CHAPTER V

LEBESGUE DECOMPOSITION WITH RESPECT TO A FUZZY MEASURE

5.0 INTRODUCTION

This chapter begins with some preliminaries taken from [5] [11] and [20]. Using this we discuss μ-continuous and μ-singular fuzzy measures and some of its properties.

5.1 PRELIMINARIES

We state the following preliminaries from [5], [11] and [20].

Definition 5.1.1

A fuzzy number \( \tilde{M} \) is a convex normalized fuzzy set \( \tilde{M} \) of the real line \( \mathbb{R} \) such that

1) There exists exactly one \( x_0 \in \mathbb{R} \) with \( \tilde{M}(x_0) = 1 \) (\( x_0 \) is called the mean value of \( \tilde{M} \))

2) \( S_{\tilde{M}}(x) \) is piece wise continuous

\[ \tilde{M}(x_0) = 1 \]

\[ S_{\tilde{M}}(x) \text{ is piece wise continuous} \]
Definition 5.1.2

A fuzzy number $\tilde{M}$ is called positive (negative) if its membership function is such that $\mu_{\tilde{M}}(x) = 0 \forall x < 0 \ (\forall x > 0)$. We denote $R^+(I)$ the set of all non-negative convex fuzzy numbers. We always assume that $L,R : [0,1] \times [0,1] \rightarrow [0,1]$ are two functions such that they are non-decreasing in both arguments, symmetric and $L(0,0)=0, R(1,1)=1$.

Definition 5.1.3

Let $X$ be a non-empty set and $R^+(I)$ be the set of all non-negative convex fuzzy numbers. Let the mapping be $\| ~\| : X \rightarrow R^+(I)$. The quadruple $(X, \| ~\|, L,R)$ is called a fuzzy normed linear space and $\| ~\|$ is a fuzzy norm, if

1. $\| x \| = \tilde{0}$, fuzzy number zero, if and only if $x = 0$

2. $\| r x \| = |r| \| x \|, x \in X, r \in R$

3. For any $x, y \in X$

   a) Whenever $s \leq \| x \|$, $t \leq \| y \|$ and $s + t \leq \| x + y \|$

   $\| x + y \| (s + t) \leq L(\| x \|(s), \| y \|(t))$

   b) Whenever $s \geq \| x \|$, $t \geq \| y \|$ and $s + t \geq \| x + y \|$

   $\| x + y \| (s + t) \leq R(\| x \|(s), \| y \|(t))$
If we take \( L(x,y)=\min(x,y) \) and \( R(x,y)=\max(x,y) \) for \( x, y \in [0,1] \) The triangular inequality is equivalent to \( \| x + y \| \leq \| x \| + \| y \| \)

**Definition 5.1.4**

Let \((x_n)\) be a sequence in a fuzzy normed linear space \((X, \| \cdot \|, L, R)\) and \(x \in X\). Then the convergence in \(X\) is defined by \( \lim_{n \to \infty} x_n = x \) if and only if \( \lim_{n \to \infty} \| x_n - x \| = 0 \), fuzzy number zero, with respect to the fuzzy measure \( \mu \).

**Definition 5.1.5**

A sequence \((x_n)\) in a fuzzy normed linear space \((X, \| \cdot \|, L, R)\) is called a cauchy sequence if \( \lim_{m,n \to \infty} \| x_m - x_n \| = 0 \) with respect to the fuzzy measure \( \mu \).

**Definition 5.1.6**

A fuzzy normed linear space is said to be complete if every cauchy sequence in \(X\) converges.

**Definition 5.1.7**

A complete fuzzy normed linear space is called a fuzzy Banach space.

**Definition 5.1.8**

Where \( \mu: C \to R \) is a fuzzy measure space. Let \( m \) be a finite fuzzy measure on \( C \). If \( m(E) = 0 \) implies \( \mu(E) = 0, E \in C \) then we say that \( \mu \) is absolutely continuous with respect to \( m \) or \( m \)-continuous.
5.2 LEBESGUE DECOMPOSITION WITH RESPECT TO A FUZZY MEASURE

In this section we show that the classical $\mu$-continuous and $\mu$-singular measures can be extended to fuzzy situation. Here we prove the classical Lebesgue decomposition theorem in the fuzzy context. Then taking two fuzzy measures $\lambda$ and $\mu$, we are establishing a few results related $\mu$-continuity and $\mu$-singularity of $\lambda$ in the fuzzy context.

