CHAPTER: 3

PROPERTIES OF DERIVATION

ALTERNATOR RINGS
In [13] and [14], Kleinfeld defined two different generalizations of alternative rings and for each of these generalizations he showed that simple rings with idempotent \( e \neq 1 \) are alternative. Both of these generalizations defined by Kleinfeld are contained in variety of derivation alternator rings. In this chapter we extend his results to prime derivation alternator rings with idempotent \( e \neq 1 \) and characteristic \( \neq 2 \). In particular, we prove these results without assuming characteristic \( \neq 3 \).

In section 3.1, we show that in a semiprime derivation alternator ring with characteristic \( \neq 2 \) every idempotent must be in the flexible nucleus. In section 3.2, we show that a prime derivation alternator ring with idempotent \( e \neq 1 \) satisfying the identity \(((y,z),w,x) = (w,x,(y,z))\) is alternative. In section 3.3, we first establish certain properties of the alternator ideal of a prime flexible derivation alternator ring. Using these properties, it is shown that a prime flexible derivation alternator ring without non zero nil ideals of index 2 is alternative.

3.1 Derivation alternator rings with idempotent:

By Theorem 2.3.1, we know that derivation alternator rings are power-asssociative. So, these rings permit the standard Albert decomposition [1] relative to \( e \). In this section, we show that in a semiprime derivation alternator ring with characteristic \( \neq 2 \) every idempotent must be in the flexible nucleus.

We know that a non associative rings with characteristic \( \neq 2 \) is called derivation alternator ring, if it satisfies the following identities:

\[
(x,x,x) = 0, \\
(yz,x,x) = y(z,x,x) + (y,x,x)z \\
\text{and } (x,x,yz) = y(x,x,z) + (x,x,y)z.
\]
Throughout this section we assume $R$ to be a derivation alternator ring with idempotent $e \neq 1$ and characteristic $\neq 2$. $R$ permits the standard Albert decomposition \[1\] relative to $e$. Thus we have $R = R_1 + R_{1/2} + R_0$ (direct sum over the ring of integers) where $R_i = \{x \in R / eox = 2ix\}$. Moreover, for $i=0$ or $1$ and $x_i, y_i \in R_i$ we have $ex_i = ix_i = xe, x_i o y_i \in R_i$ and $x_i o y_{1/2} \in R_{1/2} + R_{-1}$. Also, $x_i y_0 = 0 = y_0 x_i$ and $x_{1/2} o y_{1/2} \in R_1 + R_0$.

We know that flexible nucleus of non associative ring $R$ can be defined as $N_f(R) = \{r \in R / 0 = (r, x, r) = (r, r, x) = (r, x, y) + (y, x, r) \text{ for all } x, y \in R\}$.

**Lemma 3.1.1:** $(e, R, e) = 0$ and $(R, e, e) = (e, e, R) \subseteq R_{1/2}$

**Proof:** Since always $(e, R_1, e) = 0 = (e, R_0, e)$, we need to show that also $(e, R_{1/2}, e) = 0$. Our proof is the same as in \[13\]. Let $x \in R_{1/2}$ and $ex = y_i + y_{1/2} + y_0$ where $y_i \in R_i$. Then linearized 2.1.1 implies

$$(e, x, e) = -(e, e, x) = (xe)e + xe - ex + e(ex)$$

$$= (ex - x)e + xe - ex + e(ex) = (ex) o e - ex$$

$$= 2y_i + y_{1/2} - y_i - y_{1/2} + y_0 = y_i - y_0.$$ 

But by 2.3.4, $(e, x, e) = (e^2, x, e) = 2e(e, x, e)$, so that $y_i - y_0 = 2e(y_i - y_0) = 2y_i$.

Hence $y_i = 0 = y_0$ and $(e, R_{1/2}, e) = 0$. Since we now have $(e, R, e) = 0$, linearized 2.1.1 implies $(R, e, e) = (e, e, R)$. Also, 2.1.3 and 2.1.1 imply $(e, e, e) = (e, e, eox) = e0(e, e, x) \subseteq R_{1/2}$. Hence $(e, e, R) \subseteq R_{1/2}$. □

**Theorem 3.1.1:** Let $R$ be a derivation alternator ring with idempotent $e$ and characteristic $\neq 2$. Then $R = R_1 + R_{1/2} + R_0$ Where $R_i = \{x \in R / eox = 2ix\}$. Further more, $R_1 R_0 = 0 = R_0 R_1, R_i R_j \subseteq R$ and $R_{1/2} R_i \subseteq R$ for $i=0$ or 1.

