CHAPTER VI

FUZZY VECTOR VALUED INTEGRATION

6.0 INTRODUCTION

Finally we are coming to the concept of fuzzy vector valued integration. For which we define fuzzy semi-variation of a fuzzy vector valued measure, and discuss some of its properties.

6.1 THE FUZZY SEMI-VARIATION OF FUZZY VECTOR VALUED MEASURES

Let \( X \) be a non-empty set, \( \mathcal{C} \) be a \( \sigma \) - algebra of fuzzy subsets of \( X \), \( \mu \) be a mapping from \( \mathcal{C} \) to the vector space \( V \).

\( \mu : \mathcal{C} \rightarrow V \)

Definition 6.1.1

The fuzzy semi-variation of a fuzzy vector valued measure \( \mu \) is denoted by \( \tilde{\mu} \), defined by

\[
\tilde{\mu} (E) = \text{Sup} \left\| \sum_{i=1}^{n} \alpha_i \mu (E_i) \right\|
\]

for \( E \in \mathcal{C} \), where the supremum is taken over all finite collection of scalars with \( |\alpha_i| \leq 1 \) and \( E_i \) be partition of fuzzy set \( E \) into a finite number of disjoint fuzzy sets in \( \mathcal{C} \).
Remark 6.1.2

When $\alpha_i = 1$, $i = 1, 2, \ldots$ then

$$\tilde{\mu} (E) = \sup \left\| \sum_{i=1}^{n} \alpha_i \cdot \mu (E_i) \right\| = \|\mu (E)\|$$

where $E \in C$.

Proposition 6.1.3

Let $\mu$ be a fuzzy vector valued measure on $C$ and $\tilde{\mu} (E)$ be fuzzy semi-variation. Then

1. $\tilde{\mu} (E) \geq \|\mu (E)\| \geq 0$, $E \in C$
2. If $F \subseteq E \Rightarrow \tilde{\mu} (F) \leq \tilde{\mu} (E)$
3. If $(E_i)$ be a sequence of fuzzy sets in $C$ then $\tilde{\mu} (\oplus E_i) \leq \sum_{i=1}^{\infty} \tilde{\mu} (E_i)$

Proof

Let $E_1, E_2, \ldots, E_n$ be a partition of fuzzy sets $E \in C$, $E_i \in C$, $i = 1, 2, \ldots, n$ and $\alpha_i$'s, $i = 1, 2, \ldots, n$ be finite scalars.

We have $\tilde{\mu} (E) = \sup_{(E_i)} \sup_{(\alpha_i)} \left\| \sum_{i=1}^{n} \alpha_i \cdot \mu (E_i) \right\|$

$$\Rightarrow \sup_{(E_i)} \left\| \sum_{i=1}^{n} \alpha_i \cdot \mu (E_i) \right\| \geq \|\mu (E)\| \geq 0$$

Hence $\tilde{\mu} (E) \geq \|\mu (E)\| \geq 0$
2. \( \tilde{\mu}(F) = \sup \| \sum_{j=1}^{n} \alpha_j \mu(F_j) \| \) where \((F_j)\) is a finite sequence of disjoint fuzzy partition of \(F, \forall F_j \in C\) and the supremum is taken over all scalars \( \alpha_j, |\alpha_j| \leq 1 \)

\[
\tilde{\mu}(E) = \sup_{(E_i)} \sup_{(\beta_i) \in \{E \}} \| \sum_{i=1}^{m} \beta_i \mu(E_i) \|
\]

where \((E_i)\) is a finite sequence of disjoint fuzzy partition of \(E \in C\), Since \(F \subseteq E\), a fuzzy partition \((F_j), j=1,2 \ldots n\) of \(F\) leads to a fuzzy partition \((F_j) \oplus (E \circ F)\) of \(E\).

Thus \( \| \sum_{j=1}^{n} \alpha_j \cdot \mu(F_j) \| = \| \sum_{j=1}^{n} \alpha_j \cdot \mu(F_j) + \mu(E \circ F) \| \)

\[
= \| \sum_{j=1}^{n} \alpha_j \cdot \mu(F_j) \| + \| \mu(E \circ F) \|
\]

\[
\leq \tilde{\mu}(E)
\]

Since the fuzzy partition of \(F\) is contained in the fuzzy partition of \(E\).