**Definition 5.2.1**

Let $\lambda$ and $\mu$ be two fuzzy measures from $C$ to $R^*$. The fuzzy measure $\lambda$ is called $\mu$- absolutely continuous or simply $\mu$-continuous ($\lambda \prec \mu$) iff we have $E \in C$ and $\nu(\mu, E) = 0$ implies $\lambda(E) = 0$

**Proposition 5.2.2**

Let $\lambda$ and $\mu$ be two finite fuzzy measures. The next conditions are equivalent

1. $\lambda \prec \mu$

2. $E \in C$ and $\nu(\mu, E) = 0$ $\Rightarrow$ $\nu(\lambda, E) = 0$

3. $\nu(\mu, E) \prec \nu(\lambda, E)$

4. For any real number $\varepsilon > 0$ there exists a real number $\delta(\varepsilon) > 0$ such that $E \in C$ and $\nu(\mu, E) < \delta(\varepsilon)$ $\Rightarrow$ $\nu(\lambda, E) < \varepsilon$
Proof

(4) $\Leftrightarrow$ (2). Let us assume that (4) holds. Let $A$ be a fuzzy set in $C$ with $v(\mu, A) = 0$. Then $v(\mu, A) < \delta \left(\frac{1}{n}\right)$ and consequently $v(\lambda, A) < \frac{1}{n}$ for any $n \in \mathbb{N}$. Hence $v(\lambda, A) = 0$ so (2) holds.

Now let us assume that (2) holds and there is an $\varepsilon_0 > 0$ such that for any $\delta > 0$ a fuzzy set $E(\delta) \in C$ can be found with $v(\mu, E(\delta)) < \delta$ and $v(\lambda, E(\delta)) \geq \varepsilon_0$. Let us denote $v(\mu, E_n) \leq \frac{1}{2^n}$ and $E = \bigcup_{n \in \mathbb{N}} \lim_{k \to \infty} E_k$, where $E_k$'s are disjoint fuzzy sets in $C$.

This limit exists and it is contained in $C$. Thus we have

\[ 0 \leq v(\mu, E) = v(\mu, \lim_{n \in \mathbb{N}} \bigoplus_{k=n}^\infty E_k) \]

\[ = \lim_{n \in \mathbb{N}} v(\mu, \bigoplus_{k=n}^\infty E_k) \]

\[ \leq \lim_{n \in \mathbb{N}} \sum_{k=n}^\infty v(\mu, E_k) \]

\[ \leq \lim_{n \in \mathbb{N}} \sum_{k=n}^\infty \frac{1}{2^k} \to 0 \]

ie. \[ v(\mu, E) = 0. \]
On the other hand \( v(\lambda, E) = v(\lambda, \lim_{n \in \mathbb{N}} \bigoplus_{k=n}^{\infty} E_k) \)

\[ = \lim_{n \in \mathbb{N}} v(\lambda, \bigoplus_{k=n}^{\infty} E_k) \]

\[ \geq \lim_{n \in \mathbb{N}} \sum_{k=n}^{\infty} v(\lambda, E_k) \geq \varepsilon_0 > 0 \]

i.e. \( v(\lambda, E) > 0 \) which is a direct contradiction. Hence the proof of (4) is complete.

Similarly equivalence of other combinations of conditions holds.

**Proposition 5.2.3**

Let \( \lambda \) and \( \mu \) be two countably additive fuzzy complex or fuzzy real valued measures defined on a \( \sigma \) - algebra of fuzzy sets \( C \) and let \( \lambda \) be finite. Then \( \lambda \) is fuzzy \( \mu \) -continuous iff \( v(\mu, E) = 0 \) implies \( \lambda(E) = 0, E \in C \).

**Proof**

The necessity of the condition is obvious. To prove the sufficiency of the condition we observe that the fuzzy measure \( \lambda \) satisfies the condition if and only if the positive and negative variation of its real and imaginary parts satisfy the same condition. Thus we may assume that \( \lambda \) is non-negative. If \( \lambda \) is not \( \mu \)-continuous there is an \( \varepsilon > 0 \) and \( E_k \in C, k = 1,2,... \) with \( \lambda(E_k) \geq \varepsilon \) and

\[ v(\mu, E_k) < \frac{1}{2^i} \]
Let \( E = \lim_{n \in \mathbb{N}} \bigoplus_{k=1}^{n} E_k \). Then for each \( k = 1, 2 ... \)

\[ v(\mu, E) = \lim_{n \in \mathbb{N}} v(\mu, \bigoplus_{k=n}^{\infty} E_k) \]

\[ \leq \lim_{n \in \mathbb{N}} \sum_{k=n}^{\infty} v(\mu, E_k) \]

\[ \leq \lim_{n \in \mathbb{N}} \sum_{k=n}^{\infty} \frac{1}{2^k} \rightarrow 0 \]

ie. \( v(\mu, E) = 0 \)

On the other hand \( \lambda(E) = \lim_{n \in \mathbb{N}} \lambda\left(\bigoplus_{k=n}^{\infty} E_k\right) \geq \lim_{n \in \mathbb{N}} \sum_{k=n}^{\infty} \lambda(E_k) \geq \varepsilon_0 > 0 \)

ie. \( \lambda(E) > 0 \) and this is a contradiction.

