4.1 Introduction

In the chapter-3 we discussed about fuzzification of system of linear equations and of the simplex method in details. In this chapter we are going to consider one well-known special linear programming model, namely, transportation model. One of the earliest and most fruitful applications of linear-programming techniques has been the formulation and solution of the transportation problem as a linear-programming problem. The basic transportation problem was originally stated by Hitchcock (1941) and later discussed in detail by Koopman (1949). An earlier approach was given by Kantorovich (1942). The linear programming formulation and the associated systematic method for solution were first given by Dantzig (1951). The computational procedure is an adaptation of the simplex method applied to the system of equations of the associated linear-programming problem.

In this chapter we shall first discuss the balanced transportation problem, and then develop in terms of the revised simplex method, computational procedure for solving the problem (see e.g. Gass, p-317). Though this problem can be solved by using the simplex method, its special structure allow us to develop simplified algorithm for its solution. This model is not representative of a particular situation but may arise in many physical situations that have nothing to do with transportation. Considering this point of view we have tried to write the model in fuzzified form.

4.2 Balanced Transportation Problem

One of the earliest problems belonging to the discipline of operations research is the so-called classical(balanced) transportation problem. This is a common problem encountered in all
A homogeneous fuzzy product is to be shipped in the amounts \([a_i^{(1)}, a_i^{(2)}, a_i^{(3)}], [a_2^{(1)}, a_2^{(2)}, a_2^{(3)}], \ldots, [a_m^{(1)}, a_m^{(2)}, a_m^{(3)}]\) respectively, from each of \(m\) shipping origins and received in amounts \([b_1^{(1)}, b_1^{(2)}, b_1^{(3)}], [b_2^{(1)}, b_2^{(2)}, b_2^{(3)}], \ldots, [b_n^{(1)}, b_n^{(2)}, b_n^{(3)}]\), respectively, by each of \(n\) shipping destinations. The cost of shipping a unit amount from the \(i\)th origin to the \(j\)th destination is \(C_{ij} = [c_{ij}^{(1)}, c_{ij}^{(2)}, c_{ij}^{(3)}]\) and is known for all combinations \((i, j)\). The problem is to determine the amount \(X_{ij} = [x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}]\) to be shipped over all routes \((i, j)\) so as to minimize the total cost.

To develop the constraints of the problem, we set up Table-a. The amount shipped from origin \(i\) to destination \(j\) is \(X_{ij}\); the total shipped from origin \(i\) is \(A_i = [a_i^{(1)}, a_i^{(2)}, a_i^{(3)}] \geq [-\delta, 0, \delta]\), and the total received by destination \(j\) is \(B_j = [b_j^{(1)}, b_j^{(2)}, b_j^{(3)}] \geq [-\delta, 0, \delta]\). Here we temporarily impose the restriction that the total fuzzy amount shipped is equal to the total fuzzy amount received; that is, \(\sum_{i=1}^{m} A_i = \sum_{j=1}^{n} B_j = A\). The total fuzzy cost of shipping \(X_{ij}\) units is \(C_{ij}X_{ij}\).

Since a negative shipment has no valid interpretation for the problem as stated, we restrict each \(X_{ij} \geq [-\delta, 0, \delta]\). From the table we have the mathematical statement of the transportation problem:

Find values for the variables \(X_{ij}\) which minimize the total cost

\[
Z = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij}X_{ij} \quad \text{(here \(Z = [z^{(1)}, z^{(2)}, z^{(3)}]\))} \quad (4.2.1)
\]

Subject to the constraints

\[
\sum_{j=1}^{n} X_{ij} = A_i, \quad i = 1, 2, \ldots, m \quad (4.2.2)
\]

\[
\sum_{i=1}^{m} X_{ij} = B_j, \quad j = 1, 2, \ldots, n \quad (4.2.3)
\]

and \(X_{ij} \geq [-\delta, 0, \delta]\) for all \(i\) and \(j\) \quad (4.2.4)

Equations (4.2.2) represent the row sums of Table-a and (4.2.3) the column sums. In order for equations (4.2.2) and (4.2.3) to be consistent, we must have the sum of equations (4.2.2) equal to the sum of equations (4.2.3); that is,

\[
\sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij} = \sum_{i=1}^{m} A_i = \sum_{j=1}^{n} B_j \quad (4.2.5)
\]

