2.0 INTRODUCTION

In the previous chapter we have defined fuzzy total variation. In this chapter we are introducing the concept of fuzzy integrability of a fuzzy simple function. Further we have constructed a set function \( \alpha (E) = \int f (x_a) \, d\mu \), \( E \in C \) which is an additive fuzzy measure and which has fuzzy total variation. \( \nu (\alpha, E) = \sum \| f (x_a) \| \, d\mu \). We have proved that the fuzzy set of all fuzzy integrable simple functions on \( F(X_p) \) form a fuzzy linear subspace of \( F(X_p) \).

2.1 THE FUZZY INTEGRABILITY

Let \( X_p \) be the (crisp) collection of fuzzy points. \( Y \) be a real Banach space and \( C \) be a \( \sigma \)-algebra of fuzzy subsets of \( X_p \).

Definition 2.1.1

Let \( f \) be a function from \( X_p \) to \( Y \) which has only finite set of values \( y_1, y_2, ..., y_n \)

\[
\text{for which } f^{-1}(y_i) = \{ x_i : x_i \in X_p, f(x_i) = y_i \} \in C, \quad i=1,2,...,n.
\]
Remark 2.1.2

Consider the set of malignant cells in some part of a cancer patient and consider the set of images under X-ray photograph. Then the set of malignant cells form a collection of fuzzy sets. The images are considered to be a crisp set. Here the mapping $f$ defined from set of fuzzy sets (points) into the set of crisp set (points) is a fuzzy function.

Remark 2.1.3

Let $X_p$ be the collection of cancer infected points in some part of the body of a cancer patient and count the cancer cells. A map $f$ is defined from this set into the set of natural numbers. Then the mapping $f$ is a fuzzy function. Here we count different cancer infected points (cells) which are having different intensity of malignance by the function.

Definition 2.1.4

Any function $g$ from $X_p$ to $Y$ which differs from $f$ by a fuzzy $\mu$-null function is called a fuzzy simple function.

Definition 2.1.5

Let $f : X_p \to Y$. Then $f$ is called fuzzy measurable if for every $A \in \mathcal{C}$ with $v(\mu, A) < \infty$, the product $Af$ of $f$ with membership function $A(x)$ is measurable.

Definition 2.1.6

A fuzzy set $E$ is measurable if the membership function $E(x)$ is measurable.
Definition 2.1.7

A fuzzy simple function is integrable if it differs by a fuzzy $\mu$-null function from a function of the form $f = \sum_{i=1}^{n} y_i \cdot A_i$ where $A_1, A_2, \ldots, A_n$ form a finite family of disjoint fuzzy sets in $\mathbb{C}$ with $\bigoplus_{i=1}^{n} A_i = \varnothing$.

Remark 2.1.8

If $h$ is an integrable fuzzy simple function which differs from $f$ by a fuzzy $\mu$-null function then $|f - h| = 0$.

Definition 2.1.9 (Definition 4.2 of [I])

Let $f = \sum_{i=1}^{n} y_i \cdot A_i$ and $g = \sum_{j=1}^{m} z_j \cdot E_j$ be two fuzzy simple functions. We call the sum $f + g = \sum_{i=1}^{n} \sum_{j=1}^{m} (y_i + z_j) \cdot (A_i \cdot E_j)$ and we call the product of $f$ and $g$ be the fuzzy simple function $f \cdot g = \sum_{i=1}^{n} \sum_{j=1}^{m} (y_i \cdot z_j) \cdot (A_i \cdot E_j)$. 
Definition 2.1.10

If \( E \) is a fuzzy set in \( C \) and \( \mu \) is a non-negative finite fuzzy measure, the fuzzy integral over \( E \) of an integrable simple function \( h \) is defined by

\[
\int_E h d\mu = \int_E f d\mu = \sum_{i=1}^{n} y_i \mu(E_i E_i)
\]

where \( f \) has the form

\[
f = \sum_{i=1}^{n} y_i \cdot E_i \quad \text{with} \quad |f - h| = 0 \Rightarrow \mu(E_i E_i) = +\infty
\]

implies \( \gamma_i = 0 \).

