CHAPTER – 2

SEMIRINGS IN WHICH \((S, +)\) IS A ZEROID
2.1 INTRODUCTION:

In this chapter, we introduce the concept of zeroid in semirings. We study whether the algebraic structure of \((S, \cdot)\) may determine the order structure of \((S, +)\) and vice-versa. Throughout this chapter unless otherwise mentioned \(S\) is a semiring in which \((S, +)\) is a zeroid. The zeroid of a semiring is denoted by \(Z\). We also study the properties of zeroid semirings and ordered zeroid semirings. We prove that in a semiring every power of \(x\) is a zeroid if \(x\) is a zeroid and zeroid is a multiplicative ideal. We also prove that in a zeroid semiring which is also zero square semiring, then \(S^2 = \{0\}\).

In section 2, the required preliminaries (concepts, examples and results) are presented. In section 3, properties of semirings are discussed. We also discuss the examples of totally ordered zeroid semirings.

2.2 PRELIMINARIES:

In this section, we mention some preliminary definitions and results which will be used in this chapter and later in subsequent chapters.
**Definition 2.2.1:**

A system \((S, \cdot)\), where \(S\) is a non-empty set and \(\cdot\) is an associative binary operation on \(S\) is called a semigroup.

**Definition 2.2.2:**

A semigroup \((S, \cdot)\) is said to be partially ordered if there exists a binary relation ‘\(\leq\)’ on \(S\) satisfying the following properties:

(i) Reflexivity : \(a \leq a\), for every \(a\) in \(S\),

(ii) Antisymmetry : \(a \leq b, b \leq a\) imply \(a = b\) for all \(a,b\) in \(S\),

(iii) Transitivity : \(a \leq b, b \leq c\) imply \(a \leq c\), for all \(a,b,c\) in \(S\),

(iv) Compatibility: \(a \leq b\) implies \(ac \leq bc\) and \(ca \leq cb\), for all \(a, b, c\) in \(S\).

Partially ordered semigroup may also be denoted by \((S, \cdot, \leq)\).

**Note:** Some times we write \(a \geq b\) for \(b \leq a\). That is “\(\geq\)” is the dual relation of “\(\leq\”).

**Definition 2.2.3:**

A partially ordered semigroup \((p. o. s. g)\) in which any two elements are comparable is said to be a totally ordered semigroup \((t. o. s. g.)\) or fully ordered semigroup \((f. o. s. g.)\).
**Definition 2.2.4:**

A triple \((S, +, \cdot)\) is said to be a semiring if \(S\) is a non-empty set and \(“+,” \cdot”\) are binary operations on \(S\) satisfying that

(i) \((S, +)\) is a semigroup

(ii) \((S, \cdot)\) is a semigroup

(iii) \(a(b + c) = ab + ac\) and \((b + c)a = ba + ca\), for all \(a, b, c\) in \(S\).

**Examples of Semirings 2.2.5:**

(i) The motivating example of a semiring is the set of natural numbers \(\mathbb{N}\) under ordinary addition and multiplication. Likewise, the non-negative rational numbers and the non-negative real numbers form semiring. All these semirings are commutative.

(ii) If \(A\) is a commutative monoid, the set \(\text{End}(A)\) of endomorphisms \(F: A \rightarrow A\) form a semiring, where addition is pointwise addition and multiplication is function composition. The zero morphism and the identity are the respective neutral elements. If \(A\) is the additive monoid of natural numbers we obtain the semiring of natural numbers as \(\text{End}(A)\), and if \(A = S^n\) with \(S\) is a semiring. We obtain (after associating each morphism to a matrix) the semiring of square \(n\)-by-\(n\) matrices with coefficients in \(S\).
(iii) Let \( R = \mathbb{R} \cup \{\infty\} \). Then \((R, \min, +)\) is an additively idempotent commutative semiring in which addition is the operation of taking minimum and multiplication is ordinary addition.

(iv) Any unital, quantale is an idempotent semiring, or dioid, under join and multiplication.

(v) Any bounded distributive lattice is a commutative, idempotent semiring under join and meet.

(vi) Any ring is also a semiring.

**Definition 2.2.6:**

A semiring \((S, +, \cdot)\) is said to be totally ordered semiring (t.o.s.r.) if there exists a partially order ‘\(\leq\)’ on \(S\) such that

(i) \((S, +)\) is a t. o. s. g.

(ii) \((S, \cdot)\) is a t. o. s. g.