It should be noted that the system of equations (4.2.1) to (4.2.4) is a linear programming problem with \(m + n\) equations in \(mn\) variables (see e.g. Gass, p. 319-321).
Let us assume that the consistency condition (4.2.5) holds. Then we term the transportation problem as balanced. The problem is balanced in the sense that the total supply is equal to the total demand. It is clear from constraints (4.2.2), (4.2.3) and (4.2.4) that every component $x_{ij}$ of a fuzzy feasible solution vector $X$ is bounded, i.e.,

$$[-\delta, 0, \delta] \subseteq [x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}] \subseteq \min ([a_{1i}^{(1)}, a_{1i}^{(2)}, a_{1i}^{(3)}], [b_{ji}^{(1)}, b_{ji}^{(2)}, b_{ji}^{(3)}]).$$

Thus, the fuzzy feasible region of the problem is closed, bounded and fuzzy non-empty. Hence, there always exists a fuzzy optimal solution to the balanced fuzzy transportation problem.

Constraints (4.2.2) and (4.2.3) can be written in the matrix form as $AX = B$ with

$$X = (X_{11}, X_{12}, \ldots, X_{1n}, X_{21}, \ldots, X_{2n}, \ldots, X_{mn})^T, \quad B = (A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_n)^T,$$

and $A$ as an $(m + n) \times mn$ matrix given by

$$A = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
I & I & I & \cdots & I
\end{bmatrix},$$

where $I$ is the $1 \times n$ matrix with all the components as 1 and $I$ is the $n \times n$ identity matrix. Since the sum of $m$ equations (4.2.2) equals the sum of $n$ equations (4.2.3), the $(m + n)$ rows of $A$ are linearly dependent. This means rank $(A) \leq m + n - 1$.

The fuzzy transportation model (4.2.1)-(4.2.4) has some special structure which enables us to represent it in the form of a rectangular array, called the transportation array (see Table-a). In this array, each of the mn cells corresponds to a fuzzy variable; each row corresponds to one of the $m$ constraints (4.2.2), and each column corresponds to one of the $n$ constraints (4.2.3). Also, the $(i, j)$-th cell situated at the intersection of the $i$-th row and the $j$-th column contains cost $C_{ij}$ and variable $X_{ij}$ (see e.g., N. S. KAMBO (1997), p. 159-161). The cells in the transportation array can be classified as occupied cells and unoccupied cells. The allocated cells in the transportation array are called occupied cells and empty cells are called unoccupied cells.

(see e.g. Kalavathy, 2000, p. 125 for the table.)
Transportation array

<table>
<thead>
<tr>
<th>Table-a Destinations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2 3</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>m</td>
</tr>
</tbody>
</table>

Demand | B1 | B2 | B3 | Bn | Σi=1m Ai = Σj=1n Bj

Two basic theorems

Theorem 1. The transportation problem has a feasible solution.

Theorem 2. The transportation problem constructs a basic feasible solution. A solution of at most \( m + n - 1 \) positive \( x_{ij} \)'s exist.

4.3 Solution of a Fuzzy Transportation Problem

The solution of a fuzzy transportation problem can be obtained in two stages, namely initial solution and optimal solution.

4.3(a) Initial Basic Feasible Solution

There are methods like North West Corner Rule (NWCR), Least Cost method or Matrix Minima method, Vogel's Approximation Method (VAM) etc., for finding initial solution of a
transportation problem. VAM is preferred over the other methods, since the initial basic feasible solution obtained by this method is either optimal or very close to the optimal solution. We are going to discuss here only VAM and see effect of fuzziness on it.

4.3(a)(i) Vogel’s Approximation Method (VAM) in Fuzzified Form

The steps involved in this method for finding the fuzzy initial solution are as follows (See e.g. Kalavathy 2000, pages 133-136):

Step 1 We are to find the penalty cost namely the difference between the smallest and next smallest costs in each row and column.

Step 2 Among the penalties as found in step (1), the maximum penalty is to be chosen. If this maximum penalty is more than one (i.e. if there is a tie) then any one can be chosen arbitrarily.

Step 3 In the selected row or column as by step (2) we are to find out the cell having the least cost. Allocate to this cell as much as possible depending on the capacity and requirements.

Step 4 Next step is to delete the row or column which is fully fuzzy exhausted. Again, compute the column and row penalties for the reduced transportation table and then go to step (2). Repeat the procedure until all the requirements are fulfilled.

Note: If the column is exhausted, then there is a change in row penalty and vice versa.