Proposition 2.1.11

If \( E, A_1, A_2 \) are disjoint fuzzy sets. Then

\[
\left( E(A_1 \oplus A_2) \right)(x) = E(x) \cdot A_1(x) + E(x) \cdot A_2(x)
\]

Proof

\[
\left( E(A_1 \oplus A_2) \right)(x) = E(x) \cdot (A_1 \oplus A_2)(x)
\]

\[
= E(x) \cdot \min(1, A_1(x) + A_2(x))
\]

\[
= E(x) \cdot \left( A_1(x) + A_2(x) \right)
\]

\[
= E(x) \cdot A_1(x) + E(x) \cdot A_2(x)
\]
Proposition 2.1.12

The fuzzy integral we have defined in 2.1.10 is well defined.

Proof

Let \( f = \sum_{i=1}^{n} y_i \cdot A_i \) where \( A_1, A_2, \ldots, A_n \) is a finite family of disjoint fuzzy sets in \( C \) with \( \bigoplus_{i=1}^{n} A_i = X \) and \( y_i = f(x_i) \) be given in definition 2.1.1. Let \( g \) be another function which differs from \( h \) by a fuzzy \( \mu \)-null function and let

\[
g = \sum_{j=1}^{m} z_j \cdot E_j \text{, where } E_1, E_2, \ldots, E_m \text{ is a finite family of disjoint fuzzy sets in } C
\]

with \( \bigoplus_{j=1}^{m} E_j = X \).

By definition \( f - g = \sum_{i=1}^{n} \sum_{j=1}^{m} (y_i - z_j) \cdot (A_i \cdot E_j) \)

Now \( |f - g| = |f - h + h - g| \)

\[
\leq |f - h| + |h - g| = 0
\]

Since \( f - h \) and \( h - g \) are fuzzy \( \mu \)-null functions. Hence \( f - g \) is a fuzzy \( \mu \)-null function.

ie. \( \int f - g \, d\mu = 0 \).

This implies \( y_i - z_j = 0 \)

Therefore \( \int_{E} f \, d\mu - \int_{E} g \, d\mu = \sum_{i=1}^{n} y_i \cdot \mu(E \cdot A_i) - \sum_{j=1}^{m} z_j \cdot \mu(E \cdot E_j) \)
\[
= \sum_{i=1}^{n} \sum_{j=1}^{m} y_i \cdot \mu(l_i A_i \cdot E_j) - \sum_{i=1}^{n} \sum_{j=1}^{m} z_j \cdot \mu(E A_i \cdot E_j)
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{m} (y_i - z_j) \cdot \mu(E A_i \cdot E_j)
\]

\[
= 0. \text{ Since } y_i \cdot z_j = 0.
\]

Hence \(\int f \mu = \int g \mu\)

This means, \(\int g \mu\) is independent of the particular representing function \(f\) used.

The above argument also shows that \(\int h \mu = \int k \mu\) if \(h\) and \(k\) are both integrable fuzzy simple functions with \(|h - k| = 0\)

### 2.2. FUZZY LINEAR SUBSPACE

In this section we define fuzzy linear subspace and show that the set of all integrable fuzzy simple functions on \(F(X_p)\) form a fuzzy linear subspace of \(F(X_p)\).

**Definition 2.2.1**

A fuzzy set \(S\) in \(F(X_p)\) is called a fuzzy linear subspace of \(F(X_p)\) if \(f + g \in S\) and \(\alpha f \in S\) for every scalar \(\alpha\) and \(f, g \in S\).