**Proof:** Throughout we set $x_i, y_i \in R_i$ for $i = 0, 1$. From 2.1.2 we have $(x_i, y_i, e, e) = x_i(y_i, e, e) + (x_i, e, e)y_i = 0$. Hence 2.1.4 gives

$$(x_i, y_i, e)e = (x_i, y_i, e) + (x_i, y_i, e, e) + (x_i, y_i, e^2) - x_i(y_i, e, e) = (1 - i)(x_i, y_i, e).$$

Then using linearized 2.1.2 we have

$$(1 - i)(x_i, y_i, e) = (x_i, y_i, e)e = [(x_i, y_i, e) + (x_i, e, y_i)]e = [(x_i, y_i, e) + (x_i, e, e, y_i)] - x_i[(e, y_i, e) + (e, e, y_i)] = i[(x_i, y_i, e) + (x_i, e, y_i)] = i(x_i, y_i, e),$$

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so that \((x_i, y_i, e) = 0\). Thus \((x_i, y_i, e) = i(x_i, y_i)\). Going to the opposite ring now gives that also  
\[ e(x_i y_i, e) = i(x_i y_i) \text{, and therefore } R_i^2 \subseteq R_i. \]

Next, to prove \( R_i R_{1/2} \subseteq R_{1/2} \). We let \( x \in R_{1/2} \). Then from 2.1.4 and 2.1.3 we obtain 
\[
(x_i e, e, x) - (x_i e^2, x) + (x_i e, e x) = x_i (e, e, x) + (x_i e, e) x = x_i (e, e, x) = (e, e, x_i) x
\]
\[ = (e, e, x_i) x. \]

Thus \((i - 1)(x_i, e, x) = (e, e, x_i) x \in R_{1/2}\) by Lemma 3.1.1. Going to the opposite ring, this in turn gives  
\[(i - 1)(x_i, e, x) = (e, e, x_i) x \in R_{1/2}\]. Since \((z, e, z) = (e, z, e, z) \in R_{1/2}\) by 2.3.1, it then follows that also  
\((i - 1)(x_i, e, x) + (x_i, e, x e) = R_{1/2}\). Thus we have shown 
\[(2i - 1)(x_i, e, x) = \{ (i - 1)(x_i, e, x) + (x_i, e, x e) \} + \{ (i - 1)(x_i, e, x) + (x_i, e, x e) \} \in R_{1/2}\), so that 
(i) \((x_i, e, x) \in R_{1/2}\).

From linearized 2.1.3 we next obtain 
\[
(x_i, e, x) + (e, x_i, x) = (x_i, e, e, x) + (e, x_i, e, x)
\]
\[ = e o [(x_i, e, x) + (e, x_i, x)] + x o [(x_i, e, e) + (e, x_i, e)]
\]
\[ = e o [(x_i, e, x) + (e, x_i, x)]. \]

Thus \((x_i, e, x) + (e, x_i, x) \in R_{1/2}\), so that applying (i) we have 
(ii) \((e, x_i, x) \in R_{1/2}\). Since \((e, z, z) = e o (e, z, z) \in R_{1/2}\) by 2.1.2, this in turn gives 
\((e, x, x_i) \in R_{1/2}\).

Then going to the opposite ring we have 
(iii) \((x_i, x, e) \in R_{1/2}\).

Finally, \(x_i x = - (x_i, e, x) + (e, x_i, x) - (x_i, x, e) \in R_{1/2}\) by (i), (ii) and (iii).

Setting \(x = y_i + y_{1/2} + y_0\), this implies \(y_1 + y_0 \in R_{1/2}\) or that \(y_1 = y_0 = 0\). Thus we have established \(R_i R_{1/2} \subseteq R_{1/2}\). Since going to the opposite ring now gives \(R_{1/2} R_i \subseteq R_{1/2}\) as well, this completes the proof of the theorem. \(\Box\)

**Lemma 3.1.2:** \( Z = \{ z \in R_{1/2} / (z, R) = 0 = z R_{1/2} \} \) is a trivial ideal of \( R \).

**Proof:** Since \((Z, R) = 0\) and \( R_{1/2} Z = 0 \subseteq Z\), it suffices to show that if \( z \in Z \) and \( x_i \in R_i \), where \( i = 0 \) or \( 1 \), then \( x_i z \in Z\). We note that \( x_i z \in R_{1/2} \) by Theorem 3.1.1. Now \( 0 = (e o x_i, z) + (x_i o z, e) + (e o z, x_i) = (x_i o z, e) + (z, x_i) = 2(x_i z, e) \) by 2.3.2. Since this implies \((x_i z, e) = 0\), and since
$x, z \in R_{1/2}$, it follows that $x, z = 2e(x, z)$. Also, $(e, z) = 0$ and $z \in R_{1/2}$ imply $z = 2ez$. Now from 2.1.3 and the definition of $Z$ we have $((w, w, z), y) = ((w, w, z), y) + (z, (w, w, y)) = (w, w, (z, y)) = 0$. In particular, this implies $0 = 2((x, x, z) + (x, e, z), y) = (4i - 2)(x, z, y)$.