Therefore \( \sup \| \sum_{j=1}^{n} \alpha_j \cdot \mu(F_j) \| \leq \tilde{\mu}(E) \)

ie \( \tilde{\mu}(F) \leq \tilde{\mu}(E) \)
3. Let \((E_i)\) be a disjoint sequence of fuzzy sets in \(C\) such that \(\bigoplus_{i=1}^{\infty} E_i = E\).

We have \(\tilde{\mu} \left( \bigoplus_{i=1}^{\infty} E_i \right) = \tilde{\mu} (E)\)

\[
= \text{Sup} \| \sum_{j=1}^{k} \alpha_j \cdot \mu (F_j) \| \tag{1}
\]

Where the supreme is taken over all finite collection of scalars with \(|\alpha_j| \leq 1\) and all fuzzy partitions of \(E\) into a finite number of disjoint fuzzy sets \(F_j\) in \(C\).

\(F_1, F_2, \ldots, F_k\) are disjoint fuzzy partition of \(E = \bigoplus_{i=1}^{\infty} E_i \Rightarrow E_i, F_1, E_i, F_2, \ldots, E_i, F_k\) are disjoint fuzzy partition of \(E_i\). Also \(F_j = \bigoplus_{i=1}^{\infty} E_i, F_j, j = 1, 2, \ldots, k\)

Thus \(\mu (F_j) = \mu (\bigoplus_{i=1}^{\infty} E_i, F_j)\)

\[
= \sum_{i=1}^{\infty} \mu (E_i, F_j). \quad \text{Since } \mu \text{ is a countably additive fuzzy measure.}
\]

Thus we have \(\| \sum_{j=1}^{k} \alpha_j \cdot \mu (F_j) \| = \| \sum_{j=1}^{k} \alpha_j \cdot \sum_{i=1}^{\infty} \mu (E_i, F_j) \| \)

\[
\leq \sum_{i=1}^{\infty} \| \sum_{j=1}^{k} \alpha_j \cdot \mu (E_i, F_j) \| \leq \sum_{i=1}^{\infty} \tilde{\mu} (E_i)
\]

\[
\text{Sup} \| \sum_{j=1}^{k} \alpha_j \cdot \mu (F_j) \| \leq \sum_{i=1}^{\infty} \tilde{\mu} (E_i)
\]
Then from (1) we get

\[
\sum_{i=1}^{\infty} \tilde{\mu}(E_i) \leq \sum_{i=1}^{\infty} \tilde{\mu}(E_i)
\]

This shows that the fuzzy semi-variation is countably subadditive and is a positive fuzzy measure.

**Remark 6.1.4**

There exists a finite positive fuzzy measure \( \lambda \) defined on \( C \) such that

\[
\lambda(E) \leq \tilde{\mu}(E), \quad E \in C \quad \text{and} \quad \lim_{\lambda(E) \to 0} \tilde{\mu}(E) = 0
\]

**Proposition 6.1.5**

Let \( (\mu_n) \) be a sequence of fuzzy vector valued measures defined on \( \sigma \)-algebra of fuzzy sets in \( C \). If \( \mu(E) = \lim_{n \to \infty} \mu_n(E) \) exist for each \( E \) in \( C \), \( \mu \) is a fuzzy vector valued measure on \( C \) and the countable additivity of \( \mu_n \) is uniform in \( n = 1, 2, \ldots \).

**Proof**

By Remark 6.1.4 corresponding to every fuzzy vector measure \( \mu \), there is a finite fuzzy measure \( \lambda_n \) defined on \( C \) such that \( \lambda_n(E) \leq \tilde{\mu}_n(E), \quad E \in C \) and

\[
\lim_{\lambda_n(E) \to 0} \tilde{\mu}_n(E) = 0 \quad \text{where} \quad \tilde{\mu}_n \text{ is the fuzzy semi-variation.}
\]

Define \( \lambda(E) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\lambda_n(E)}{1 + \lambda_n(X_p)} \) \hspace{1cm} (1)
Each $\mu_n$ is $\lambda$-continuous, ie. $\lambda (E) \rightarrow 0$ implies $\| \mu_n (E) \| \rightarrow 0$