**Proposition 2.2.2**

Let \(S\) be the fuzzy set of all integrable fuzzy simple functions on \(F(X_p)\). This \(S\) is a fuzzy linear subspace of \(F(X_p)\).
Proof

Let \( h \) be an integrable fuzzy simple function which differs by a fuzzy \( \mu \)-null function of the form \( f = \sum_{i=1}^{n} y_i A_i \) where \( A_1, A_2, \ldots, A_n \) are finite family of disjoint fuzzy sets with \( \bigoplus_{i=1}^{n} A_i = \emptyset \) and \( y_i = 0 \) if \( \nu(\mu, A_i) = \infty \). Similarly let \( k \) be an integrable fuzzy simple function which differs by a fuzzy \( \mu \)-null function of the form \( g = \sum_{j=1}^{m} z_j E_j \) where \( E_1, E_2, \ldots, E_m \) are disjoint family of fuzzy sets with \( \bigoplus_{j=1}^{m} E_j = X \) and \( z_j = 0 \) if \( \nu(\mu, E_j) = \infty \).

\[
|f + g - (h + k)| \leq |f - h| + |g - k| = 0
\]

Since \( f - h \) and \( g - k \) are fuzzy \( \mu \)-null functions. Hence \( h + k \) differs by a fuzzy \( \mu \)-null function from a function of the form.

\[
f + g = \sum_{i=1}^{n} \sum_{j=1}^{m} (y_i + z_j) \cdot (A_i \oplus E_j)
\]

where \( A_i \cdot E_j \) are disjoint fuzzy sets in \( X \) and \( \nu(\mu, A_i, E_j) < \infty, y_i \neq 0, z_j \neq 0 \) and \( \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} A_i \cdot E_j = \emptyset \).

Thus if \( h \in S, k \in S \), then \( h + k \in S \). Now \( |\alpha \cdot f - \alpha \cdot h| = |\alpha| \cdot |f - h| = 0 \) since \( f - h \) is a fuzzy \( \mu \)-null function. Here \( \alpha \cdot h \) differs by a fuzzy \( \mu \)-null function from a function of the form \( \alpha f = \alpha \sum_{i=1}^{n} y_i A_i = \sum_{i=1}^{n} \alpha y_i A_i \). Hence \( \alpha \cdot h \in S \). Thus \( S \) form a fuzzy linear subspace of \( F(X_{\mu}) \).
Proposition 2.2.3

Let $S$ be the set of all integrable fuzzy simple functions on $\mathbb{F}(X_p)$ and $\mu$ be an additive fuzzy measure. Then the fuzzy integral

$$\int_E h \, d\mu = \int_E f \, d\mu = \sum_{i=1}^{n} y_i \, \mu(E_i)$$

is a linear mapping from $S$ to $Y$, where $Y$ is a real Banach space.

Proof

Let $h$ be an integrable fuzzy simple function which differs by a fuzzy $\mu$-null function of the form $f = \sum_{i=1}^{n} y_i \, A_i$ where $A_1, A_2, \ldots, A_n$ are finite family of disjoint fuzzy sets with $\bigcup_{i=1}^{n} A_i = \emptyset$. Similarly let $k$ be an integrable fuzzy simple function which differs by a fuzzy $\mu$-null function of the form $g = \sum_{j=1}^{m} z_j \, E_j$ where $E_1, E_2, \ldots, E_m$ are disjoint family of fuzzy sets with $\bigcup_{j=1}^{m} E_j = \emptyset$.

We have $\int_E f \, d\mu = \sum_{i=1}^{n} y_i \, \mu(E_i)$ and $\int_E g \, d\mu = \sum_{j=1}^{m} z_j \, \mu(E_j)$, $E \in C$.

