \infty$
Hence $Tf_n \to Tf$ in fuzzy measure

\[
\lim_{m,n \to \infty} \int_X \|T f_m(x) - Tf_n(x)\| dv \leq \lim_{m,n \to \infty} \int_X \|f_m(x) - f_n(x)\| dv
\]

\[
= 0 \quad \text{since } T \text{ is bounded}
\]

Hence $Tf$ is fuzzy integrable

Now

\[
\lim_{n \to \infty} \int_E Tf_n(x) \, d\mu = \lim_{n \to \infty} \int_E f_n(x) \, d\mu
\]

\[
= T \int_E \lim_{n \to \infty} f_n(x) \, d\mu \quad \text{for } E \in \mathcal{C}
\]

Hence

\[
\int_E Tf(x) \, d\mu = T \int_E f(x) \, d\mu
\]

Proposition 3.1.9

Let $g$ be fuzzy integrable and for $E \in \mathcal{C}$ let $G(E) = \int g(x) \, d\mu$. Then $G(E)$ is an additive fuzzy measure on $\mathcal{F}$ and has fuzzy total variation

\[
v(G,E) = \int_E \|g(x)\| dv \quad E \in \mathcal{C}
\]
Proof

Let $g$ be fuzzy integrable. Then there exists a sequence of integrable fuzzy simple functions $(g_n)$ converging to $g$ in fuzzy measure and

$$\lim_{m,n \to \infty} \int_{\mathbb{P}} |g_m(x) - g_n(x)| \, dv = 0$$

and we have

$$\int_E g(x) \, d\mu = \lim_{n \to \infty} \int_E g_n(x) \, d\mu$$

for all $E \subseteq C$

Let $G_n(E) = \int_E g_n(x) \, d\mu \quad \forall \in C$

and its fuzzy total variation $\nu(G_n, E) = \int_E \|g_n(x)\| \, dv$. First we prove that $\int_{bC} g \, d\mu = 4$

$G_n(E)$ is an additive fuzzy measure on $C$

Let $E_1, E_2 \subseteq C$

Then

$$G_n(E_1 \oplus E_2) = \int_{E_1 \oplus E_2} g_n(x) \, d\mu$$

$$= \sum_{i=1}^n y_i \mu(A_i \oplus E_2)$$

$$= \sum_{i=1}^n y_i (\mu(A_i E_1) + \mu(A_i E_2))$$

$$= \sum_{i=1}^n y_i (\mu(A_i E_1) + \sum_{i=1}^n y_i \mu(A_i E_2))$$

$$= \int_{E_1} g_n(x) \, d\mu + \int_{E_2} g_n(x) \, d\mu$$

$$= G_n(E_1) + G_n(E_2)$$
Thus \( \lim_{n \to \infty} G_n(E_1 \oplus E_2) = \lim_{n \to \infty} G_n(E_1) + \lim_{n \to \infty} G_n(E_2) \)

ie. \( G(E_1 \oplus E_2) = G(E_1) + G(E_2) \)

If \( E_1, E_2, \ldots, E_n \) are disjoint fuzzy sets in \( C \), then \( G(\oplus E_i) = \sum G(E_i) \)

Since \( G(E) = \int g(x_\lambda) \, d\mu = \lim_n \int g_r(x_\lambda) \, d\mu = \lim_n G_n(E) \), hence each \( G_n(E) \) is

additive so \( G(E) \) is additive

\[
G(\oplus E_n) = G \left( \lim_{n \to N} \oplus E_i \right)_{i=1}^n
\]

\[
= \lim_{n \to N} \sum_{i=1}^n G(E_i)
\]

\[
= \sum_{i=1}^\infty G(E_i)
\]

Clearly \( G(\emptyset) = 0 \) and \( G(E) \geq 0 \)

Hence \( G(E) = \int_{E} g(x_\lambda) \, d\mu \) is an additive fuzzy measure.
By proposition 2.2.4 if \( G(E) = \int g(x_\lambda) \, d\mu \), then total variation is

\[ v(G,E) = \int \| g(x_\lambda) \| \, dv \]

Let \( E_1, E_2, \ldots, E_k \) be disjoint fuzzy sets in \( C \) such that \( \bigoplus_{i=1}^{k} E_i = X \)

\[ \sum_{i=1}^{k} \| \int \mu_i(x_\lambda) \, d\mu - \sum_{i=1}^{k} \| \int g_n(x_\lambda) \, d\mu \| \]

\[ \leq \sum_{i=1}^{k} \| \int g(x_\lambda) \, d\mu - \int g_n(x_\lambda) \, d\mu \| \]

\[ = \sum_{i=1}^{k} \| g(x_\lambda) - g_n(x_\lambda) \| \, d\mu \]

\[ \leq \sum_{i=1}^{k} \| g(x_\lambda) - g_n(x_\lambda) \| \, dv \]

\[ \leq \int \| g(x_\lambda) - g_n(x_\lambda) \| \, dv \quad \text{by theorem 2.2.4} \]