Thus we have shown $(RZ, R) = 0$.

Next let $y \in R_{1/2}$. Since $Z R_{1/2} = 0$ and by Theorem 3.1.1, $R_{1/2} R_i \subseteq R_{1/2}$ for $i = 0$ or 1, we have $(z, y, x_i) = 0$. Hence from linearized 2.1.2 we obtain

$$(z, x_i, y) = (ez, x_i, y) + (ez, y, x_i) = 0((z, x_i, y) + (z, y, x_i)) + zo((e, x_i, y) + (e, y, x_i))$$

$$= 0(z, x_i, y).$$

This means $(x, z, y) = (z, x_i, y) = 0(z, x_i, y) = 0((x, z, y)) \in R_{1/2}$. But then $(RZ, R) = 0$ and $xoy \in R_1 + R_0$ for $x, y \in R_{1/2}$ imply $2(z, y) = (z, y) \in R_{1/2} \cap (R_1 + R_0) = 0$. Thus we have shown $(RZ)R_{1/2} = 0$, which completes the proof that $Z$ is an ideal of $R$. Moreover, $Z$ is trivial since $Z^2 \subseteq ZR_{1/2} = 0$. $\square$

**Lemma 3.1.3**: Let $H = \{h \in R_{1/2} : (e, h) = 0\}$. Then $HR_i \subseteq H$ and $R_i H \subseteq H$ for $i = 0, 1$.

**Proof**: Let $h \in H$ and $x_i \in R_i$ for $i = 0, 1$. From 2.1.2 and definition of $H$ we have

$$0 = ((e, h), w, w) = ((e, w, w), h) + (e, (h, w, w))$$

for all $w \in R$. In particular,

$$0 = ((e, x_i) + (e, (h, x_i) + (h, x_i, e)) = (e, (h, x_i) + (h, x_i, e)).$$

Since $(h, e, x_i) + (h, x_i, e) \in R_{1/2}$ by Theorem 3.1.1, it thus follows $(h, e, x_i) + (h, x_i, e) \in H$. Now $h = 2he$, so expansion of $2[(h, e, x_i) + (h, x_i, e)]$ gives $(1 - 4i)hx_i + 2(hx_i)e \in H$. Hence we obtain

(iv) $(4i - 1)hx_i = 2(hx_i)e \mod H$.

Then, since $0 = (e, h) = 2(e, he)$ implies $He \subseteq H$, multiplying (iv) through on the right by $2e$ gives $(8i - 2)h \equiv 4(hx_i)e \mod H$. But using 2.1.2 and $4(h, e, e) = 2he - 4he = -h$, we have $4[(hx_i)e] = 4hx_i, e) + 4(hx_i)e = [4h(x_i, e) + (h, e, e)x_i] + 4(hx_i)e = -hx_i + 4(hx_i)e$. Thus we arrive at $(8i - 2)(hx_i)e \equiv -hx_i + 4(hx_i)e \mod H$, or

(v) $hx_i \equiv (3 - 4i)2(hx_i)e \mod H$.

Finally, combining (iv) and (v) leads to $mod H$ to $hx_i \equiv (3 - 4i)2(hx_i)e \equiv (3 - 4i)(4i - 1)hx_i \equiv -3hx_i$ for $i = 0$ or 1. But then $4hx_i \equiv 0 \mod H$, so that $HR_i \subseteq H$. Going to opposite ring now gives $R_i H \subseteq H$ as well. $\square$
**Lemma 3.1.4:** \((e,x,y_{1/2}) + (y_{1/2},x,e) \in Z\) for \(y_{1/2} \in R_{1/2}\) and \(x \in R_.

**Proof:** First linearization of 2.3.4 and 2.3.5 together with Lemma 3.1.1 gives \((e_{2},x,y_{1/2}) + (e_{0}y_{1/2},x,e) = 2e[(e,x,y_{1/2}) + (y_{1/2},x,e)]\) and \((x_{1/2})e + (e,x,e_{0}y_{1/2}) = 2[(e,x,y_{1/2}) + (y_{1/2},x,e)]e_.\) Adding these two equations and dividing by 2, we find that \((e,x,y_{1/2}) + (y_{1/2},x,e) = e_{0}[(e,x,y_{1/2}) + (y_{1/2},x,e)]e\). Also, subtracting one equation from another, we obtain \((e,(e,x,y_{1/2}) + (y_{1/2},x,e)) = 0\). In particular, this shows 

\((vi) (e,x,y_{1/2}) + (y_{1/2},x,e) \in H_.\)