Then

$$\int_E h \, d\mu = \int_E f \, d\mu = \sum_{i=1}^{n} y_i \, \mu(E_i)$$

$$\int_E k \, d\mu = \int_E g \, d\mu = \sum_{j=1}^{m} z_j \, \mu(E_j).$$
But by definition

\[ f + g = \sum_{i=1}^{n} \sum_{j=1}^{m} (y_i + z_j) \cdot (A_i \cup E_j) \]

Therefore

\[ \int_{E} (f + g) \, d\mu = \sum_{i=1}^{n} \sum_{j=1}^{m} (y_i + z_j) \cdot \mu (E_i \cup E_j) \]

Hence

\[ \int_{E} (f + g) \, d\mu = \int_{E} (h + k) \, d\mu = \sum_{i=1}^{n} \sum_{j=1}^{m} (y_i + z_j) \cdot \mu (E_i \cup E_j) \]

Now

\[ \int_{E} f \, d\mu + \int_{E} g \, d\mu = \sum_{i=1}^{n} y_i \cdot \mu (E_i) + \sum_{j=1}^{m} z_j \cdot \mu (E_j) \]

Since \( E_1, E_2, \ldots, E_m \) and \( A_1, A_2, \ldots, A_n \) are disjoint fuzzy sets in \( X \) and

\[ \bigoplus_{i=1}^{n} A_i = X \]

\[ \bigoplus_{j=1}^{m} E_j = X \]

Hence

\[ \int_{E} f \, d\mu + \int_{E} g \, d\mu = \sum_{i=1}^{n} \sum_{j=1}^{m} (y_i + z_j) \cdot \mu (E_i \cup E_j) \]

\[ = \sum_{i=1}^{n} \sum_{j=1}^{m} (y_i + z_j) \cdot \mu (E_i \cup E_j) \]

\[ = \int_{E} (f + g) \, d\mu \]
Let $\alpha$ be a scalar.

Then $\int \alpha f \, d\mu = \int \alpha h \, d\mu = \alpha \int h \, d\mu = \alpha \int f \, d\mu$.

Hence the mapping $\int f \, d\mu$ from $S$ to $Y$ is linear.

**Theorem 2.2.4**

If $f$ is an integrable fuzzy simple function then $\| \int f(x) \, d\mu \| \leq \int \| f(x) \| \, dv$.

The set function $\alpha(E) = \int f(x) \, d\mu$ is an additive fuzzy measure on $C$ whose total variation $\nu(\alpha, E) = \int f(x) \, dv$. Also $\nu(\mu, E) \to 0$ as $\int f(x) \, d\mu = 0$.

**Proof**

First we prove that $\alpha(E)$ is an additive fuzzy measure.

Let $A_1, A_2 \in C$, then $\alpha(A_1 \oplus A_2) = \int f(x) \, d\mu$.

$$= \sum_{i=1}^{n} y_i \mu(A_1 \oplus A_2, E_i)$$

$$= \sum_{i=1}^{n} y_i \left[\mu(A_1, E_i) + \mu(A_2, E_i)\right]$$

$$= \sum_{i=1}^{n} y_i \mu(A_1, E_i) + \sum_{i=1}^{n} y_i \mu(A_2, E_i)$$

$$= \int f(x) \, d\mu + \int f(x) \, d\mu$$

$$= \alpha(A_1) + \alpha(A_2)$$
Similarly $\alpha \left( \bigoplus_{i=1}^{n} A_i \right) = \sum_{i=1}^{n} \alpha(A_i)$.

Now $\alpha \left( \bigoplus_{n \in N} A_n \right) = \alpha \left( \lim_{n \in N} \bigoplus_{k=1}^{n} A_k \right)$

$$= \lim_{n \in N} \alpha \left( \bigoplus_{k=1}^{n} A_k \right)$$

$$= \lim_{n \in N} \sum_{k=1}^{n} \alpha(A_k)$$

$$= \sum_{n=1}^{\infty} \alpha(A_n)$$

Clearly $\alpha(\emptyset) = 0$ and $\alpha(E) \geq 0$. Hence $\alpha(E)$ is an additive fuzzy measure on $C$.