It is usually denoted by \((S, +, \cdot, \leq)\).

**Definition: 2.2.7:**

An element ‘\(x\)’ in a totally ordered semiring (t. o. s. r.) \((S, +, \cdot)\) is said to be

(i) an additive identity if \(a + x = x + a = a\).

(ii) a multiplicative identity if \(ax = xa = a\).
(iii) an additive zero if \( a + x = x + a = x \) and

(iv) a multiplicative zero if \( ax = xa = x \), for every ‘a’ in S.

Note:

(1) When we say a semiring with zero (or) a t.o.s.r. with zero mean multiplicative zero.

(2) ‘x’ is said to be left(right) additive identity if \( x + a = a \) (\( a + x = a \)) for every ‘a’ in S.

(3) ‘x’ is said to be left (right) additive zero if \( x + a = x \) (\( a + x = x \)) for every ‘a’ in S.

Definition 2.2.8:

A non - empty subset A of a semiring \( (S, +, \cdot) \) is said to be a subsemiring of S if \( (A, +, \cdot) \) is a semiring by itself.

[Note: A non-empty t.o. subset A of a totally ordered semiring \( (S, +, \cdot, \leq) \) is a totally ordered subsemiring of S, if \( (A, +, \cdot, \leq) \) is a totally ordered semiring by itself].
Examples of totally ordered semirings 2.2.9:

(i) The set of natural numbers under the usual addition, multiplication and ordering.

(ii) Consider the set \( S = \{0,1,2,3,\ldots\} \) with \( m + n = \max (m, n) \) or \( \min(m, n) \), \( mn = m + n \). where the addition in the multiplication is the usual addition, for all \( m, n \) in \( S \) and the order being the usual order relation.

Then \( (S, +, \cdot, \leq) \) is a totally ordered semiring.

(iii) If \( (X, \leq) \) is a totally ordered set and \( + \) and \( \cdot \) are min and max operations, then \( (X, +, \cdot, \leq) \) is a t.o.s.r. If \( S = X \cup \{z\} \) and \( +, \cdot \) and \( \leq \) are extended to \( S \) by defining \( s + z = z = z + s, s \cdot z = z = z \cdot s \) and \( s \leq z \) for all \( s \) in \( S \), then \( (S, +, \cdot, \leq) \) is a t.o.s.r.

Definition 2.2.10:

In a totally ordered semiring \((S, +, \cdot, \leq)\)

(i) \((S, +, \leq)\) is positively totally ordered (p.t.o.), if \( a + b \geq a, b \) for all \( a, b \) in \( S \) and

(ii) \((S, \cdot, \leq)\) is positively totally ordered (p.t.o.), if \( ab \geq a, b \) for all \( a, b \) in \( S \).
**Definition 2.2.11:**

A totally ordered semiring \((S, +, \cdot, \leq)\) is said to be a positively ordered in the strict sense if both \((S, +, \leq)\) and \((S, \cdot, \leq)\) are positively ordered in the strict sense.

**Definition 2.2.12:**

In a t. o. s. r. \((S, +, \cdot, \leq)\)

(i) \((S, +)\) is said to be right naturally totally ordered (r. n. t. o) if \((S, +)\) is positively ordered in the strict sense and if \(a < b\) implies \(b = a + c\) for some \(c\) in \(S\), and

(ii) \((S, \cdot)\) is said to be r. n. t. o. if \((S, \cdot)\) is positively ordered in the strict sense and if \(a < b\) implies \(b = ac\) for some \(c\) in \(S\).

**Definition 2.2.13:**

\((S, +, \cdot, \leq)\) is said to be a weak partially ordered semiring (w.p.o.s.r.) if \((S, +, \cdot)\) is a semiring and ‘\(\leq\)’ is a partial order relation on \(S\) such that \((S, +, \cdot, \leq)\) is a p.o.s.r. In case ‘\(\leq\)’ is a total order relation (full order relation) the weak p.o.s.r. is said to be a weak t.o.s.r. (or weak f.o.s.r.).
Definition 2.2.14:

A semigroup \((S, \cdot)\) with zero (usually we denote the zero element by the symbol ‘0’) is said to have no zero-divisors if \(xy = 0\) implies \(x = 0\) (or) \(y = 0\) for all \(x, y\) in \(S\).

Definition 2.2.15:

An element ‘\(x\)’ in a semigroup \((S, +)\) is said to be an additive idempotent if \(x + x = x\).

Note:

\(E (+)\) denotes the set of all additive idempotents in \((S, +)\).