Next let \(x_{1/2} \in R_{1/2}\). Then using Theorem 3.1.1 and fact \(x_{1/2} \in R_{1} + R_{0},\) in the Albert decomposition, we have 

\[(e,x_{1/2},y_{1/2}) + (y_{1/2},x_{1/2},e) =\]

\[= (ex_{1/2})y_{1/2} - e(x_{1/2},y_{1/2}) + (y_{1/2},x_{1/2},e) - y_{1/2}(x_{1/2},e)\]

\[= (ex_{1/2})oy_{1/2} - e(x_{1/2},oy_{1/2})\]

\[+ (y_{1/2},x_{1/2})oe - y_{1/2},x_{1/2} \in R_{1} + R_{0}\]

But this together with \((vi)\) implies

\[(vii) (e,x_{1/2},y_{1/2}) + (y_{1/2},x_{1/2},e) \in (R_{1} + R_{0}) \cap R_{1/2} = 0.\]

Let \(x_{i} \in R_{i}\) for \(i = 0, 1_.\) Then from linearized 2.3.1, Theorem 3.1.1 and \((vii)\) we have 

\[[(e,x_{i},y_{1/2}) + (y_{1/2},x_{i},e)]x_{i} = [(e,x_{i},1/2,y_{1/2}) + (y_{1/2},x_{i},x_{1/2},e)] - x_{i}[(e,x_{i},1/2,y_{1/2}) + (y_{1/2},x_{i},e)] = 0.\] Thus in conjunction with \((vii)\) we have shown 

\[[(e,x_{i},y_{1/2}) + (y_{1/2},x_{i},e)]R_{1/2} = 0.\]

If we now also go to opposite ring, it follows that

\[(viii) R_{1/2} [(e,x_{i},y_{1/2}) + (y_{1/2},x_{i},e)] = 0 = [(e,x_{i},y_{1/2}) + (y_{1/2},x_{i},e)] R_{1/2} .\]

Lastly, let \(h = (e,x_{i},y_{1/2}) + (y_{1/2},x_{i},e)\) and again \(x_{i} \in R_{i}\) for \(i = 0, 1_.\) Then \(0 = (eox_{i},h) + (x_{i},oh,e) + (hoe,x_{i}) = (2i-1)(x_{i},h) + (x_{i},oh,e)\) by 2.3.2. But \((x_{i},oh,e) = 0 by (vi) and Lemma 3.1.3, so that \((2i-1)(x_{i},h) = 0.\) Hence we obtain

\[(ix) (R_{i} (e,x_{i},y_{1/2}) + (y_{1/2},x_{i},e)) = 0 \text{ for } i = 0, 1.\]

The lemma now follows from \((vi),(viii)\) and \((ix)\). \(\square\)

**Theorem 3.1.2:** If \(R\) is a semiprime derivation alternator ring with idempotent \(e\) and characteristic \(\neq 2,\) then \(e \in N_{f}(R).\)
**Proof:** It will suffice to show \((e,x,y) + (y,x,e) = 0\) for all \(x,y \in R\), since then also \((x,e,x) = -\{(e,x,x) + (x,x,e)\} = 0\) by linearized 2.1.1. Now since \(R\) is semiprime, from Lemmas 3.1.2 and 3.1.4 we have

\[
(x) (e,x,y_{1/2}) + (y_{1/2},x,e) = 0 \quad \text{for} \quad x \in R \quad \text{and} \quad y_{1/2} \in R_{1/2}.
\]

Also, from Theorem 3.1.1 we have

\[
(xi) \quad (e,x_j,y_i) = 0 = (y_i,x_j,e) \quad \text{for} \quad x_j \in R_j \quad \text{and} \quad y_i \in R_i \quad \text{where} \quad i,j = 0,1.
\]

We next consider \(x \in R_{1/2}\) and \(y \in R_i \) for \(i = 0,1\). From linearization of 2.3.4 and 2.3.5 together with Lemma 3.1.1 we obtain

\[
(e^2,x,y_i) + (eoy,x,e) = 2e[(e,x,y_i) + (y_i,x,e)] \quad \text{and} \quad (y_i,x,e^2) + (e,x,eoy_i) = 2[(e,x,y_i) + (y_i,x,e)].
\]

Since \(w = (e,x,y_i) + (y_i,x,e) \in R_{1/2}\) by Theorem 3.1.1, adding these last two equations gives \((2i + 1)w = 2eow = 2w\). Hence \((2i - 1)w = 0\), so that

\[
(xii) (e,x_{1/2},y_i) + (y_i,x_{1/2},e) = 0 \quad \text{for} \quad x_{1/2} \in R_{1/2} \quad \text{and} \quad y_i \in R_i \quad \text{where} \quad i = 0,1.
\]

Since \((x),(xi)\) and \(xii)\) show that in fact \((e,x,y) + (y,x,e) = 0\) for all \(x,y \in R\). This completes the proof of the theorem.