Let $E \in C$ and $\bigoplus_{i=1}^{n} E_i = E$ where $E_1, E_2 \ldots E_n$ are disjoint fuzzy sets in $C$.

Then $\|\alpha(E)\| = \|\alpha \left( \bigoplus_{i=1}^{n} E_i \right)\|$

$$\leq \sum_{i=1}^{n} \|\alpha(E_i)\|$$

$$\leq \sup \left( \sum_{i=1}^{n} \|\alpha(E_i)\| \right) \quad (E_i \in C)$$

$$\leq \nu(\alpha, E)$$
Hence $\| \alpha (E) \| \leq \nu (\alpha, E)$

\[ \| f (x, \lambda) \| d\mu \leq \int_{E} \| f (x, \lambda) \| d\nu (x, \lambda) \]

Since $f$ is an integrable fuzzy simple function it differs by a fuzzy $\mu$-null function from a function of the form $h = \sum_{i=1}^{n} y_i E_i$ where $E_1, E_2, \ldots, E_n$ are disjoint fuzzy sets in $C$ with $\bigoplus_{i=1}^{n} E_i = X$ and $y_i = 0$ if $\nu (\mu, E_i) = \infty$. Then $| f - h | = 0$

ie. $f = h$ on $C$

We have $\alpha (E) = \sum_{E_{i} = 1}^{n} y_i E_i$.\[ \mu (E, E_i) \]

Now $h = \sum_{i=1}^{n} y_i E_i$ where $E_i$ is the membership function representing the fuzzy sets $E_i$.

\[ \| h \| = \sum_{i=1}^{n} \| y_i \| E_i \]

Hence $\| f (x, \lambda) \|$ is an integrable fuzzy simple function and is real valued. Now $\nu (\mu, E)$ is an additive fuzzy measure. $f$ is real valued integrable fuzzy simple function. $\| f \|$ is also real valued fuzzy simple function.

Then $\int_{E} \| f (x, \lambda) \| d\nu = \sum_{i=1}^{n} \| y_i \| \nu (\mu, E, E_i)$. \[ (1) \]
Let $E \in C$ and let $A_j, j = 1, 2, \ldots, m$ be disjoint fuzzy sets in $C$ with $A = \bigoplus_{j=1}^{m} A_j \subseteq E$

Then $\alpha (A_j) = \int x_i \, d\mu = \sum_{i=1}^{n} y_i \cdot \mu (A_j, E_i)$

Therefore $\| \alpha (A_j) \| = \sum_{i=1}^{n} \| y_i \| \cdot | \mu (A_j, E_i) |$

ie. $\sum_{j=1}^{m} \| \alpha (A_j) \| = \sum_{j=1}^{m} \sum_{i=1}^{n} \| y_i \| \cdot | \mu (A_j, E_i) |$

But $| \mu (A_j, E_i) | \leq \nu (\mu, A_j, E_i)$

Therefore $\sum_{j=1}^{m} \| \alpha (A_j) \| \leq \sum_{j=1}^{m} \sum_{i=1}^{n} \| y_i \| \cdot \nu (\mu, A_j, E_i)$

\[\leq \sum_{i=1}^{n} \| y_i \| \cdot \sum_{j=1}^{m} \nu (\mu, A_j, E_i)\]

\[\leq \sum_{i=1}^{n} \| y_i \| \cdot \nu (\mu, \bigoplus_{j=1}^{m} A_j, E_i)\]

\[\leq \sum_{i=1}^{n} \| y_i \| \cdot \nu (\mu, A, E_i)\]

\[\leq \sum_{i=1}^{n} \| y_i \| \cdot \nu (\mu, E, E_i)\]

