## APPENDIX I

The basic equations ( 4.1 ) - ( 4.6 ), ( 4.11 ) and ( 4.10 ) before normalization are

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial z}(n_e v_{ez}) = 0 \qquad (4.i)$$

$$\frac{\partial v_{ez}}{\partial t} + v_{ez}\frac{\partial v_{ez}}{\partial z} = \frac{e}{m_e}\left(\frac{\partial \psi}{\partial z} - \frac{T_e/e}{n_e}\frac{\partial n_e}{\partial z}\right) \qquad (4.ii)$$

for the electrons

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i v_{ix}) + \frac{\partial}{\partial z}(n_i v_{iz}) = 0 \qquad (4.iii)$$

$$\frac{\partial v_{ix}}{\partial t} + v_{ix}\frac{\partial v_{ix}}{\partial x} + v_{iz}\frac{\partial v_{ix}}{\partial z} = -\frac{e}{m_i}\left(\frac{\partial \phi}{\partial x} - B_0 v_{iy}\right) \qquad (4.iv)$$

$$\frac{\partial v_{iy}}{\partial t} + v_{ix}\frac{\partial v_{iy}}{\partial x} + v_{iz}\frac{\partial v_{iy}}{\partial z} = -\frac{eB_0}{m_i}v_{ix} \qquad (4.v)$$

$$\frac{\partial v_{iz}}{\partial t} + v_{ix}\frac{\partial v_{iz}}{\partial x} + v_{iz}\frac{\partial v_{iz}}{\partial z} = -\frac{e}{m_i}\frac{\partial \psi}{\partial z} \qquad (4.vi)$$

for the ions

$$\frac{\partial^4(\phi-\psi)}{\partial x^2 \partial z^2} = \mu_0 e\left[\frac{\partial^2}{\partial t\,\partial z}\left(n_i v_{iz}\right) + \frac{\partial^2 n_e}{\partial t^2}\right], \mu_0 = \frac{4\pi}{c^2} \qquad (4.vii)$$

for the integrated Maxwell's equations ( derived in **chapter 4** ) and

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = -4\pi e(n_i - n_e) \qquad (4.viii)$$

the Poisson equation respectively.

Linearizing equations ( 4.i ) – ( 4.viii ) with

$n_e = n_o + n_{e1}$ , $v_e = v'_e + v_{e1}$, $n_i = n_o + n_{i1}$, $v_{ix} = 0 + v_{ix}$ etc., $\phi = 0 + \phi_1$ , $\psi = 0 + \psi_1$

where $v'_e$ is the initial drift velocity of the electrons in z - direction , we have the set of equations

$$\frac{\partial n_{e1}}{\partial t} + n_o\frac{\partial v_{e1}}{\partial z} + v'_e\frac{\partial n_{e1}}{\partial z} = 0 \qquad (Ap.i)$$

$$n_o\frac{\partial v_{e1z}}{\partial t} + v'_e n_o\frac{\partial v_{e1z}}{\partial z} = \frac{e}{m_e} n_o \frac{\partial \psi_1}{\partial z} - \frac{T_e}{m_e}\frac{\partial n_{e1}}{\partial z} \quad \text{(Ap.ii)}$$

$$\frac{\partial n_{i1}}{\partial t} + n_o\frac{\partial v_{i1x}}{\partial x} + n_o\frac{\partial v_{i1z}}{\partial z} = 0 \quad \text{(Ap.iii)}$$

$$\frac{\partial v_{i1x}}{\partial t} = \frac{e}{m_i}\left(-\frac{\partial \phi_1}{\partial x} + B_o v_{i1y}\right) \quad \text{(Ap.iv)}$$

$$\frac{\partial v_{i1y}}{\partial t} = -\frac{eB_o}{m_i} v_{i1x} \quad \text{(Ap.v)}$$

$$\frac{\partial v_{i1z}}{\partial t} = -\frac{e}{m_i}\frac{\partial \psi_1}{\partial z} \quad \text{( Ap.vi)}$$

$$\frac{\partial^4}{\partial x^2 \partial z^2}(\phi_1 - \psi_1) = \mu_o e\left(n_o\frac{\partial^2 v_{i1z}}{\partial t \partial z} + \frac{\partial^2 n_{e1}}{\partial t^2}\right) \quad \text{(Ap.vii)}$$

$$\frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \psi_1}{\partial z^2} = -4\pi e(n_{i1} - n_{e1}) \quad \text{( Ap.viii)}$$

