CHAPTER – 5

SOME SPECIAL CLASSES OF SEMIRINGS
5.1 INTRODUCTION

In this chapter, we discuss cancellative semirings, zero sum semirings and semirings in which \((S, \cdot)\) is p.t.o.

5.2 PRELIMINARIES:

In this section, we give some definitions that are needed in this chapter.

Definition 5.2.1:

A semiring \((S, +, \cdot)\) with additive zero is said to be zero - sum semiring if \(x + x = 0\) for all \(x \in S\).

Definition 5.2.2:

A semiring \((S, +, \cdot)\) is said to be a Boolean semiring if \((S, \cdot)\) is a band.

Definition 5.2.3:

An element \(x\) with zero is said to be nilpotent if \(x^n = 0\) for some natural number \(n\). If every element of \(S\) is nilpotent, then \(S\) is called a nilsemigroup.
5.3 SOME SPECIAL CLASSES OF SEMIRINGS:

In this section, the author explores some special properties of semirings.

**Theorem 5.3.1:** Let \((S, +, \cdot)\) be a semiring in which \((S, +)\) is cancellative and \((S, \cdot)\) is a band, then \((S, +)\) is commutative.

**Proof:** Let \(a, b \in S\). Then \(a = a^2\) and \(b = b^2\)

Consider, \(ab + b^2 + a^2 + ba = (a + b)b + (a + b)a\)

\[= (a + b)(b + a)\]

\[= a(b + a) + b(b + a)\]

\(ab + b^2 + a^2 + ba = ab + a^2 + b^2 + ba\)

\(b^2 + a^2 = a^2 + b^2\) (since \((S, +)\) is cancellative)

\(b + a = a + b\) (since \((S, \cdot)\) is a band)

\[\therefore (S, +)\) is commutative\]

**Theorem 5.3.2:** Let \((S, +, \cdot)\) be a semiring with multiplicative identity and \((S, +)\) be cancellative, then \((S, +)\) is commutative.

**Proof:** \(a + b + a + b = a.1 + b.1 + a.1 + b.1\)

\[= (a + b).1 + (a + b).1\]
\[(a + b)(1 + 1)\]
\[= a(1 + 1) + b(1 + 1)\]
\[= a + a + b + b\]
\[b + a = a + b \text{ (Since } (S, +) \text{ is cancellative.)}\]
\[\therefore (S, +) \text{ is commutative.}\]

**Theorem 5.3.3:** Let \((S, +, \cdot)\) be a semiring and \(E[+] = \phi\). If for every \(a, b \in S\), \(a < b\) implies \(b = a + c\) for some \(c \in S\) and if \((S, +)\) is left cancellative, then \((S, \cdot)\) is cancellative.

**Proof:** Suppose \((S, \cdot)\) is not cancellative

Then \(\exists a, b, c \in S\) such that \(ab = ac\) with \(b < c\)

\(b < c \Rightarrow c = b + t\), for some \(t \in S\)

\[\Rightarrow ac = a (b + t)\]

\[\Rightarrow ac = ab + at\]

\[\Rightarrow ac + at = ab + at + at\]

\[\Rightarrow at = at + at\], which is a contradiction.

\[\therefore (S, \cdot) \text{ is cancellative.}\]
**Theorem 5.3.4:** Let \((S, +, \cdot)\) be a semiring with multiplicative identity which is also an additive idempotent, then \((S, +)\) is a band.

**Proof:** By hypothesis \(1 + 1 = 1\)

\[
\Rightarrow a \cdot (1 + 1) = a \cdot 1 \quad \forall \ a \in S
\]

\[
\Rightarrow a \cdot 1 + a \cdot 1 = a \cdot 1
\]

\[
\Rightarrow a + a = a, \quad \forall \ a \in S
\]

\(\therefore (S, +)\) is a band.

**Theorem 5.3.5:** Every Boolean semiring \(S\) in which \((S, +)\) is cancellative has the following properties.

(iii) \(S = \{a, 2a\} \cup \{b, 2b\} \cup \ldots \) for all \(a, b \ldots \in S\).

