CHAPTER – 3

PROPERTIES OF C – SEMIRINGS AND PRDs
3.1 INTRODUCTION

In this Chapter, we study some properties of C-semirings. We prove that if \((S, +, \cdot)\) is a C-semiring and \((S, +)\) is left cancellative, then \((S, \cdot)\) is a band. We also discuss the structure of Positive Rational Domains (PRDs). Sidney S.Mitchell and Kyungpook Porntip Sinutoke[23] studied the properties of PRDs. This motivated us to study the structure of ordered PRDs. We prove that in a PRD \((S, +, \cdot)\) if \((S, +)\) is cancellative, then \(|E[+]| = 0\). We also study some properties of PRDs and ordered PRDs.

3.2 PRELIMINARIES:

In this section we study the concepts (and results) which are not mentioned in the earlier chapters and which are needed for the study of main theorems of this chapter. We also discuss the properties of ordered C-semirings.
Definition 3.2.1:

A C–semiring is a semiring in which

(i) \((S,+)\) is a commutative monoid

(ii) \((S, \cdot)\) is a commutative monoid

(iii) \(a(b+c) = ab + ac\) and \((b+c)a = ba + ca\) for every \(a, b, c \in S\)

(iv) \(a.0 = 0.a = 0\)

(v) \((S, +)\) is a band and 1 is the absorbing element of ‘+’.

Definition 3.2.2:

A semiring \((S, +, \cdot)\) is said to be a Positive Rational Domain (PRD) if and only if \((S, \cdot)\) is an abelian group.

Definition 3.2.3:

A semigroup \((S, +)\) is said to be a band if \(a + a = a\) for all \(a \in S\).
3.3 PROPERTIES OF C-SEMIRINGS:

In this section, the properties of C-semirings are studied. We prove that if $(S, +, \cdot)$ is a C-semiring, then $a^2 = a + a^2$, $\forall a \in S$ if and only if $(S, \cdot)$ is a band.

Example 3.3.1:

Let $S$ be a C-semiring. Define $nx + 0 = nx = 0 + nx$, $x + 1 = 1 = 1 + x$ $\forall x \in A$, then $(S, \cdot)$ is a band.

\[
\begin{array}{c|cccc}
+ & 1 & x & y & 0 \\
\hline
1 & 1 & 1 & 1 & 1 \\
x & x & y & x & x \\
y & y & y & y & y \\
0 & 0 & x & y & 0 \\
\end{array}
\quad
\begin{array}{c|cccc}
\cdot & 1 & x & y & 0 \\
\hline
1 & 1 & x & y & 0 \\
x & x & 0 & 0 & 0 \\
y & y & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Theorem 3.3.2: Let $(S, +, \cdot)$ be a C-semiring and $(S, +)$ be left cancellative, then $(S, \cdot)$ is a band.

Proof: Consider $a + a^2 = a.1 + a^2$

\[= a(1 + a)\]

\[= a.1\]
Therefore \( a + a^2 = a \) ----(1)

Also \( a + a = a \) ----(2)

From (1) and (2), \( a + a^2 = a + a \)

\[ \Rightarrow a^2 = a \ (\text{since } (S, +) \text{ is left cancellative}) \]

\( \therefore \ (S, \ast) \text{ is a band.} \)

**Theorem 3.3.3:** Let \((S, +, \ast)\) be a C-semiring, then \( a^2 = a + a^2, \forall \ a \in S \)

if and only if \((S, \ast)\) is a band.

**Proof:** Suppose \( a^2 = a + a^2 \)

\[ = a.1 + a^2 \]

\[ = a(1 + a) \ (\text{Since } 1 \text{ is an absorbing element w.r.to.'}+) \]

\[ = a.1 \]

\[ a^2 = a \]

\( \therefore \ (S, \ast) \text{ is a band} \)

Conversely suppose \((S, \ast)\) is a band

i.e., \( a^2 = a, \forall \ a \in S \)

\[ a^2 = a.1 \]

\[ = a(1 + a) \ (\text{Since } 1 \text{ is an absorbing element w.r to.'}+) \]

\[ = a + a^2, \forall \ a \in S \]
\[ a^2 = a + a^2 \]

**Theorem 3.3.4:** Let \((S, +, \cdot)\) be a \(C\)–semiring. If \((S, \cdot)\) is regular, then \((S, \cdot)\) is a band.