Similar to classical measure theory we introduce the following definition.

**Definition 5.2.4**

Let \( \lambda \) and \( \mu \) be two finite fuzzy measures. The fuzzy measure \( \lambda \) is said to be \( \mu \)-singular, denoted by \( \lambda \perp \mu \) iff there is fuzzy set \( E \) in \( \mathcal{C} \) such that \( v(\mu, E) = 0 = v(\lambda, E') \).

**Remark 5.2.5**

Let \( \lambda \) and \( \mu \) be finite fuzzy measures. Then

1. \( \lambda \perp \mu \iff \mu \perp \lambda \)

2. \( \lambda \perp \mu \) and \( \lambda' \perp \mu \) implies \( \lambda + \lambda' \perp \mu \) and \( \lambda \perp \lambda' + \mu \).
Proposition 5.2.6

If \( \lambda \) and \( \mu \) are finite fuzzy measure, then the next conditions are equivalent.

1. \( \lambda \perp \mu \)

2. There exists a set \( E \in C \) such that \( v(\mu, E) = 0 = v(\lambda, E') \)

3. \( \lambda^+ \perp \mu \) and \( \lambda^- \perp \mu \)

4. \( v(\lambda, E) \perp v(\mu, E) \) where \( E \in C \).

The above results are clearly true from definition.

Proposition 5.2.7

If \( \lambda \) and \( \mu \) are finite fuzzy measures such that \( \lambda \ll \mu \) and \( \lambda \perp \mu \), then

\( \lambda(E) = 0 \) for any \( E \) in \( C \).

Proof

Since \( \lambda \) is \( \mu \)-continuous, for any positive integer \( n \), there exists \( \delta (\frac{1}{n}) > 0 \) such that for \( B \in C \) and \( v(\mu, B) < \delta (\frac{1}{n}) \) gives \( v(\lambda, B) < \frac{1}{n} \) since \( \lambda \) is \( \mu \)-singular.

A fuzzy set \( A \in C \) such that \( v(\mu, A) = 0 = v(\lambda, A') \). If \( v(\mu, A) = 0 < \delta (\frac{1}{n}) \) for any \( n \) there is a fuzzy set \( A \in C \) such that \( v(\lambda, A) \to 0 \) so \( v(\lambda, A) = 0 \) for

\( C(A) = \{ E: E \in C \text{ and } E \subseteq A \} \).

Also \( E \in C \) then \( A \in C(A) \) and \( A, E' \in C(A) \) such that \( A.E \oplus A.E' = A \).
If $E \in C$, $\nu (\lambda, E) = \nu (\lambda, A.E) + \nu (\lambda, A. E')$

$$= \nu (\lambda, A.E \oplus A.E')$$

$$= \nu (\lambda, A)$$

$$= 0$$

So $\lambda^+ (E) = 0$ and $\lambda^- (E) = 0$

Hence $\lambda^+ (E) = 0$ for any $E$ in $C$

**Proposition 5.2.8**

If $\lambda$ and $\mu$ are two finite countably additive fuzzy measures defined on $C$, then $\lambda$ is uniquely representable as a sum $\lambda = \lambda_c + \lambda_s$ where $\lambda_c$ is $\mu$-continuous and $\lambda_s$ is $\mu$-singular.

**Proof**

First we prove the uniqueness. There exists at most one pair of fuzzy measures $(\lambda_c, \lambda_s)$ which satisfy the property $\lambda = \lambda_c + \lambda_s$. Let $(\alpha_c, \alpha_s)$ be another pair of finite fuzzy measures which satisfies the property $\lambda = \alpha_c + \alpha_s$. Then $\lambda_c + \lambda_s = \alpha_c + \alpha_s$ i.e. $\lambda_c - \alpha_c = \lambda_s - \alpha_s$. Here $\lambda_c - \alpha_c$ is $\mu$-continuous and $\lambda_s - \alpha_s$ is $\mu$-singular. Hence by proposition 5.2.7 $\lambda_c - \alpha_c = 0$ and $\lambda_s - \alpha_s = 0$. So $\lambda_c = \alpha_c$ and $\lambda_s = \alpha_s$. Hence $\lambda = \lambda_c + \lambda_s$ is a unique representation.