\(|E (+)|\) denotes the cardinal number of the set \(E [+]\).

Definition 2.2.16:

An element \(x\) in a semigroup \((S, \cdot)\) is said to be multiplicative idempotent if \(x^2 = x\).

Note:

\(E (\cdot)\) denotes the set of all multiplicative idempotents in \((S, \cdot)\).

\(|E (\cdot)|\) denotes the cardinal number of the set \(E [\cdot]\).

Definition 2.2.17:

(i) A semigroup \((S, \cdot)\) is said to be a band if every element in \(S\) is an idempotent.
(ii) A commutative band is called a semilattice.

**Definition 2.2.18:**

An element \( x \) in a p.o.s.g \((S, \cdot, \leq)\) is non-negative (non-positive) if \( x^2 \geq x \) \((x^2 \leq x)\).

**Definition 2.2.19:**

A p.o.s.g. \((S, \cdot, \leq)\) is non-negatively (non-positively) ordered if every element of \( S \) is non-negative (non-positive).

**Definition 2.2.20:**

An element \( x \) in a t.o.s.r is minimal (maximal) if \( x \leq a \) \((x \geq a)\) for every \( a \) in \( S \).

**Definition 2.2.21:**

In a semigroup \((S, \cdot)\), a non-empty subset \( A \) of \( S \) is called

(i) a left ideal, if \( sa \in A \), for every \( s \in S \) and for every \( a \in A \)

(ii) a right ideal, if \( as \in A \), for every \( a \in A \) and for every \( s \in S \)

(iii) an ideal, if \( A \) is both a left ideal as well as a right ideal

(iv) a completely prime ideal, if it is an ideal and if \( ab \in A \) for any \( a, b \) in \( S \), then either \( a \in A \) or \( b \in A \) and

(v) completely semiprime, if it is an ideal and if \( a^2 \in A \) for any \( a \) in \( S \), then \( a \in A \).
Definition 2.2.22:

A semigroup \((S, \cdot)\) with zero is called o-simple if

(i) \(\{0\}\) and \(S\) are the only ideals and

(ii) \(S^2 \neq \{0\}\)

Definition 2.2.23:

An element \(x\) in a semigroup \((S, \cdot)\) is said to be

(i) left cancellable, if \(xa = xb\) for any \(a, b\) in \(S\) implies \(a = b\);

(ii) right cancellable if \(ax = bx\) for any \(a, b\) in \(S\) implies \(a = b\);

(iii) cancellable if it is both left as well as right cancellable.

Definition 2.2.24:

A semigroup \((S, \cdot)\) with all of its elements are left (right) cancellable is said to be left (right) cancellative semigroup.

Definition 2.2.25:

A semigroup \((S, \cdot)\) is weakly commutative, if for any \(x, y\) in \(S\), \((xy)^n \in ySx\) for some positive integer ‘n’.

Definition 2.2.26:

An element ‘a’ in a t. o. s. r. \((S, \cdot, \leq)\) is said to be

(i) left positive if \(ax \geq x\) for every \(x\) in \(S\).

(ii) right positive if \(xa \geq x\) for every \(x\) in \(S\).
(iii) positive if it is both left as well as right positive.

(iv) left negative if $ax \leq x$ for every $x$ in $S$.

(v) right negative if $xa \leq x$ for every $x$ in $S$.

(vi) negative if it is both left as well as right negative.

**Note:** ‘≥’ is the dual of ‘≤’.

**Definition 2.2.27:**

Two distinct elements $a, b$ in a t.o.s.g $(S, \cdot, \leq)$ are said to form an anomalous pair if $a^n < b^{n+1}$ and $b^n < a^{n+1}$ where $a, b$ are positive (or) $a^n > b^{n+1}$ and $b^n > a^{n+1}$ for all $n > 0$ where $a, b$ are negative.

**Definition 2.2.28:**

An element $x$ different from the identity in a non-negatively ordered semigroup $(S, \cdot, \leq)$ is said to be o-Archimedean if for every $y$ in $S$ there exists a natural number ‘$n$’ such that $x^n \geq y$.