\(\text{Since } A = \bigoplus_{j=1}^{m} A_j \subseteq E\)
Taking the supremum over all partitions $A_i$ of $E$ and the R.H.S is independent of $A_i$

$$v(\alpha, E) \leq \sum_{i=1}^{n} \| y_i \| \cdot v(\mu, E.E_i)$$

$$v(\alpha, E) \leq \int \| f(x) \| \, dv$$

Now we are going to show that $v(\alpha, E) \geq \int \| f(x) \| \, dv$

Let $\varepsilon > 0$, there exists $F_p \in C$, $p = 1, 2, \ldots$, be the disjoint fuzzy subsets of $E$ with

$$\sum_{p=1}^{P_i} \| \mu, (E.F_p) \| \geq v(\mu, E.E_i) - \frac{\varepsilon}{\sum_{i=1}^{n} \| y_i \|}$$

We have $\alpha(E.F_p) = \int f(x) \, d\mu$

$$= y_i \cdot \mu(E.F_p). \quad \text{Since } f \text{ takes the value } y_i \text{ on } E. F_p$$

$$\| \alpha(E.F_p) \| = \| y_i \| \cdot | \mu(E.F_p) |$$

Now $v(\alpha, E.E_j) = \sup \sum_{p=1}^{P_i} \| \alpha(E.F_p) \|$

$$\geq \sum_{p=1}^{P_i} \| \alpha(E.F_p) \|$$
\[
\sum_{j=1}^{n} v(\alpha, E, E_j) \geq \sum_{j=1}^{n} \sum_{p=1}^{p_j} \| \alpha (E, F) \| \\
\nu(\alpha, E) \geq \sum_{j=1}^{n} \sum_{p=1}^{p_j} \| \nu_{ij} \| \mu (E, F) \\
\geq \sum_{j=1}^{n} \| y_i \| \sum_{p=1}^{p_j} \| \mu (E, F) \| \\
\geq \sum_{j=1}^{n} \| y_i \| \left( v(\mu, E, E_j) - \frac{\varepsilon}{\sum_{j=1}^{n} \| y_j \|} \right) \\
\geq \sum_{j=1}^{n} \| y_i \| \nu(\mu, E, E_j) - \varepsilon \\
\geq \int_{E} \| f(x, \lambda) \| dv - \varepsilon \\
\nu(\alpha, E) \geq \int_{E} \| f(x, \lambda) \| dv 
\] (using 1)

But \( \varepsilon \) is arbitrary

\[
\nu(\alpha, E, E_j) \geq \int_{E} \| f(x, \lambda) \| dv 
\] (4)

From (2) and (4)

\[
\nu(\alpha, E, E_j) = \int_{E} \| f(x, \lambda) \| dv 
\]

We have seen that \( \| \alpha (E) \| \leq \nu(\alpha, E) \) for \( E \in C \)

ie. \[ \int_{E} \| f(x, \lambda) \| d\mu \leq \int_{E} \| f(x, \lambda) \| dv, \quad E \in C \]
To prove the last result consider the following.

\[ \int_{E} f(x_{\lambda}) \, d\mu = \sum_{j=1}^{n} y_{i} \cdot \mu(E.E_{i}) \quad \text{where } E_{i}'s \text{ are disjoint fuzzy sets in } C \text{ and } \]

\[ \bigoplus_{i=1}^{n} E_{i} = X \]

Then \[ \| \int_{E} f(x_{\lambda}) \, d\mu \| \leq \sum_{i=1}^{n} \| y_{i} \| \cdot | \mu(E.E_{i})| \leq \sup_{1 \leq i \leq n} \| y_{i} \| \cdot \sum_{i=1}^{n} | \mu(E.E_{i})| \]

\[ \leq \sup_{1 \leq i \leq n} \| y_{i} \| \cdot v(\mu, E.) \]

Hence \[ \lim_{\mathcal{X}(\lambda,E) \to 0} \int_{E} f(x_{\lambda}) \, d\mu = 0 \]