**Proof:** Consider \(x(1+1)x = x\)

\[ \Rightarrow x^2 + x^2 = x \]

But \(x^2 + x^2 = x^2\)

\[ \therefore x = x^2 \]

\[ \therefore (S, \cdot) \text{ is a band.} \]

**Theorem 3.3.5:** Let \((S, +, \cdot)\) be a \(C\)–semiring. Then \(S\) contains two elements \(a\) and \(b\) such that \(ab = a + b + ab\) if and only if \(ab = a + b = a = b\).

**Proof:** \(ab = a + b + ab\)

\[ = a + (1 + a)b \]

\[ ab = a + b \]

\[ ab = a + b + ab \]

\[ ab + b = a + b + ab + b \]

\[ (a + 1)b = a + b + (a + 1)b \]
\[ \begin{align*}
1 \cdot b &= a + b + 1 \cdot b \\
\Rightarrow 1 \cdot b &= a + b + b \\
b &= a + b \ (\text{since } b + b = b) \\
ab &= a + b + ab \\
\Rightarrow ab + a &= a + b + ab + a \\
a(b + 1) &= a + b + a(b + 1) \\
a \cdot 1 &= a + b + a \cdot 1 \ (\text{since } a + a = a) \\
a &= a + b + a \\
a &= a + b \\
\text{conversely, } ab &= a + b \\
&= a + b + b \ (\text{since } b + b = b) \\
&= a + b + ab \\
\therefore ab &= a + b + ab
\end{align*} \]
3.4 PROPERTIES OF ORDERED C – SEMIRINGS:

In this section, we study properties of ordered C – semirings. It is proved that if \((S, +, \cdot)\) is a t.o. C – semiring in which \((S, +)\) is p.t.o.(n.t.o.), then for any \(x, y\) in \(S\), \(x + y = x\) or \(y\).

Examples 3.4.1:

The following are the examples of ordered C – semirings.

(i)

\[
\begin{array}{cccc}
+ & 1 & x & 0 \\
1 & 1 & 1 & 1 \\
x & 1 & x & x \\
0 & 1 & x & 0 \\
\end{array}
\quad
\begin{array}{cccc}
\cdot & 1 & x & 0 \\
1 & 1 & x & 0 \\
x & x & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

(ii)

\[
\begin{array}{cccc}
+ & 1 & x & 0 \\
1 & 1 & 1 & 1 \\
x & 1 & x & x \\
0 & 1 & x & 0 \\
\end{array}
\quad
\begin{array}{cccc}
\cdot & 1 & x & 0 \\
1 & 1 & x & 0 \\
x & x & x & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]
(iii) In this example, \((S, +)\) is p.t.o. and \(0 < x < 1\).

\[
\begin{array}{c|ccc}
+ & 0 & x & 1 \\
\hline
0 & 0 & x & 1 \\
x & x & x & 1 \\
1 & 1 & 1 & 1 \\
\end{array}
\]
\[
\begin{array}{c|cccc}
\cdot & 0 & x & 1 \\
\hline
0 & 0 & 0 & 0 \\
x & 0 & 0 & x \\
1 & 0 & x & 1 \\
\end{array}
\]

**Theorem 3.4.2:** Let \((S, +, \cdot)\) be a t.o. C-semiring in which \((S, +)\) is p.t.o. (n.t.o.), then for every \(x, y\) in \(S\), \(x + y = x\) or \(y\).

**Proof:** Suppose \((S, +)\) is p.t.o.

Suppose \(x < y\), then \(x + y \leq y + y\)

\[x + y \leq y \quad \text{(since } y + y = y\text{)} \tag{1}\]

By hypothesis \((S, +)\) is p.t.o., i.e., \(x + y \geq y \quad \text{----(2)}\)

from (1) \& (2) \(x + y = y\)

If \(y < x\)

\[x + y \leq x + x\]

\[x + y \leq x\]

Since \((S, +)\) is p.t.o.

\[x + y \geq x\]

\[\therefore x + y = x\]
Suppose \((S, +)\) is n.t.o.