**Definition 2.2.29:**

A non-negatively ordered semigroup $(S, \cdot, \leq)$ is said to be o-Archimedean if every one of its elements different from its identity (if exists) is o-Archimedean.
Definition 2.2.30:

A t.o.s.g \((S, \cdot, \leq)\) is o-isomorphic to a t.o.s.g \((T, \cdot, \leq_1)\) if there exists a mapping \(f: S \to T\) such that

(i) \(f\) is one-to-one and onto map

(ii) whenever \(a \leq b\) for any \(a, b\) in \(S\), then \(f(a) \leq_1 f(b)\) (that is, \(f\) preserves the order) and

(iii) \(f(ab) = f(a) f(b)\) (that is, \(f\) preserves the multiplication)

Definition 2.2.31:

A t.o.s.g \((S, +, \cdot, \leq)\) is o-isomorphic to a t.o.s.r \((T, \oplus, \odot, \leq_1)\) if there exists a mapping \(f : S \to T\) such that

(i) \(f\) is one-to-one and onto map

(ii) whenever \(a \leq b\) for any \(a, b\) in \(S\), then \(f(a) \leq_1 f(b)\) (that is, \(f\) preserves the order)

(iii) \(f(a + b) = f(a) \oplus f(b)\) (that is, \(f\) preserves the addition) and

(iv) \(f(ab) = f(a) \odot f(b)\) (that is, \(f\) preserves the multiplication).

Definition 2.2.32:

Let \(a \in S\). The least element of the set \(\{x \in N: (\text{there exists } y \in N) \quad xa = ya, x \neq y\}\) is called the index of \(a\) and is denoted by \(m\), where \(N\) is the set of natural numbers.
Definition 2.2.33:

The least element of the set \( \{ x \in \mathbb{N} : (m + x)a = ma \} \) is called the period of \( a \) and is denoted by \( r \).
2.3 PROPERTIES OF SEMIRINGS IN WHICH (S, +) IS A ZEROID:

In this section, the structure of zeroid semiring and its properties are studied.

Definition 2.3.1:

Zeroid of a semiring \((S, +, \cdot)\) is the set of all \(x\) in \(S\) such that \(x + y = y\) or \(y + x = y\) for some \(y\) in \(S\) we may also term this as the zeroid of \((S, +)\).

Definition 2.3.2:

A semiring \((S, +, \cdot)\) with multiplicative zero is said to be zero-square semiring if \(x^2 = 0\) for all \(x \in S\).

Examples 2.3.3:

In the following examples (i) to (vii) \(S\) is a semiring in which \((S, +)\) is a zeroid and 0 is the additive zero. In (iii), (vii) and (viii) 0 is also a additive identity.
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**Theorem 2.3.4:** Let \((S, +, \cdot)\) be a zero - square semiring with additive identity \(0\). If \((S, +)\) is a zeroid then \(S^2 = \{0\}\).

**Proof:** Since \((S, +)\) is a zeroid

\[x + y = y \text{ or } y + x = y\]

Since \(S\) is a zero - square semiring

\[x^2 = 0, \ y^2 = 0, \ \forall \ x, y \in S\]
$x + y = y$ implies $(x + y)y = y^2$

$xy + y^2 = y^2$

$xy + 0 = 0$

$xy = 0$

$x + y = y$ implies $y(x + y) = y^2$

$yx + y^2 = y^2$

$yx + 0 = 0$

$yx = 0$

If $y + x = y$, then $y(y + x) = y^2$

$\Rightarrow y^2 + yx = y^2$

$\Rightarrow 0 + yx = 0$

$\Rightarrow yx = 0$

Also $y + x = y$ implies $(y + x)y = y^2$

$\Rightarrow y^2 + xy = y^2$

$\Rightarrow 0 + xy = 0$

$xy = 0$

$\therefore xy = yx = 0$

Hence, $S^2 = \{0\}$
**Lemma 2.3.5**: Let \((S, +, \cdot)\) be a semiring. If \(x \in Z\), where \(Z\) is the zeroid of semiring, then every power of \(x\) is a zeroid.

**Proof**: Let \(x \in Z\). Then by definition there exists some \(y\) in \(S\) such that

\[ x + y = y \text{ or } y + x = y \tag{1} \]

\[ \Rightarrow x(x + y) = xy \]

\[ \Rightarrow x^2 + xy = xy \]

\[ \Rightarrow x^2 + s = s, \text{ where } xy = s \in S \]

\[ \Rightarrow x^2 \text{ is a zeroid} \]

From (1), \(x^2(x + y) = x^2y\)

\[ \Rightarrow x^3 + x^2y = x^2y \]

\[ \Rightarrow x^3 + s^1 = s^1, \text{ where } s^1 = x^2 y \in S \]

\[ \therefore x^3 \in Z \text{ is a zeroid} \]

Continuing in this way, every power of \(x\) is in \(Z\).