(iv) \(a = a + ab + ba\) and \(b = b + ba + ab\)

**Proof:** (i) Since \((S, \cdot)\) is a band

\[
(a + a)^2 = a + a
\]

\[
(a + a) \cdot (a + a) = a + a
\]

\[
\Rightarrow a^2 + a^2 + a^2 + a^2 = a + a
\]

\[
\Rightarrow a + a + a + a = a + a
\]

\[
\Rightarrow a + a + a = a
\]

\[
\Rightarrow 3a = a
\]
\[ 4a = 2a \quad \ldots \ldots \]

This proves the theorem.

(ii) \( (a + b)^2 = a + b \)

\[ \Rightarrow (a + b) (a + b) = a + b \]

\[ \Rightarrow a^2 + ab + ba + b^2 = a + b \]

\[ \Rightarrow a + ab + ba + b = a + b \]

\[ \Rightarrow a + ab + ba = a \quad \text{using right cancellation law} \]

and \( ab + ba + b = a \quad \text{using left cancellation law.} \)

**Theorem 5.3.6:** Let \((S, +, \cdot)\) be a semiring and satisfy \(ab = a + b + ab\), for all \(a, b \in S\) and \((S, +)\) be right cancellative. If \((S, \cdot)\) is commutative, then \((S, +)\) is commutative.

**Proof:** \(ab = a + b + ab\)

\[ ba = b + a + ba \]

Since \(ab = ba\),

\[ a + b + ab = b + a + ba \]

\[ = b + a + ab \]

Since \((S, +)\) is right cancellative.

\[ a + b = b + a. \]

\[ \therefore (S, +) \text{ is commutative.} \]
5.4 SOME SPECIAL CLASSES OF ORDERED SEMIRINGS:

In this section, the author explores some special properties of ordered semirings.

Theorem 5.4.1: Let \((S, +, \cdot)\) be a totally ordered semiring with multiplicative zero. If \((S, \cdot)\) is a nilsemigroup and non-negatively ordered, then \((S, \cdot)\) is p.t.o.

Proof: Suppose \(xy < x \quad (1)\)

\[\Rightarrow xy^2 \leq xy < x\]

continuing like this, \(xy^m \leq xy\)

But \(xy^m \geq xy\) (since \((S, \cdot)\) is non-negatively ordered)

\[\Rightarrow xy^m = xy\]

\[\Rightarrow xy = 0\] (Since \((S, \cdot)\) is a nilsemigroup, \(x^m = 0\))

From (1), \(0 < x \quad (2)\)

We first prove that 0 is the maximum element

Let \(x \in S\). Then \(x^n = 0\),

If \((S, \cdot)\) is non-negatively ordered, then \(x^2 \geq x, \forall x \in S\)

\[\Rightarrow x^n \geq x^{n-1} \geq \ldots \geq x^2 \geq x\]

\[\Rightarrow 0 \geq x, \forall x \in S\) (since \(x^n = 0\)),\]
which is a contradiction to (2)

\[ \therefore xy \geq x \]

If \( xy < y \) -----(3)

\[ \Rightarrow x^n y \leq xy \]

But \( x^n y \geq xy \) (since \( (S, \cdot) \) is non-negatively ordered)

\[ \Rightarrow x^n y = xy \]

\[ \Rightarrow xy = 0 \text{(Since } (S, \cdot) \text{ is a nilsemigroup, } x^n = 0) \]

From (3), \( 0 < y \), which is a contradiction, since 0 is the maximum element

\[ \therefore xy \geq y \]

i.e., \( xy \geq x, y \)

i.e., \( (S, \cdot) \) is p.t.o.

**Theorem 5.4.2:** Let \( (S, +, \cdot) \) be a t.o.s.r. If \( (S, \cdot) \) is p.t.o., and \( S \) contains multiplicative identity 1, then \( (S, +) \) is non-negatively ordered.

**Proof:** Let \( x \in S \). Since \( (S, \cdot) \) is p.t.o., using result 3.5.1,

1 is the minimum element

\[ \therefore 1 + 1 \geq 1 \]

\[ \Rightarrow x(1 + 1) \geq x.1 \]

\[ \Rightarrow x + x \geq x \]

i.e., \( (S, +) \) is non-negatively ordered
Note: In the above theorem, if we drop the multiplicative identity, then
(S, +) is not necessarily non-negatively ordered. This is evident from the
following examples.