Next we have to prove the existence of the pair $(\lambda_c, \lambda_s)$ for non-negative measures only. Indeed if $\mu$ is any finite fuzzy measure, then it suffices to prove
the proposition for \( \nu (\mu) \) instead of \( \mu \) because of propositions 5.2.2 and 5.2.6. If the proposition is proved for any non-negative finite fuzzy measures \( \mu \) and \( \lambda \) and we consider \( \lambda_1 \) to any finite fuzzy measure, then using the proposition for the non-negative finite fuzzy measure \( \lambda_1^+ \) and \( \lambda_1^- \) we obtain the fuzzy measures.

\[
\lambda_c = \lambda_1^+ - \lambda_1^- \quad \text{and} \quad \lambda_s = \lambda_1^+ - \lambda_1^- \quad \text{which accomplish for } \lambda_1 \text{ instead of } \lambda.
\]

Hence \( \lambda = \lambda_c + \lambda_s \)

**Definition 5.2.9**

Let \((\mu_n)\) be a sequence of countably additive fuzzy measures on \( \sigma \)-algebra of fuzzy sets in \( C \) and let \((E_m)\) be a disjoint sequence of fuzzy sets in \( C \) such that \( \lim \mu_n (E_m) = 0 \) uniformly for \( n=1,2,... \) then countable additivity of \( \mu_n \) is said to be uniform.

**Proposition 5.2.10**

Let \((\mu_n)\) be a sequence of countably additive fuzzy measures on \( \sigma \)-algebra of fuzzy sets in \( C \) if \( \mu (E) = \lim \mu_n (E) \) exist for each \( E \in C \), then \( \mu \) is countably additive fuzzy measure on \( C \).
Proof

We have \( \mu(\varphi) = \lim_{n} \mu_{n}(\varphi) = 0 \)

Let \((E_{n})\) be a sequence of disjoint fuzzy sets in \( C \)

\[
\mu\left( \bigoplus_{n=1}^{\infty} E_{n} \right) = \lim_{n} \mu_{n}\left( \bigoplus_{n=1}^{\infty} E_{n} \right)
\]

\[
= \lim_{n} \sum_{n=1}^{\infty} \mu_{n}(E_{n}) \quad \text{since } \mu_{n} \text{ is a countably additive fuzzy measure.}
\]

\[
= \sum_{n=1}^{\infty} \lim_{n} \mu_{n}(E_{n})
\]

Therefore \( \mu(E) = \lim_{n} \mu_{n}(E_{n}) \)

Hence \( \mu \) is a countably additive fuzzy measure on \( C \).

Proposition 5.2.11

Let \( \mu \) be a non-negative countably additive fuzzy measure defined on a \( \sigma \)-algebra \( C \). For each \( A \subseteq X \) and \( \mu(A) = \inf \sum_{n=1}^{\infty} \mu(E_{n}) \) where the infimum is taken over all sequence \((E_{n})\) of fuzzy sets in \( C \) with \( \bigoplus_{n=1}^{\infty} E_{n} \subseteq A \), then \( A \subseteq \bigoplus_{n=1}^{\infty} E_{n} \).
(1) \( \overline{\mu} (\emptyset) = 0 \)

(2) \( A \subseteq B \Rightarrow \overline{\mu} (A) \leq \overline{\mu} (B), \quad \forall A, B \in \mathcal{C} \)

(3) \( \overline{\mu} \left( \bigoplus_{n=1}^{\infty} E_n \right) \leq \sum_{n=1}^{\infty} \overline{\mu} (E_n), \quad (E_n) \subseteq \mathcal{C} \)

\textbf{Proof}

Evidently \( \overline{\mu} (\emptyset) = 0 \) and \( A \subseteq B \Rightarrow \overline{\mu} (A) \leq \overline{\mu} (B) \) by the definition of \( \overline{\mu} \).

Let \( E = \bigoplus_{n=1}^{\infty} E_n \), where \( (E_n) \) is the sequence of fuzzy sets in \( X \). Let \( \varepsilon > 0 \) and for each \( n = 1, 2, \ldots \) and the \((E_{m,n})\) has the properties.

\[
E_{m,n} \in \mathcal{C}, E_n \subseteq \bigoplus_{n=1}^{\infty} E_{m,n} \quad \text{and} \quad \sum_{n=1}^{\infty} \overline{\mu} (E_{m,n}) \leq \overline{\mu} (E_n) + \frac{\varepsilon}{2^n}
\]

Now \( \overline{\mu} (E) = \overline{\mu} \left( \bigoplus_{n=1}^{\infty} E_n \right) \)

\[
\leq \overline{\mu} \left( \bigoplus_{m,n=1}^{\infty} E_{m,n} \right) \\
\leq \sum_{m,n=1}^{\infty} \overline{\mu} (E_{m,n}) \\
\leq \sum_{n=1}^{\infty} \left( \overline{\mu} (E_n) + \frac{\varepsilon}{2^n} \right) \\
\leq \sum_{n=1}^{\infty} \overline{\mu} (E_n) + \varepsilon
\]

Since \( \varepsilon \) is arbitrary \( \overline{\mu} \left( \bigoplus_{n=1}^{\infty} E_n \right) \leq \sum_{n=1}^{\infty} \overline{\mu} (E_n) \)