**Theorem 2.3.6**: Let \((S, +, \cdot)\) be a semiring and \(Z\) be a zeroid, then \(Z\) is a multiplicative ideal.

**Proof**: Let \(x \in Z\)

Then there exists \(y \in S\) such that \(x + y = y\) or \(y + x = y\)

Let \(s \in S\)
Then \( sy = s(y + x) \)

\[ = sy + sx \]

\[ \Rightarrow sx \in Z \]

Similarly \( xs \in Z \)

\[ \therefore Z \text{ is a multiplicative ideal.} \]

**Theorem 2.3.7:** Let \((S, +, \cdot)\) be a semiring and \((S, +)\) be commutative.

Then \((Z, +)\) is a subsemigroup of \((S, +)\).

**Proof:** Let \( x, y \in Z \)

\( x \in Z \Rightarrow \) there exists some \( p \) in \( S \) such that \( x + p = p \) or \( p + x = p \)

\( y \in Z \Rightarrow \) there exists some \( q \) in \( S \) such that \( y + q = q \) or \( q + y = q \)

Now \( (x + p) + (y + q) = p + q \)

\[ \Rightarrow (x + y) + (p + q) = p + q \text{ (Since } (S, +) \text{ is commutative)} \]

\[ \Rightarrow (x + y) + s = s \text{, where } p + q = s \]

\[ \Rightarrow (x + y) \in Z \]

\[ \therefore Z \text{ is a subsemigroup of } (S, +). \]

**Theorem 2.3.8:** Let \((S, +, \cdot)\) be a semiring with IMP in which \((S, +)\) is a zeroid. If \((S, +)\) is cancellative, then \((S, \cdot)\) is a band.
**Proof:** Since \((S, +)\) is a zeroid, and \(x \in S\).

Then \(x + y = y\) or \(y + x = y\) for some \(y\) in \(S\).

Suppose \(x + y = y\)

\[x + x + y = x + y\]

\[2x + y = y\]

Continuing like this, \(nx + y = x + y\)

This implies \(nx = x\)---(1) since \((S, +)\) is right cancellative

If \(y + x = y\)

\[y + x + x = y + x\]

\[y + 2x = y + x\]

Continuing like this, \(y + nx = y + x\)

This implies \(nx = x\)---(2) since \((S, +)\) is left cancellative

\(S\) satisfies IMP, \(x^2 = nx\)---(3)

\[\therefore \text{ From (1), (2) and (3) } x^2 = x\]

\[\therefore (S, \cdot) \text{ is a band.}\]
2.4. EXAMPLES OF ORDERED SEMIRINGS IN WHICH (S, +) IS A ZEROID:

In this section, we construct some examples of ordered semirings in which (S, +) is a zeroid.

**Definition 2.4.1:**

A semiring S is said to be mono semiring if \( a + b = a \cdot b \) for all \( a, b \) in S.

**Examples 2.4.2:**

The following are the examples of totally ordered semirings in which (S, +) is a zeroid. Also (S, \( \cdot \)) is a zeroid since (S, +, \( \cdot \)) is a mono semiring.
(i) \[ a < b < 2a < a + b < b + a < 2b < c \]

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
+ , \cdot & a & b & 2a & a + b & b + a & 2b & c \\
\hline
a & 2a & a + b & c & c & c & c & c \\
\hline
b & b + a & 2b & c & c & c & c & c \\
\hline
2a & c & c & c & c & c & c & c \\
\hline
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . \\
\hline
. & . & . & . & . & . & . & . \\
\hline
\end{array}
\]

(S, +) is non-commutative.

A commutative version is obtained identifying \( a + b \) and \( b + a \).

In examples (ii) and (iii) (S, +) is commutative.
(ii) \[ a < 2a < c < b < a + b < d \]

\[
\begin{array}{cccccc}
+ & \times & a & 2a & c & b & a + b & d \\
\hline
a & 2a & c & c & a + b & d & d \\
2a & c & c & c & d & d & d \\
c & c & c & c & d & d & d \\
b & d & d & d & d & d & d \\
a + b & d & d & d & d & d & d \\
d & d & d & d & d & d & d \\
\end{array}
\]

(iii) \[ 2a < a < b \]

\[
\begin{array}{ccc}
+ & \times & 2a & a & b \\
\hline
2a & 2a & 2a & b \\
a & 2a & 2a & b \\
b & b & b & b \\
\end{array}
\]