### 3.2 Prime derivation alterntor rings with idempotent:

In [6], Hentzel and Smith proved that if \(R\) is a prime derivation alternator rings with idempotent \(e^1\) and without non zero nil ideals of index 2, then \(R\) is alternative. In this section, we show that a prime derivation alternator ring with idempotent \(e^1\) satisfying the identity \(((y,z),w,x) = (w,x,(y,z) is alternative.

Throughout this section, we assume \(R\) to be a derivation alternator ring with idempotent \(e^1\) and characteristic \(\neq 2\).

**Lemma 3.2.1:** If \(R\) is a semi prime derivation alternator ring with idempotent, then

\[I_i = \{x \in R / xR_{1/6} = 0 \iff R_{1/6} \}$

is an ideal of \(R\) for \(i = 0,1\).

**Proof:** Let \(e^1\) be an idempotent. Since \(R\) is semi prime, by theorem 3.1.2 any idempotent in \(R\) belongs to the flexible nucleus \(N_F(R)\) of \(R\). Also, from Lemma 6 in [6] \((ex)^2 = 0 = (xe)^2\) for \(x \in R_{1/6}\). Thus \((e+ex)^2 = e + ex\) using \(ex \in R_{1/6}\), which implies the
Idempotent e+ex is in NF(R), so that ex = (e + ex) - e e NF(R). Similarly (e+ex)^2 = e + ex implies xee NF(R), so that then x = ex + xe e NF(R). Hence we have shown

R_1/2 \subseteq NF(R).

We next make a complete linearization of 2.3.4 to obtain

\[(xoz,y,w)+(zow,y,x)+(wox,y,z)=2\{w[(x,y,z)+(z,y,x)]+x[(z,y,w)+(w,y,z)]+z[(w,y,x)+(x,y,w)]\}\]

............................................... 3.2.1

Then we set x = e in 3.2.1, while letting z \in R_1/2 and y_i, w_i \in R_i where i = 0,1.

Thus (eoz,y_i,w_0) + (zow_i,y_i,e) + (w_i,oe, y_i, z) = 2\{ w_i [(e, y_i, z)+(z, y_i, e)] + e[(z, y_i, w_i) + (w_i, y_i, z)] + z[(w_i, y_i, e)+(e, y_i, w_i)]\}. But since e,z \in NF(R), the right-hand side of the 3.2.1 is zero. Hence using eoz = z, z \in NF(R), and w_i, oe = 2iw_i, we arrive at

\[(2i-1) (w_i y_i, z)+ (zow_i, y_i, e) = 0 \text{ for } i=0,1 \]

............................................... 3.2.2

Now if w_i \in I_i, since z \in R_1/2 it is clear (zow_i,y_i,e) = 0. Also, since zow_i \in R_1/2 and ey_i = iy_i for i = 0 or 1, if y_i \in I_i it follows that (zow_i,y_i,e) = 0. Thus 3.2.2 leads to

\[(I_i R_i R_{1/2}) = 0 = (R_i I_i R_{1/2}) \text{ for } i=0,1. \]

............................................... 3.2.3

Using R_1/2 \subseteq NF(R), this means too that

\[(R_{1/2}, R_i I_i) = 0 = (R_{1/2}, I_i R_i) \text{ for } i=0,1. \]

............................................... 3.2.4

At this point, R_1 R_0 = R_0 R_1, the definition of I_i, the fact that R_i^2 \subseteq R_i R_i R_{1/2} \subseteq R_{1/2}, R_{1/2} R_i \subseteq R_{1/2} for i = 0 or 1 and 3.2.3 - 3.2.4 are all utilized in a straightforward fashion to conclude that both I_i and I_0 are ideals of R.

**Theorem 3.2.1:** A prime derivation alternator ring R with idempotent e ≠ 1 is flexible.

**Proof:** From [13] we know that if for each idempotent e ≠ 1 the ideal

B = \{b \in R_{1/2} / bR_{1/2} \subseteq R_{1/2} \text{ and } R_{1/2} b \subseteq R_{1/2} \} is zero, then R is alternative. Of course in this case R is flexible. Thus we may assume B ≠ 0 for some idempotent. Since the characteristic ≠ 2, to prove R is flexible it suffices to show that relative to this idempotent (x_i y_j z_k) + (z_k y_j x_i) = 0 for every x_i \in R_i, y_j \in R_j, z_k \in R_k and i,j,k = 0,1/2,1.