As $\lambda (E) \rightarrow 0$, from (1). $\lambda_n (E) \rightarrow 0$ and $\lim_{\lambda_n (E) \rightarrow 0} \mu_n (E) = 0$ so that

$$\mu_n (E) \rightarrow 0 \quad \text{as} \quad \lambda_n (E) \rightarrow 0$$

But by proposition 6.1.3 $\mu_n (E) \geq \| \mu_n (E) \|$

Therefore $\| \mu_n (E) \| \rightarrow 0$ as $\lambda (E) \rightarrow 0$ by proposition 5.2.10

$$\mu (E) = \lim_{n} \mu_n (E)$$

is countably additive. If $(E_n)$ is a sequence of disjoint fuzzy sets in $C$

$$\lim_{n \rightarrow \infty} \lambda (E_n) = 0$$

By definition 5.2.9 $\lim_{n} \mu_n (E) = 0$ uniformly for $n=1,2,...$

**Definition 6.1.6**

Let $\mu$ be a fuzzy vector valued measure on $C$. A subset of a fuzzy set $E \in C$ such that the fuzzy semi-variation $\mu (E) = 0$ is called a fuzzy $\mu$-null set.

Let $\lambda$ be a finite positive fuzzy measure on $C$ related to $\mu$ as in remark 6.1.4.

**Remark 6.1.7**

A fuzzy $\mu$-null set is same as fuzzy $\lambda$-null set.
Proof

By remark 6.1.4, $\lambda (E) \leq \mu (E)$, $E \in C$. If $E_0$ is a fuzzy $\mu$-null set $\mu (E_0) = 0$ so that $\lambda (E_0)=0$. Hence $E_0$ is a fuzzy $\mu$-null set implies $E_0$ is a fuzzy $\lambda$-null set. Also by remark 6.1.4 $\lim_{\lambda (E) \to 0} \mu (E) = 0$, $E \in C$.

For a given $\varepsilon > 0$, there exists $\delta > 0$ such that $\mu (E) < \varepsilon$, $E \in C$ with $\lambda (E) < \delta$. $E_0$ is a fuzzy $\lambda$-null set such that $\lambda (E_0) = 0$ implies $\mu (E_0) < \varepsilon$ $\forall \varepsilon$ implies $\mu (E_0)=0$. Therefore $E_0$ is fuzzy $\lambda$-null set implies $E_0$ is fuzzy $\mu$-null set.

6.2 INTEGRATION WITH RESPECT TO FUZZY VECTOR VALUED MEASURES

In the previous chapters we have taken $\mu$ as an extended real valued fuzzy measure. Here we are taking $\mu$ as a fuzzy vector valued measure. Then we are giving some definitions and results using fuzzy vector valued measure.

Definition 6.2.1

A function $f; X_\mu \to Y$ is said to be fuzzy $\mu$-simple if it can be written in the form $f = \sum_{i=1}^{n} \alpha_i A_i$ where $A_1, A_2, \ldots A_n$ are finite family of disjoint fuzzy sets in $C$ with $\bigcup_{i=1}^{n} A_i = X_\mu$ and $\alpha_1, \alpha_2, \ldots \alpha_n \in \mathbb{R}^n$.
Definition 6.2.2

If $f$ is fuzzy $\mu$-simple function of the form $f = \sum_{i=1}^{n} \alpha_i \cdot E_i$ where $E_1, E_2, \ldots, E_n$ are disjoint fuzzy sets in $\mathcal{C}$, then the fuzzy integral of $f$ is defined by

$$\int f \, d\mu = \sum_{i=1}^{n} \alpha_i \cdot \mu(E_i)$$

Proposition 6.2.3

The above definition 6.2.2 of the fuzzy integral is unique.