Given \( \varepsilon > 0 \) there exists an integer \( N_\varepsilon \) such that

\[ n > N_\varepsilon \Rightarrow \int \| g(x_\lambda) - g_n(x_\lambda) \| \, dv < \varepsilon \]

Now \[ \sum_{i=1}^{k} \| \int g(x_\lambda) \, d\mu \| - \sum_{i=1}^{k} \| \int g_n(x_\lambda) \, d\mu \| \]

\[ \leq \int \| g(x_\lambda) - g_n(x_\lambda) \| \, dv < \varepsilon, \quad n > N_\varepsilon \quad \text{which is independent of the choice} \]

\( E_1, E_2, \ldots, E_k \)
Next we will show that $v(G_n, E) \to v(G, E)$ as $n \to \infty$

$$v(G, E) = \sup_{\mathcal{E}, i} \sum_{i=1}^k \| G(E_i) \|_E$$

Thus for a given $\varepsilon > 0$ there exists a fuzzy partition $A_1, A_2, ..., A_n \in \mathcal{C}$ such that

$$v(G, E) - \frac{m}{\sum_{i=1}^m} \| G(A_i) \| < \varepsilon$$

i.e. $$v(G, E) - \sum_{i=1}^m \left\| \int_{A_i} g(x_\lambda) d\mu \right\| < \varepsilon$$

Similarly $B_1, B_2, ..., B_p \in \mathcal{C}$ such that

$$v(G_n, E) - \frac{p}{\sum_{j=1}^p} \| G_n(B_j) \| < \varepsilon$$

i.e. $$v(G_n, E) - \sum_{j=1}^p \left\| \int_{B_{ij}} g_n(x_\lambda) d\mu \right\| < \varepsilon$$

Let $E_1, E_2, ..., E_k$ be the family of intersection of all fuzzy sets $A_i$ and $B_j$. Then

$$v(G, E) - \frac{k}{\sum_{i=1}^k} \left\| \int_{E_i} g(x_\lambda) d\mu \right\| < \varepsilon$$

and $$v(G_n, E) - \frac{k}{\sum_{j=1}^k} \left\| \int_{E_{ij}} g_n(x_\lambda) d\mu \right\| < \varepsilon$$
Therefore \[ |\nu(G,E) - \nu(G_n,E)| \]

\[ \leq |\nu(G_1,E) - \sum_{i=1}^{k} \int_{E_i} g_i(x_i) \, d\mu| + |\nu(G_n,E) - \sum_{i=1}^{k} \int_{E_i} g_n(x_i) \, d\mu| \]

\[ < \varepsilon + \varepsilon + \varepsilon < 3 \varepsilon \quad \forall n \in \mathbb{N} \quad \text{But } \varepsilon \text{ is arbitrary} \]

\[ |\nu(G,E) - \nu(G_n,E)| \to 0 \text{ as } n \to \infty \]

ie.

\[ \nu(G,E) = \lim_{n} \nu(G_n,E), \quad E \in \mathcal{C} \text{ and by theorem 2.2.4} \]

\[ \nu(G_n,E) = \int_{E} \|g_n(x_i)\| \, dv \]

Since \((g_n)\) is a sequence of integrable fuzzy simple functions which determines \(g\), \((\|g_n\|)\) is a sequence of integrable fuzzy simple functions determining \(\|g\|\)

Then \[ \lim_{n \to \infty} \int_{E} \|g_n(x_i)\| \, dv = \int_{E} \|g(x_i)\| \, dv \text{ and } \nu(G_n,E) \to \nu(G,E) \]

\[ \nu(G,E) = \int_{E} \|g(x_i)\| \, dv, \quad E \in \mathcal{C} \]
Proposition 3.1.10

Let $g$ be fuzzy integrable. For $E \in \mathcal{C}$ let $G(E) = \int_E g(x) \, d\mu$ and its fuzzy total variation

$$v(G,E) = \int \|g(x)\| \, dv$$

Then $\lim_{\nu(\mu,E) \to 0} v(G,E) = 0$

Proof

Let $(g_n)$ be a sequence of integrable fuzzy simple functions determining $g$ such that

$$\lim_{m,n \to \infty} \int_p \| g_n(x) - g(x) \| \, dv = 0$$

Since $g_n$ converges to $g$ in fuzzy measure, for a given $\varepsilon > 0$ there exists a positive integer $N$ such that

$$\int_p \| g_n(x) - g(x) \| \, dv < \varepsilon$$

where $g_n$ is an integrable fuzzy simple function.

Therefore exists a fuzzy simple function $h$ such that $\| g_n - h \| = 0$

Take $h = g_k$ which is finite.