Now as in the proof of Lemma 3.2.1 above, we know R_{1/2} to be contained in flexible nucleus of R. Hence (x_i y_j z_k) + (z_k y_j x_i) = 0 if any of x_i, y_j, or z_k is in R_{1/2}. Thus we are reduced to the cases where i,j,k are each either 0 or 1. Also, since R_1 R_0 = 0 = R_0 R_1 and

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R_i^2 \subseteq R_i for i=0 or 1, we may further assume i = j = k. At this point, let us utilize 2.3.6 and the fact that R_i R_{i/2} \subseteq R_{i/2} to see that \( (x_i,y_i,x_i) R_{i/2} \subseteq (x_i,y_i R_{i/2},x_i) + y_i (x_i R_{i/2},x_i) \subseteq (x_i R_{i/2},x_i) = 0 \). Similarly \( R_{i/2}(x_i,y_i,x_i) = 0 \), which shows \((x_i,y_i,x_i) \in I_i\). Now the ideal \( B \) is by definition contained in \( R_{i/2} \), so \( I_i B = 0 = B I_i \). But since \( I_1 \) and \( I_0 \) are ideals by Lemma 3.2.1, \( B \neq 0 \), and \( R \) is prime by assumption, this means \( I_i = 0 \) for \( i = 0,1 \). Thus \((x_i,y_i,x_i) = 0\), so that also \((x_i,y_i,z_i) + (z_i,y_i,x_i) = 0\). This exhausts all possible cases, and thereby proves \( R \) is flexible. □

**Corollary 3.2.1:** A simple derivation alternator ring \( R \) with idempotent \( e \neq 1 \) is alternative.

Now we prove the following Theorem.

**Theorem 3.2.2:** Let \( R \) be a prime derivation alternator ring with idempotent \( e \neq 1 \) and characteristic \( \neq 2,3 \) satisfying the identity \((y,z),w,x) = (w,x,(y,z))\). Then \( R \) is alternative.

**Proof:** From 2.1.1 we obtain
\[
(x,x,y) + (x,y,x) + (y,x,x) = 0
\]

Since \( R \) is flexible by Theorem 3.2.1, this implies that
\[
(x,x,y) + (y,x,x) = 0
\]
By taking commutator \((x,y)\) for \( y \), we get
\[
(x,x,(x,y)) + ((x,y),x,x) = 0.\text{But} (y,z),w,x) = (w,x,(y,z)).
\]
Since \((x,x,(x,y)) = ((x,y),x,x)\). Thus \( 2(x,x,(x,y)) = 0 \).
Since \( R \) is of characteristic \( \neq 2,(x,x,(x,y)) = 0 \).
In particular, for any idempotent \( e \) we have \((e,e,(e,y)) = 0 \).

Thus from the Albert decomposition, 3.2.6 and 2.1.3, it follows that \((e,e,x) = (e,e,ex) = 2(e,e,x)e\). Iteration then gives \( 2(e,e,x)e = 4 ((e,e,ex)e) e = 4(e,e,ex)e, so that 2(e,e,x)e = 0 \). This in turn means \((e,e,x) = 2(e,e,x)e = 0 \) for any idempotent \( e \). At this point the argument given in section 3 of [19] so that \( A \) is alternative, which completes the proof of the Theorem. □
In this section, we establish certain properties of alternator ideal of a prime flexible derivation alternator ring. Using these properties, it is shown that a prime flexible derivation alternator ring without nonzero nil ideals of index 2 is alternative.

A linear mapping $D$ from a ring to itself is called a derivation provided $D(xy) = D(x)y + xD(y)$. Using the standard notation for the commutator, $(x,y) = xy - yx$, for any derivation $D$ one has $D((x,y)) = (D(x)y + (x,D(y))$. Since for a derivation alternator ring $R$, $D(y) = (x,x,y)$ is by 2.1.3 a derivation for each fixed $x \in R$, this implies

$$\text{(x,x,y)(z,w) = 0.}$$

**Lemma 3.3.1:** In a flexible derivation alternator ring $(x,x,y)(z,w) = 0$.

**Proof:** We first use twice 2.3.12 and 2.3.9, then linearized 2.1.3, followed by 2.3.12 and 2.3.9 with some cancellation of terms, to compute the identity:

$$2(x,x,y)ow = \{(wox,x,yz) - xo(w,x,yz)\} + \{(x,wox,yz) - xo(x,w,yz)\}
= \{(wox,x,yz) + (x,wox,yz)\} - \{(w,x^2,yz) + (x^2,w,yz)\}
= y[(wox,x,z) + (x,wox,z) - (w,x^2,z) - (x^2,w,z)]
+ [(wox,x,y) + (x,wox,y) - (w,x^2,y) - (x^2,w,y)]z
= 2 \{y[wo(x,x,z)] + [wo(x,x,y)]z\}.