Assuming disturbances to vary as $e^{i(kr-\omega t)}$ we use the following in the equations (Ap.i) – (Ap.viii),

$$\frac{\partial}{\partial t} \equiv -i\omega, \quad \frac{\partial}{\partial x} \equiv ik_x, \quad \frac{\partial}{\partial z} \equiv ik_z, \quad \frac{\partial^2}{\partial x^2} \equiv i^2k^2_x,$$

$$\frac{\partial^2}{\partial z^2} \equiv i^2k^2_z, \quad \frac{\partial^2}{\partial t^2} \equiv i^2\omega^2.$$

Now

$$\text{( Ap.i )} \Rightarrow \qquad v_{e1z} = \frac{\omega - v'_e k_z}{n_o k_z} n_{e1}$$

$$\text{( Ap.ii )} \Rightarrow \quad n_{e1} = \frac{e n_o k^2_z}{T_e k^2_z - m_e(\omega - v'_e k_z)^2}\psi_1 \quad \text{( Ap.A )}$$

From ( Ap. iv ) and ( Ap.v )

$$v_{i1x} = \frac{ek_x \, \omega m_i}{(m_i^2\omega^2 - e^2B_o^2)}\phi_1 \quad \text{( Ap.B )}$$

From ( Ap.vi ), $v_{iz} = \dfrac{ek_z}{m_i\omega}\,\psi_1$

These values of $v_{ix}$ and $v_{iz}$, with the help of ( Ap.iii ) give

$$n_{i1} = en_0\left[\frac{m_i k^2_x}{(m^2_i\omega^2 - e^2B^2_o)}\phi_1 + \frac{k^2_z}{m_i\omega^2}\psi_1\right] \qquad \text{( Ap.C )}$$

Using ( Ap.B ) and ( Ap.A ) for $v_{iz}$ and $n_{e1}$ in ( Ap.vii), the linearized Maxwell's equations we have

$$\phi_1 = \left[1 + \frac{4\pi e^2 n_0}{c^2 m_i k^2_x} - \frac{4\pi e^2 n_o}{c^2 k^2_x\left\{T_e k^2_z - m_e(\omega - v'_e k_z)^2\right\}}\right]\psi_1 .$$

Using this result and (Ap.A ) and ( Ap.C )for values of $n_{e1}$ and $n_{i1}$ respectively , in ( Ap.viii ), the linearized Poisson's equation, we have

$$k^2_x\left[1 - \frac{4\pi e^2 n_o m_i}{(\omega^2 m^2_i - e^2B^2_o)}\right]\left[1 + \frac{4\pi e^2 n_0}{c^2 m_i k^2_x} - \frac{4\pi e^2 n_o}{c^2 k^2_x\left\{T_e k^2_z - m_e(\omega - v'_e k_z)^2\right\}^2}\right]\psi_1$$

$$= k^2_z\left[\frac{4\pi e^2 n_o}{m_i\omega^2} - \frac{4\pi e^2 n_o}{\left\{T_e k^2_z - m_e(\omega - v'_e k_z)^2\right\}} - 1\right]\psi_1$$

or,

$$k^2_x\left[1 - \frac{4\pi e^2 n_o m_i}{(\omega^2 m^2_i - e^2B^2_o)}\right]\left[1 + \frac{4\pi e^2 n_0}{c^2 m_i k^2_x} - \frac{4\pi e^2 n_o}{c^2 k^2_x\left\{T_e k^2_z - m_e(\omega - v'_e k_z)^2\right\}^2}\right]\psi_1$$