If \(x < y\), then \(x + x \leq x + y\)

\[ \Rightarrow x \leq x + y \]

\[ \Rightarrow x + y \geq x \]

But \(x + y \leq x\) (since \((S, +)\) is n.t.o.)

\[ \therefore x + y = x \]

If \(y < x\) then \(y + y \leq x + y\)

\[ \Rightarrow y \leq x + y \]

\[ \Rightarrow x + y \geq y \quad \text{(1)} \]

But \(x + y \leq y \quad \text{(2)} \)

From (1) and (2), \(x + y = y\).

**Theorem 3.4.3**: Let \((S, +, \cdot)\) be a t.o. C–semiring in which \((S, \cdot)\) is p.t.o.

Then 0 is the maximum element.

**Proof**: \(a.0 = 0.a = 0\) since 0 is the multiplicative zero

Since \((S, \cdot)\) is p.t.o., 0 is the maximum element.
3.5 POSITIVE RATIONAL DOMAINS (PRDs):

In this section, properties of positive rational domain semirings and ordered positive rational domain semirings are studied.

Result 3.5.1: [Proposition 2.1(iii), 28]

If a positively ordered semigroup $S$ contains identity 1, then 1 is the minimum element.

Result 3.5.2: [Proposition 6, 36]

If a t.o.s.g. $(S, \cdot, \leq)$ is non-negatively ordered, then it is positively ordered in the strict sense if any of the following conditions is satisfied.

(i) $(S, \cdot, \leq)$ is o-Archiepedean

(ii) $(S, \cdot)$ is left cancellative semigroup without idempotents

(iii) $(S, \cdot)$ is cancellative.

Result 3.5.3: [Proposition 1, 31]

If a totally ordered semiring $(S, +, \cdot)$ contains 1, then $(S, +)$ is non-negatively or non-positively ordered.
Result 3.5.4: [Theorem 11, 42]

Let \((S, +, \cdot)\) be a totally ordered semiring and \(x, y \in S\) such that \(x \leq x + x\) and \(y + y \leq y\). Then \(xy, yx \in E[+]\).

Theorem 3.5.5: If \(|S| > 1\) in a totally ordered PRD in which \((S, +)\) is cancellative semigroup, then one of the following is true.

(i) \((S, +)\) is positively ordered in strict sense

(ii) \((S, +)\) is negatively ordered in strict sense

Proof: Since \(S\) is a PRD, \(S\) contains multiplicative identity. Now using result 3.5.3, \((S, +)\) is either non-negatively ordered or non-positively ordered

Now using result 3.5.2, \((S, +)\) is p.t.o. or n.t.o.

Theorem 3.5.6: Let \((S, +, \cdot)\) be a PRD and \((S, +)\) be cancellative. Then \(|E[+]| = 0.\)

Proof: Since \((S, +)\) is cancellative, it contains at most one idempotent

Suppose \(S\) has an idempotent \(x\).

Now \(x + x = x\)

\(\Rightarrow x.(1 + 1) = x.1\)
\[ (1 + 1) = 1 \text{ (since } (S, \cdot) \text{ is a group}) \]

Now for every \( y \in S \), \( y.(1 + 1) = y.1 \)

\[ \Rightarrow y.1 + y.1 = y.1 \]

\[ \Rightarrow y + y = y \]

\[ \Rightarrow (S, +) \text{ is a band, which is a contradiction} \]

\[ \therefore |E[+]| = 0 \]

**Theorem 3.5.7:** Let \((S, +, \cdot)\) be a t.o. PRD and \( x \notin x + S \) and \( x \notin S + x \) for every \( x \in S \). Then \((S, +)\) is positively ordered in the strict sense or negatively ordered in the strict sense.

**Proof:** Since PRD contains multiplicative identity and using result 3.5.3, \((S, +)\) is either non-negatively ordered or non-positively ordered.

Suppose \((S, +)\) is non-negatively ordered.

If \( x + y < x \) for some \( y \) in \( S \), then \( x + 2y \leq x + y \)

Also \( y \leq 2y \implies x + y \leq x + 2y \)

\[ \therefore x + y = x + 2y \]

\[ \therefore x + y = x + 2y \in x + y + S, \text{ which is a contradiction} \]

Similarly we can prove that \((S, +)\) is negatively ordered in the strict sense if \((S, +)\) is non-positively ordered.