Since characteristic $\neq 2$, we thus arrive at

$$\text{(x,x,yz)ow = y[wo(x,x,z)] + [wo(x,x,y)]z.}$$

On the other hand, by 2.1.3 we have

$$(x,x,yz)ow = [y(x,x,z)]ow + [(x,x,y)z]ow.
\text{.......................... 3.3.4}$$

If we now subtract 3.3.3 from 3.3.4, we see that

$$0 = (y,(x,x,z),w) - (w,(x,x,y),z) + ((x,x,y),z,w) - (w, y,(x,x,z)).$$

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We then use flexibility to rewrite 3.3.5 as

\[0 = (y, (x, x, z), w) + (z, (x, x, y), w) - (w, z, (x, x, y)) - (w, y, (x, x, z)) + (y, w, (x, x, z)) + (z, w, (x, x, y)) + (x, x, y)(z, w) + (w, y)(x, x, z)\]

Next we interchange \(y\) and \(z\) in 3.3.6, and then subtract the resulting equation from 3.3.6. This gives

\[0 = (x, x, y)o(z, w) + (x, x, z)o(w, y),\]

which we record as

\[(x, x, y)o(z, w) = (x, x, z)o(y, w).\]

Finally, making repeated use of 3.3.7, we conclude

\[(x, x, y)o(z, w) = (x, x, z)o(y, w) = -(x, x, z)o(w, y) = - (x, x, w)o(z, y) = (x, x, w)o(y, z) = (x, x, y)o(w, z).\]

Thus \(2(x, x, y)o(z, w) = 0\), so that \((x, x, y)o(z, w) = 0\). This completes the proof of the lemma. □

For non-empty subsets \(S\) and \(T\) of a ring \(R\), suppose we now let \(S \triangle T\) denote the linear span of all elements of the form \(sot\), where \(s \in S\) and \(t \in T\). As a result of lemma 3.3.1 one then has

**Corollary 3.3.1:** If \(R\) is a flexible derivation alternator ring, then \((\text{AoA}, R) = 0\).

**Proof:** By 3.3.2, \((\text{AoA}, R) \subseteq \text{Ao}(A, R) = 0\). □

**Lemma 3.3.2:** In a flexible derivation alternator ring \(R\), the ideal generated by \(\text{AoA}\) is \(J = \text{AoA} + R(\text{AoA})\).

**Proof:** By the preceding corollary, \((\text{AoA})R = R(\text{AoA})\). This with 2.3.12 and the fact \(A\) is an ideal gives

\[\begin{align*}
[R(\text{AoA})]R & = [(\text{AoA})R]R \subseteq (\text{AoA}, R, R) + (\text{AoA})R \\
& \subseteq \text{AoA}(R, R) + R(\text{AoA}) \subseteq \text{AoA} + R(\text{AoA}).
\end{align*}\]

Similarly 2.3.13 shows \([R(\text{AoA})] \subseteq (R, R, \text{AoA}) + R(\text{AoA}) \subseteq \text{Ao}(R, R, A) + R(\text{AoA}) \subseteq \text{AoA} + R(\text{AoA}),\]

Which completes the proof \(J\) is an ideal of \(R\). □

For a non-empty subset \(I\) of \(R\), we next set \(\text{Ann } (I) = \{x \in R | xI = 0 = Ix\}\). Then one can prove
Lemma 3.3.3: Let $R$ be a flexible derivation alternator ring. If $I$ is an ideal of $R$, then $I^1 = \{x \in \text{Ann}(I)/(x,R,I) = (R,x,I)\}$ is also an ideal of $R$.

Proof: Using the definition of $I^1$, the fact $I$ is an ideal, and flexibility, we have 

$$(I^1 R)I \subseteq (I^1,R,I) + I^1 (RI) + I^1 (R I) \subseteq I^1 = 0,$$

and $I (I^1 R) \subseteq (I^1,R,I) + (I^1)R \subseteq (R,I^1, I) = 0$.

Similarly one sees $(R I^1)I = 0 = I (R I^1)$, so that

$I^1 R \subseteq \text{Ann}(I)$ and $RI^1 \subseteq \text{Ann}(I)$.

Next the definition of $I^1$ and $I$ an ideal can be utilized with 2.3.12 to show 

$$(R I^1,R,I) \subseteq Ro (I^1,R,I) + I^1 o (R,R,I) \subseteq I^1 o I = 0.$$ 

Thus $(Ro I^1,R,I) = 0$.