Then $\int_p \| g(x) - g_k(x) \| \, dv = \int_p \| g(x) - g_k(x) \| + \| g_k(x) - h(x) \| \, dv$

$$\leq \int_p \| g(x) - g_k(x) \| \, dv + \int_p \| g_k(x) - h(x) \| \, dv$$

$$< \varepsilon \quad \text{since} \quad \int_p \| g_k(x) - h(x) \| \, dv = 0$$
Therefore there exists a set $A \in C$ with $\nu(\mu, A) < \infty$ and $\|g_k(x_\lambda)\| < M$ for every $x_\lambda \in A$.

where $M$ is a constant and $g_k(x_\lambda) = 0$ for every $x_\lambda \in A'$.

\[ \nu(G, E) = \int_E \|g(x_\lambda)\| \, d\nu \]

\[ = \int_E \|g(x_\lambda) - g_k(x_\lambda) + g_k(x_\lambda)\| \, d\nu \]

\[ \leq \int_E \|g(x_\lambda) - g_k(x_\lambda)\| \, d\nu + \int_E \|g_k(x_\lambda)\| \, d\nu \]

\[ \leq \varepsilon + \int_{A,E} \|g_k(x_\lambda)\| \, d\nu \]

\[ \leq \varepsilon + M \int_{A,E} \, d\nu \]

\[ \leq \varepsilon + M \nu(\mu, A,E) \]

\[ \leq \varepsilon + M \nu(\mu, E) \quad \text{since } \nu(\mu, A,E) < \nu(\mu, E) \]

\[ < \varepsilon + M \frac{\varepsilon}{M} \quad \text{where } \nu(\mu,E) = \frac{\varepsilon}{M} \]

\[ < 2\varepsilon \]

Thus given $\varepsilon > 0$ there exists $\delta > 0$ such that $\nu(\mu, E) < \delta$ whenever $\nu(\mu, E) < \delta$.

Therefore $\lim_{\nu(\mu,E) \to 0} \nu(G, E) = 0$. 

Theorem 3.1.1

Let \( g \) be fuzzy integrable.

Let \( G(E) = \int_{E} g(x) \, d\mu \quad \forall \ E \in \mathcal{C}. \) \( G(E) \) is an additive fuzzy measure whose fuzzy total variation \( v(G,E) = \int_{E} \| g(x) \| \, dv \)

Then \( \int_{E} \| g(x) \| \, dv = 0 \) iff \( g \) is a fuzzy \( \mu \)-null function

Proof

Let \( \int_{E} \| g(x) \| \, dv = 0 \) and let \((g_n)\) be a sequence of integrable fuzzy simple functions determining \( g \) and \( \lim_{m,n \to \infty} \int_{E} \| g_m(x) - g_n(x) \| \, dv = 0 \)

We have to show that \( g \) is a fuzzy \( \mu \)-null function

Let \( \delta > 0, \) Define

\( E_n(\delta) = \{ x_\lambda \in X_\mu : \| g_n(x_\lambda) \| > \delta \} \)

\( F_n(\delta) = \{ x_\lambda \in X_\mu : \| g(x_\lambda) - g_n(x_\lambda) \| > \delta \} \)
Then $E_n(\delta) \in C$. Since $g_n$ is an integrable fuzzy simple function

$$\int_{x} ||g_n(x_{\lambda})|| \, dv \geq \int_{E_n(\delta)} ||g_n(x_{\lambda})|| \, dv$$

$$\geq \delta \int_{E_n(\delta)} dv$$

$$\geq \delta \, v(\mu, E_n(\delta))$$

Since $\lim_{n \to \infty} \int_{x} ||g_n(x_{\lambda})|| \, dv = 0$ and $\int_{x} v(\mu, E_n(\delta)) = 0$

Since $g_n \to g$ in fuzzy measure $\lim_{n \to \infty} v^*(\mu, F_n(\delta)) = 0$ by Proposition 1.5.

But $||g(x_{\lambda})|| = ||g(x_{\lambda}) - g_n(x_{\lambda}) + g_n(x_{\lambda})||$

$$\leq ||g(x_{\lambda}) - g_n(x_{\lambda})|| + ||g_n(x_{\lambda})||$$

$$\leq \delta + \delta = 2\delta \text{ where } x_{\lambda} \in E_n(\delta) \oplus F_n(\delta)$$

This means $X_p(\{ |g(x_{\lambda})| > \delta \}) \subseteq E_n(\delta) \oplus F_n(\delta)$

ie. $\lim_{n \to \infty} v^*(\mu, E_n(\delta) \oplus F_n(\delta)) = 0$ where $\delta$ is arbitrary

Therefore $\lim_{n \to \infty} v^*(\mu X_p | g(x_{\lambda})| > \delta) \leq \lim_{n \to \infty} v^*(\mu, E_n(\delta) \oplus F_n(\delta)) = 0$

where $\delta$ is arbitrary

ie. $v^*(\mu X_p | g(x_{\lambda})| > \delta) = 0 \quad \forall \delta > 0$

Therefore $g$ is a fuzzy $\mu$-null function.

Conversely, let $g$ be a fuzzy $\mu$-null function then $|g| = 0$ ie. $g$ is a fuzzy $\mu$-null function.

Hence $\int_{x} ||g_n(x_{\lambda})|| \, dv = 0$