Example 5.4.3: Let (S, +) be an infinite cyclic semigroup generated by x
adjoined by 0. Define every product to be 0;
0 + 0 = 0, mx + 0 = 0 + mx = mx. (S, +, •) is a totally ordered semiring by
ordering: ........ < 3x < 2x < x < 0.

Example 5.4.4: Let (S, +) be a commutative semigroup generated by x
and y adjoined with 0 subject to the relations
3x = 0 + 0 = x + 0 = 0 + x = y + 0 = 0 + y = 0.
Define the product of every two elements to be 0. (S, +, •) is a totally
ordered semiring by defining the order:

........< 2y < y < x < ..... < ny + x < ......< y + x

< 2x < ..... < ny + 2x < ......< y + 2x
< 3x = 0
Example 5.4.5: The following are the examples in which $(S, \cdot)$ is p.t.o. and $S$ contains multiplicative identity and $(S, +)$ is non-negatively ordered.

(i) $S = \{1/3, 2/3, 1\}$  $1/3 < 2/3 < 1$

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(ii) $\mathbb{N} = \{1, 2, 3, . . . \}$ with usual addition, multiplication and ordering

Theorem 5.4.6: Let $(S, +, \cdot)$ be a t.o.s.r. and $E[+] = \emptyset$. Then $(S, +)$ is either strictly non-negatively ordered or strictly non-positively ordered.

Proof: If $x$ is a non-negatively ordered element and $y$ is a non-positively ordered element, then using theorem 11[30] $xy, yx \in E[+]$, which is a contradiction

$\therefore (S, +)$ is either strictly non-negatively ordered or strictly non-positively ordered.
Lemma 5.4.7: Let \((S, +, \cdot)\) be a t.o.s.r. such that for every \(a \in S\), \(a^2 \geq na\) for some positive integer \(n\). If \((S, +)\) is a band, then \((S, \cdot)\) is non-negatively ordered.

Proof: Since \((S, +)\) is a band

\[
a = a + a = 2a = 3a = \cdots = na
\]

Now \(a^2 \geq na = a\)

\[
\therefore (S, \cdot) \text{ is non-negatively ordered}
\]

Theorem 5.4.8: Let \((S, +, \cdot)\) be a t.o.s.r. such that for every \(a \in S\), \(a^2 \geq na\), for some positive integer \(n\). If \((S, +)\) is a band, then \((S, \cdot)\) is p.t.o. if one of the following conditions is satisfied.

(iv) \((S, \cdot)\) is O-Archimedean

(v) \((S, \cdot)\) is left cancellative without idempotents

(vi) \((S, \cdot)\) is cancellative

Proof: Using lemma 5.4.7, \((S, \cdot)\) is non-negatively ordered

Now using proposition 6[36], \((S, \cdot)\) is p.t.o.
**Theorem 5.4.9:** Let $(S, +, \cdot, \leq)$ be a t.o.s.r. in which $(S, +)$ is o-Archimedean. Then $(S, +)$ is either non-negatively ordered or non-positively ordered.

**Proof:** Suppose $x$ is a non-negatively ordered element and $y$ is a non-positively ordered element.

Using theorem 1[30] $xy, yx \in E [+], which is a contradiction to the hypothesis that $(S, +)$ is o-Archimedean.

$\therefore (S, +)$ is either non-negatively ordered or non-positively ordered.

**Theorem 5.4.10:** Let $(S, +, \cdot)$ be a t.o. zero sum semiring. If $(S, +)$ is non-negatively (non-positively) ordered, then zero is the maximum (minimum) element.

**Proof:** Since $S$ is a zero sum semiring.

$x + x = 0, \forall x \in S$

If $(S, +)$ is non-negatively ordered, then $x + x \geq x, \forall x \in S.$

$\Rightarrow 0 \geq x, \forall x \in S$

If $(S, +)$ is non-positively ordered, then $x + x \leq x, \forall x \in S.$

$\Rightarrow 0 \leq x, \forall x \in S$