Analogously linearized 3.3.1 and flexibility lead to


$$\subseteq (R,I,(I^1,R,I)) + (I^1,R,I) + (I^1,R,R) \subseteq (R,I,(R,I^1)) + (R,(I^1,R,I)).$$

But since $I$ is an ideal, $(I^1,I,R) = 0$ and $(R,I,(R,I^1)) = 0$ using 3.3.8 as well. This means

$$(I,R,(R,I^1)) \subseteq (R,I,(R,I^1)) + (R,I^1,R) = 0,$$

so by flexibility

$$(R,I^1,R,I) = 0.$$ 

Now since characteristic $\neq 2$, 3.3.9 and 3.3.10 together imply

$$(R^1,R,I) = 0 = (I^1,R,R,I)$$

If we next use 3.3.8 repeatedly, flexibility, and 3.3.11, we see $(R,RI^1,I) = (R(RI^1))I \subseteq (R,R,I^1)I + (I^1,R,R,I) \subseteq (I^1,R,R,I) + (I^1,R,R,I) \subseteq (I^1,R,R,I) = 0$. Then this, the definition of $I^1$, and linearized 2.3.9 show likewise $(R, I^1 R, I) \subseteq (R,R,I^1,R) + Ro(R,I^1,R,I) + I^1 o (R,R,I) \subseteq I^1 o I = 0$. Thus $(R,R,I^1,R) = 0 = (R, I^1 R, I)$......3.3.12

Together 3.3.8,3.3.11 and 3.3.12 establish that $I^1$ is an ideal of $R$. □

Lemma 3.3.4: If $R$ is a flexible derivation alternative ring, then $(R,R) \subseteq J^1$ for the ideal $J = AoA + R(AoA)$.

Proof: First, because $A$ is an ideal, it follows $(AoA)\circ(R,R) \subseteq Ao(R,R) = 0$ by Lemma 3.3.1. But by the corollary 3.3.1, we also know $(AoA,(R,R)) \subseteq (AoA,R) = 0$. Since characteristic $\neq 2$, $(AoA)\circ(R,R) = 0 = (AoA,(R,R))$ implies

$(AoA)(R,R) = 0 = (R,R)(AoA).$ 

3.3.12
Let us next use 2.3.13 and 2.3.12 to show \(((R,R),R,AoA) \subseteq Ao((R,R),R,A) \subseteq (Ao(R,R),R,A) + (R,R)o(A,R,A) = 0\), since \(Ao(R,R) = 0\) by Lemma 3.3.1 and \((A,R,A) \subseteq A\). In fact, using 2.3.12, 2.3.9 and 2.3.13 and completely analogous reasoning will establish that associators of the form
\[ ((R,R),R,AoA) = 0 \]  
for any permutation of the entries.

If we now use \((AoA,R) = 0, 3.3.13\) and 3.3.12, we find \((R,R)(R(AoA)) = (R,R)((AoA)R) = (R,(R,R)(AoA))R = 0\). Likewise \((R(AoA))(R,R) = R((AoA)(R,R)) = 0\). These last two equations with 3.3.12 prove \((R,R) \subseteq \text{Ann}(J)\). 

We can also use 3.3.13 and 3.3.12, with 3.3.13 and \(A\) an ideal, to show \(((R,R),R, Ro(AoA)) \subseteq Ro((R,R),R,AoA) + (AoA)o((R,R),R,R) = (AoA)o((R,R),R,R) \subseteq (Ao((R,R),R,R) + (R,R)oA by Lemma 3.3.1. In completely analogous fashion, 3.3.13 and 3.3.9 \((R,(R,R),Ro(AoA)) = 0\). But since \((AoA,R) = 0\) and characteristic \(\neq 2\), it thus follows
\[ ((R,R),R,R(AoA)) = 0 = (R,(R,R),R(AoA)). \]  
Together 3.3.14, 3.3.13 and 3.3.15 prove \((R,R) \subseteq J^1\).

**Theorem 3.3.1:** In a prime flexible derivation alternator ring \(R\), the alternator ideal \(A\) is nil of index 2.

**Proof:** Since by Lemma 3.3.2 and 3.3.3, \(J\) and \(J^1\) are ideals of \(R\) such that \(JJ^1 = 0 = J^1J\), either \(J^1 = 0\) or \(J = 0\). Now if \(J^1 = 0\), then by Lemma 3.3.4 \((R,R) \subseteq J^1 = 0\). This means \(R\) is commutative, and so alternative by Theorem 3 in [7]. In this case \(A = 0\) is trivially a nil ideal of index 2. On the other hand, if \(J = 0\), then \(AoA \subseteq J = 0\). Of course this also means \(A\) is nil of index 2, and therefore completes the proof of the theorem.

**Corollary 3.3.2:** A prime flexible derivation alternator ring without nonzero nil ideals of index 2 is alternative.

**Corollary 3.3.3:** A simple flexible derivation alternator ring is either alternative or anti commutative.