Proof

Suppose that the above representation of $f$ is not unique. That is $f$ has the form $f = \sum_{j=1}^{m} \beta_j \cdot A_j$ where $A_1, A_2, \ldots, A_m$ are disjoint fuzzy sets in $\mathcal{C}$ with $\bigcup_{j=1}^{m} A_j = X$

$$0 = f - f = \sum_{i=1}^{n} \alpha_i \cdot E_i - \sum_{j=1}^{m} \beta_j \cdot A_j$$

$$= \sum_{i=1}^{n} \alpha_i \cdot E_i - \sum_{j=1}^{m} \beta_j \cdot A_j - \sum_{j=1}^{m} \beta_j \cdot \sum_{i=1}^{n} E_i$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i \cdot \beta_j) \cdot (E_i \cdot A_j)$$

$$= 0 \quad \text{Hence} \quad \alpha_i \cdot \beta_j = 0$$

$(E_i, A_j)$ is a disjoint collection of fuzzy sets in $\mathcal{C}$, since $E_1, E_2, \ldots, E_n$ and $A_1, A_2, \ldots, A_m$ are disjoint.
\[
\sum_{i=1}^{n} \alpha_i \cdot \mu(E.E_i) - \sum_{j=1}^{m} \beta_j \cdot \mu(E.A_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \cdot \mu(E.E_i, A_j) - \sum_{i=1}^{n} \sum_{j=1}^{m} \beta_j \cdot \mu(E.E_i, A_j)
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i - \beta_j) \cdot \mu(E.E_i, A_j)
\]

\[
= 0 \quad \text{Since } \alpha_i - \beta_j = 0
\]

Hence \( \sum_{i=1}^{n} \alpha_i \cdot \mu(E.E_i) = \sum_{j=1}^{m} \beta_j \cdot \mu(E.A_j) \)

So \( \int_E f \, d\mu = \sum_{j=1}^{m} \alpha_i \cdot \mu(E.E_i) \)

Hence the representation is unique

**Remark 6.2.4**

The fuzzy integral of a fuzzy \( \mu \)-simple function over \( E \) is a countably additive fuzzy measure.

**Remark 6.2.5**

If \( f \) is a fuzzy \( \mu \)-simple function such that \( \| f(x_\lambda) \| \leq M \) for each \( x_\lambda \in E \),

then \( \| \int_E f(x_\lambda) \, d\mu \| \leq ( \sup_{x_\lambda \in E} \| f(x_\lambda) \| ) \bar{\mu}(E), \quad E \in \mathcal{C} \) and \( \bar{\mu}(E) \) is the fuzzy semi-variation.
Proof

\[ \int_{E} f(x_{\lambda}) \, d\mu = \sum_{i=1}^{n} \alpha_{i} \cdot \mu(E_{i}) \]

\[ \| \int_{E} f(x_{\lambda}) \, d\mu \| = \| \sum_{i=1}^{n} \alpha_{i} \cdot \mu(E_{i}) \| \]

\[ = \| \sum_{i=1}^{n} \frac{\alpha_{i}}{M} \cdot \mu(E_{i}) \| \]

\[ \leq M \| \sum_{i=1}^{n} \frac{\alpha_{i}}{M} \cdot \mu(E_{i}) \| \]

\[ \leq M \tilde{\mu}(E) \]

Hence \[ \| \int_{E} f(x_{\lambda}) \, d\mu \| \leq \left( \sup_{x_{\lambda} \in E} \| f(x_{\lambda}) \| \right) \tilde{\mu}(E) \]

Definition 6.2.6

The fuzzy \( \mu \)-essential suprmeum of the function \( f \) on the fuzzy set \( E \in C \)

\[ \text{defined as } \inf \{ \lambda \in E : \| f(x_{\lambda}) \| > A \} \text{ is a fuzzy } \mu \text{-null set} \]

If fuzzy \( \mu \)-essential supremum \( \| f(x_{\lambda}) \| < \infty \) we say that \( f \) is fuzzy \( \mu \)-essentially bounded on the fuzzy set \( E \in C \).

Proposition 6.2.7

Fuzzy \( \mu \)-essential \( \sup_{x_{\lambda} \in E} \| f(x_{\lambda}) \| = \) fuzzy \( \lambda \)-essential \( \sup_{x_{\lambda} \in E} \| f(x_{\lambda}) \| \) and

that \( f \) is fuzzy \( \mu \)-essentially bounded iff \( f \) is fuzzy \( \lambda \)-essentially bounded.
Proof

A fuzzy set is fuzzy $\mu$-null set implies it is fuzzy $\lambda$-null set.