$$= k^2_z\left[\frac{4\pi e^2 n_o}{m_i\omega^2} - \frac{4\pi e^2 n_o}{\left\{T_e k^2_z - m_e(\omega - v'_e k_z)^2\right\}} - 1\right]\psi_1$$

or

$$k^2_x\left[1 - \frac{c^2/v^2_A}{\omega^2/\Omega^2_{ci} - 1}\right]\left[1 + \frac{\omega^2_{pi}}{c^2k^2_x} + \frac{\omega^2_{pe}\omega^2}{c^2k^2_x\left\{(\omega - v'_e k_z)^2 - c^2_s k^2_z/Q\right\}}\right]$$

$$= k^2_z\left[\frac{\omega^2_{pi}}{\omega^2} + \frac{\omega^2_{pe}}{\left\{(\omega - v'_e k_z)^2\ c^2_s k^2_z/Q\right\}} - 1\right]$$

Here

$$v_A = cB_o/4\pi n_o m_i \qquad \Omega_{ci} = eB_o/m_i \qquad c_s^2 = T_e/m_i,$$

$$\omega^2_{pi} = 4\pi e^2 n_o/m_i \qquad \omega^2_{pe} = 4\pi e^2 n_o/m_e \qquad \omega^2_{pi}/\Omega^2_{ci} = c^2/v_A^2$$

$Q = m_e/m_i$ (electron to ion mass ratio).

Further simplifications give

$$k_x^2\left[c^2 - v_A^2\left(\frac{\omega^2}{\Omega^2_{ci}} - 1\right)\right]\left[1 + \frac{\omega^2_{pi}}{c^2k_x^2} + \frac{\omega^2_{pe}\omega^2}{c^2k_x^2\lambda^2}\right]$$

$$= k_z^2 v_A^2\left(\frac{\omega^2}{\Omega^2_{ci}} - 1\right)\left[1 - \frac{\omega^2_{pi}}{\omega^2} - \frac{\omega^2_{pe}}{\lambda^2}\right]$$

where

$$\lambda^2 = \left\{(\omega - v_e' k_z)^2 - c_s^2 k_z^2/Q\right\}$$

or,

$$\left[1 - \frac{v_A^2}{c^2}\left(\cancelto{0}{\frac{\omega^2}{\Omega^2_{ci}}} - 1\right)\right]\omega^2_{pe}\left[\frac{k_x^2}{\omega^2_{pe}} + \cancelto{0}{\frac{Q}{c^2}} + \frac{\omega^2}{c^2\lambda^2}\right]$$

$$= k_z^2\frac{v_A^2}{c^2}\left(\cancelto{0}{\frac{\omega^2}{\Omega^2_{ci}}} - 1\right)\omega^2_{pe}\left[\frac{1}{\omega^2_{pe}} - \cancelto{0}{\frac{Q}{\omega^2}} - \frac{1}{\lambda^2}\right]$$

Under the considerations $\omega \ll \Omega_{ci}$, $v_A < c$, this becomes

$$\left(1 + \frac{v_A^2}{c^2}\right)\left[\frac{k_x^2}{\omega^2_{pe}} + \frac{\omega^2}{c^2\lambda^2}\right] = -k_z^2\frac{v_A^2}{c^2}\left[\frac{1}{\omega^2_{pe}} - \frac{1}{\lambda^2}\right]$$

or,

$$\frac{1}{\lambda^2}\left[\frac{\omega^2}{c^2}\left(1 + \frac{v_A^2}{c^2}\right) - k_z^2\frac{v_A^2}{c^2}\right] = -\frac{1}{\omega^2_{pe}}\left[k_z^2\frac{v_A^2}{c^2} + k_x^2\left(1 + \frac{v_A^2}{c^2}\right)\right]$$

or,

$$k_z^2 v_A^2 - \omega^2\left(1 + \frac{v_A^2}{c^2}\right) = \frac{\lambda^2 c^2}{\omega^2_{pe}}\left[k_z^2\frac{v_A^2}{c^2} + k_x^2\left(1 + \frac{v_A^2}{c^2}\right)\right].$$

This is the linear dispersion relation for Kinetic Alfvén waves in cold plasma.