**Proposition 2.2.5**

If \((f_{n}^{1}), (f_{n}^{2})\) are sequences of integrable fuzzy simple functions both converging in fuzzy measure on \(X_{\lambda}\) to the same limit and if

\[ m, n \to \infty \int_{X_{\lambda}} \| f_{n}^{i} (x_{\lambda}) - f_{n}^{i} (x_{\lambda}) \| dv = 0, \quad i=1, 2 \]

Then \[ n \to \infty \int_{E} f_{n}^{i} (x_{\lambda}) d\mu, i=1,2 \text{ exist uniformly with respect to } E \text{ in } C \text{ and are equal.} \]
Proof

\[
\| \int_E f^i_n(x_\lambda) \, d\mu - \int_E f^i_m(x_\lambda) \, d\mu \| = \| \int_E (f^i_n(x_\lambda) - f^i_m(x_\lambda)) \, d\mu \| \\
\leq \int_E \| f^i_n(x_\lambda) - f^i_m(x_\lambda) \| \, d\nu \quad \text{by Proposition 2.2.4}
\]

But \[\lim_{m, n \to \infty} \int_{\chi_p} \| f^i_n f(x_\lambda) - f^i_m(x_\lambda) \, d\mu \| \, d\nu = 0, \quad i = 1, 2\]

This means that \[\lim_{m, n \to \infty} \int_{\chi_p} \| f^i_n - f^i_m \, (x_\lambda) \, d\mu \| \, d\nu \to 0 \quad \text{as} \ m, n \to \infty \quad i = 1, 2\]

Thus \[\int_E f^i_n(x_\lambda) \, d\mu - \int_E f^i_m(x_\lambda) \, d\mu \] are Cauchy sequences. The values of these sequences are in Banach space which converge uniformly with respect to \( E \) in \( C \).

Next we have to show that these limits are equal. Let us denote.

\[
\mathcal{P}_n(E) = \| f^i_n(x_\lambda) - f^{i'}_n(x_\lambda) \| \quad \text{and} \quad P_n(E) = \int_{\chi_p} p_n \, d\mu
\]

\[
\| p_n(x_\lambda) - p_m(x_\lambda) \| = \| f^i_n(x_\lambda) - f^{i'}_n(x_\lambda) - (f^i_m(x_\lambda) - f^{i'}_m(x_\lambda)) \| \\
\leq \| f^i_n(x_\lambda) - f^i_m(x_\lambda) \| + \| f^{i'}_n(x_\lambda) - f^{i'}_m(x_\lambda) \|
\]

\[
\lim_{m, n \to \infty} \int_{\chi_p} \| p_n(x_\lambda) - p_m(x_\lambda) \| \, d\nu \leq \lim_{m, n \to \infty} \int_{\chi_p} \| f^i_n(x_\lambda) - f^i_m(x_\lambda) \| \, d\nu
\]

\[+ \lim_{m, n \to \infty} \int_{\chi_p} \| f^{i'}_n(x_\lambda) - f^{i'}_m(x_\lambda) \| \, d\nu = 0 \quad \text{given}\]
Hence \( \lim_{m,n \to \infty} \int_E \left| p_n(x, \lambda) - p_m(x, \lambda) \right| \, d\nu = 0 \) since \( E \subseteq X_\nu \). Hence \( \{\mathcal{P}_n(E)\} \) is a Cauchy sequence and the limit \( \lim_{n \to \infty} \mathcal{P}_n(E) = \mathcal{P}(E) \) exists uniformly with respect to \( E \) in \( C \). By Proposition 2.2.4

\[
\left\| \int_E f_n^1(x, \lambda) \, d\mu - \int_E f_n^2(x, \lambda) \, d\mu \right\| := \int_E \left\| f_n^1(x, \lambda) - f_n^2(x, \lambda) \right\| \, d\nu = \int_E p_n \, d\nu
\]

\[
= P_n(E) \to 0 \quad E \in C
\]