Fuzzy $\mu$-essentially $\sup_{x_{\lambda}\in E}||f(x_{\lambda})||$

$= \inf \{ A. \{ x_{\lambda}\in E. \sup ||f(x_{\lambda})|| > A \} \text{ is a fuzzy } \mu \text{-null set } \}$

$= \inf \{ A. \{ x_{\lambda}\in E. \sup ||f(x_{\lambda})|| > A \} \text{ is a fuzzy } \lambda \text{-null set } \}$

$= \text{fuzzy } \lambda \text{-essential supremum of } f$

Definition 6.2.8

A scalar valued fuzzy measurable function $f$ is said to be fuzzy integrable if there exists a sequence $(f_n)$ of fuzzy simple functions such that:

1. $f_n(x_{\lambda}) \rightarrow f(x_{\lambda})$, $\mu$-almost everywhere.

2. The sequence $(\int f_n(x_{\lambda}) d\mu)$ converges in the norm of the space $E$ for each $E \in C$.

Proposition 6.2.9

If $f$ and $g$ are scalar valued fuzzy integrable functions, if $a$ and $b$ are scalars and if $E \in C$, then

$$\int_E (a f(x_{\lambda}) + bg(x_{\lambda})) \, d\mu = a \int_E f(x_{\lambda}) \, d\mu + b \int_E g(x_{\lambda}) \, d\mu$$
Proof

Let \((f_n)\) and \((g_n)\) be sequence of fuzzy simple functions converging to \(f\) and \(g\) respectively \(\mu\)-almost everywhere and

\[
\lim_{n \to \infty} \int_E f_n(x_\lambda) \, d\mu = \int_E f(x_\lambda) \, d\mu \quad \text{and} \quad \lim_{n \to \infty} \int_E g_n(x_\lambda) \, d\mu = \int_E g(x_\lambda) \, d\mu
\]

\[
\int_E (a f(x_\lambda) + b g(x_\lambda)) \, d\mu = \lim_{n} \int_E (a f_n(x_\lambda) + b g_n(x_\lambda)) \, d\mu
\]

\[
= \lim_{n} \int_E (a f_n(x_\lambda) \, d\mu + \lim_{n} \int_E b g_n(x_\lambda) \, d\mu
\]

Since \(f_n(x_\lambda)\) and \(g_n(x_\lambda)\) are fuzzy \(\mu\)-simple functions.

\[
= a \lim_{n} \int_E f_n(x_\lambda) \, d\mu + b \lim_{n} \int_E g_n(x_\lambda) \, d\mu
\]

\[
= a \int_E f(x_\lambda) \, d\mu + b \int_E g(x_\lambda) \, d\mu
\]

Hence

\[
\int_E (a f(x_\lambda) + b g(x_\lambda)) \, d\mu = a \int_E f(x_\lambda) \, d\mu + b \int_E g(x_\lambda) \, d\mu
\]

Proposition 6.2.10

If \(f\) is a scalar valued fuzzy integrable function, then the fuzzy integral

\[
\int_E f(x_\lambda) \, d\mu
\]

is countably additive fuzzy measure on \(C\) to \(Y\).
Proof

Since \( f \) is a scalar valued fuzzy integrable function, then there exists \((f_n)\), a sequence of fuzzy simple functions converging to \( f \), \( \mu \) almost everywhere and \( \left( \int_{E} f_n(x, \lambda) \, d\mu \right) \) converges in the space \( Y \) for each \( E \in \mathcal{C} \). We also have the fuzzy simple function is countably additive fuzzy measure on \( C \).

Let \( G_n(E) = \int_{E} f_n(x, \lambda) \, d\mu \)

\( E = \bigoplus_{i=1}^{\infty} E_i \) and \( E_1, E_2 \ldots \) are disjoint fuzzy partition of \( E \).

Also suppose \( G(E) = \int_{E} f(x, \lambda) \, d\mu \)

Then \( G(E) = \lim_{n} G_n(E) \)

\[ = \lim_{n} G_n \left( \bigoplus_{i=1}^{\infty} E_i \right) \]

\[ = \lim_{n} G_n(E_1) + \lim_{n} G_n(E_2) + \ldots \]

\[ = \sum_{i=1}^{\infty} \lim_{n} G_n(E_i) \]

\[ = \sum_{i=1}^{\infty} \int_{E_i} f(x, \lambda) \, d\mu \]

\[ = \sum_{i=1}^{\infty} G(E_i) \]
Proposition 6.2.11

If \( f \) is a fuzzy integrable function and \( E \in C \), then

\[
\lim_{\mu(E) \to 0} \int_E f(x_\lambda) \, d\mu = 0
\]
where \( \mu(E) \) is the fuzzy semi-variation

\[\mu(E)\]

\( \mu \)

Proof

Since \( f \) is fuzzy integrable, there exists \((f_n)\) a sequence of fuzzy simple functions converges to \( f \) \( \mu \) almost everywhere and \((f_n(x_\lambda))\) converges in the space \( Y \) for each \( E \in C \).

Also \( \int_E f(x_\lambda) \, d\mu = \lim_{n \to \infty} \int_E f_n(x_\lambda) \, d\mu \)

Since \( f_n(x_\lambda) \) is fuzzy simple function,

\( \mu \)

\( \mu \)

\( \mu \)

By result 5.4.5 we have \( \| \int_E f(x_\lambda) \, d\mu \| \leq (\sup_{x_\lambda \in E} \| f(x_\lambda) \|) \mu(E) \)

ie. \( \lim_{\mu(E) \to 0} \| \int_E f_n(x_\lambda) \, d\mu \| = 0 \)

Hence \( \lim_{\mu(E) \to 0} \int_E f_n(x_\lambda) \, d\mu = 0 \)

Thus \( \lim_{\mu(E) \to 0} \int_E f(x_\lambda) \, d\mu = \lim_{n \to \infty} \lim_{\mu(E) \to 0} \int_E f_n(x_\lambda) \, d\mu = 0 \)
Proposition 6.2.12

If $T$ be a bounded linear operator from $Y$ into a Banach space $Z$, then

$T \mu : \mathbb{C} \rightarrow Z$, $T \mu$ is a fuzzy vector valued measure and for any fuzzy integrable function $f$ and $E \in \mathbb{C}$, we have

$$T \left( \int_{E} f(x) \, d\mu \right) = \int_{E} f(x) \, T(d\mu)$$

Proof

$T$ is a bounded linear operator from the Banach space $Y$ to the Banach space $Z$. If $f_n$ is a fuzzy simple function given by

$$f_n = \sum_{i=1}^{n} \alpha_i A_i$$

where $A_1, A_2 \ldots A_n$ are disjoint fuzzy sets in $\mathbb{C}$ with $\bigoplus_{i=1}^{n} A_i = X$

Then

$$\int_{E} f_n(x) \, d\mu = \sum_{i=1}^{n} \alpha_i \mu(E.A_i)$$

$$T \left( \int_{E} f_n(x) \, d\mu \right) = T \left( \sum_{i=1}^{n} \alpha_i \mu(E.A_i) \right)$$

$$= \sum_{i=1}^{n} \alpha_i T(\mu(E.A_i)) \quad \text{Since } T \text{ is a linear operator}$$

$$= \int_{E} f_n(x) \, T(d\mu)$$
Since $f$ is the limit of the sequence $(f_n)$ of fuzzy integrable simple function

We get \( \lim_{n} T\left( \int f_n(x) \, d\mu \right) = \lim_{n} \int f_n(x) \, T(d\mu) \)

\[ T\left( \int f(x) \, d\mu \right) = \int f(x) \, T(d\mu) \]

**Concluding Remarks**

In this thesis we have taken a fuzzy measure whose domain is fuzzy whereas the range is non-fuzzy. The function $f$ we have considered to have fuzzy domain but with non-fuzzy range.

There is ample scope for extending this theory. For example taking the domain to be non-fuzzy and the range to be fuzzy in the case of measure $\mu$ as well as in the case of the function $f$. This is a very relevant situation we come across in our day to day life. Consider the set of square plots suitable for house construction. Define $\mu$ to be the area of a square plot. When the area is calculated by taking measurements of a plot by different individuals the values will vary. Thus the range is fuzzy but the domain is non-fuzzy. The theory can be extended by making both domain and range to be fuzzy. Another possible extension is by considering fuzzy vector valued function and a fuzzy scalar valued measure and vice versa.