Thus \( \left\| \int_E f_n^1(x, \lambda) \, d\mu - \int_E f_n^2(x, \lambda) \, d\mu \right\| \to 0 \)

ie. \( \lim_{n \to \infty} \int_E f_n^1(x, \lambda) \, d\mu - \int_E f_n^2(x, \lambda) \, d\mu = 0 \)

Thus \( \lim_{n \to \infty} \int_E f_n^1(x, \lambda) \, d\mu = \int_E f_n^2(x, \lambda) \, d\mu \)

Next we show that \( \mathcal{P}(E) = 0 \)

Let \( \{f_n^1(x, \lambda)\} \) and \( \{f_n^2(x, \lambda)\} \) converges in fuzzy measure to \( f \). Then

\[
n \to \infty \quad | f_n^1(x, \lambda) - f(x, \lambda) | = 0 \quad \text{and} \quad n \to \infty \quad | f_n^2(x, \lambda) - f(x, \lambda) | = 0
\]
\[
\begin{align*}
\lim_{n \to \infty} | f_n(x_1) - f^2(x_1) | &= \lim_{n \to \infty} | f_n(x_1) - f(x_1) + f(x_1) - f^2(x_1) | \\
&\leq \lim_{n \to \infty} | f_n(x_1) - f(x_1) | + \lim_{n \to \infty} | f_n^2(x_1) - f(x_1) | \\
&= 0
\end{align*}
\]

Hence \( \lim_{n \to \infty} | f_n(x_1) - f_n^2(x_1) | = 0 \) \ ...(1)

Consider \( \lim_{\|x, E\| \to 0} \Phi(E) = \lim_{\|x, E\| \to 0} \int F_n(x_1) \, d\nu = 0 \) by proposition 2.2.4

Thus for \( \varepsilon > 0 \) there exists a \( \delta > 0 \) and an integer \( n_0 \) such that

\( P(E) < \varepsilon \) for \( \nu(\mu, E) < \delta \) \ ...(2)

Since \( P(E) \to 0 \) as \( \nu(\mu, E) \to 0 \) and \( |\Phi(E) - p_n(E)| < \varepsilon \) for \( n \geq n_0, E \in C \) \ ...(3)

Since \( p_{n_0}(x_1) \) is fuzzy simple function \( p_{n_0}(x_1) = 0 \) for \( x_1 \in A' \) for the fuzzy set \( A \in C \) with \( \nu(\mu, A) < \infty \)

Therefore \( \int_{A'} \| f_{n_0}(x_1) - f_{n_0}^2(x_1) \| \, d\nu = 0 \)

Hence \( p_{n_0}(A') = 0 \)

Since from (3) \( |P(A') - P_{n_0}(A')| < \varepsilon \)

ie. \( |P(A') - 0| < \varepsilon \) implies \( \Phi(A) < \varepsilon \) - \ ...(4)
Hence $\delta_0 (E) \to 0$ in fuzzy measure on $X_p$ by using (1) and thus there exists an integer $n_1 \geq n_0$ and a fuzzy set $B \in C$ such that $\nu(\mu, B') < \delta$ implies

$$p_{n_1}(x_\lambda) < \frac{\epsilon}{\nu(\mu, A) + 1} \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (5)$$

for $x_\lambda \in B$

From (3) and (5)

$$P(A \cup B) \leq \int_{A \cup B} p_{n_1}(x_\lambda) d\nu + \epsilon \leq \epsilon + \epsilon < 2\epsilon \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (6)$$

Since $\nu(\mu, A \cup B') \leq \nu(\mu, B')$

$$P(X_p) \leq P(A \cup B) + P(A \cup B') + P(A') \leq 2\epsilon + \epsilon + \epsilon < 4\epsilon \quad \text{using (2), (4) and (6) since } \epsilon \text{ is arbitrary}$$

Therefore $0 \leq P(E) \leq P(X_p)$

Hence $P(E) = 0 \ \forall \quad E \in C$