4.3(a)(ii) A Numerical Example

We are now going to obtain the fuzzy initial basic feasible solution by VAM of a transportation problem whose cost and requirement table is given below.
Here $\sum A_i = [949.85, 950, 950.15]$ and $\sum B_j = [949.80, 950, 950.20]$. Since $\Sigma A_i$ and $\Sigma B_j$ are fuzzy equal (i.e., differing by fuzzy zero, [-.35, 0, .35]) the problem is fuzzy balanced and there exists a fuzzy feasible solution to the problem.

First we find the row and column penalty $P_i$ as the difference between the fuzzy least and next fuzzy least cost. In our problem the maximum penalty is $[4.9, 5, 5.1]$ (see 1 allocation). So, in the first allocation we choose the first column arbitrarily as there are two maximum penalties. In this column the cell having the fuzzy least cost is (1, 1). Allocate to this cell with minimum magnitude (i.e., $\min([249.95, 250, 250.05], [199.95, 200, 200.05]) = [199.95, 200, 200.05]$). This exhausts the first column by fuzzy zero, $[-.1, 0, .1]$ and supply is reduced to $([249.95, 250, 250.05](-)[199.95, 200, 200.05]=) [49.9, 50, 50.1]$.

<table>
<thead>
<tr>
<th>Destination</th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>$D_4$</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>Origin O1</td>
<td>[10.95, 11, 11.05]</td>
<td>[12.95, 13, 13.05]</td>
<td>[16.95, 17, 17.05]</td>
<td>[13.95, 14, 14.05]</td>
<td>[249.95,250,250.05]</td>
</tr>
<tr>
<td>Origin O2</td>
<td>[15.95, 16, 16.05]</td>
<td>[17.95, 18, 18.05]</td>
<td>[13.95, 14, 14.05]</td>
<td>[9.95, 10, 10.05]</td>
<td>[299.95,300,300.05]</td>
</tr>
<tr>
<td>Origin O3</td>
<td>[20.95, 21, 21.05]</td>
<td>[23.95, 24, 24.05]</td>
<td>[12.95, 13, 13.05]</td>
<td>[9.95, 10, 10.05]</td>
<td>[399.95,400,400.05]</td>
</tr>
<tr>
<td>Demand</td>
<td>[199.95,200, 200.05]</td>
<td>[224.95,225, 225.05]</td>
<td>[274.95,275, 275.05]</td>
<td>[249.95,250, 250.05]</td>
<td>[949.85,950,950.15]</td>
</tr>
</tbody>
</table>

(row sum) $[949.80, 950, 950.20]$ (column sum)
I allocation

<table>
<thead>
<tr>
<th></th>
<th>D₁</th>
<th>D₂</th>
<th>D₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>O₁</td>
<td>[10.95, 11, 11.05]</td>
<td>[12.95, 13, 13.05]</td>
<td>[16.95, 17, 17.05]</td>
</tr>
<tr>
<td></td>
<td>(199.95, 200, 200.05)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>O₂</td>
<td>[15.95, 16, 16.05]</td>
<td>[17.95, 18, 18.05]</td>
<td>[13.95, 14, 14.05]</td>
</tr>
<tr>
<td>O₃</td>
<td>[20.95, 21, 21.05]</td>
<td>[23.95, 24, 24.05]</td>
<td>[12.95, 13, 13.05]</td>
</tr>
<tr>
<td>Demand</td>
<td>[199.95, 200, 200.05]</td>
<td>[224.95, 225, 225.05]</td>
<td>[274.95, 275, 275.05]</td>
</tr>
<tr>
<td>P₁</td>
<td>[4.9, 5, 5.1]</td>
<td>[4.9, 5, 5.1]</td>
<td>[9, 1, 1.1]</td>
</tr>
</tbody>
</table>

So we delete the first column. Since column is deleted there is a change in row penalty \(P_{II}\) in the II allocation

<table>
<thead>
<tr>
<th></th>
<th>D₂</th>
<th>D₃</th>
<th>D₄</th>
<th>Supply</th>
<th>P₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>O₁</td>
<td>[12.95, 13, 13.05]</td>
<td>[16.95, 17, 17.05]</td>
<td>[13.95, 14, 14.05]</td>
<td>[49.9, 50, 50.1]</td>
<td>[1.9, 2, 2.1]</td>
</tr>
<tr>
<td></td>
<td>(49.9, 50, 50.1)</td>
<td></td>
<td></td>
<td>[49.9, 50, 50.1]</td>
<td>[-2, 0, 0.2]</td>
</tr>
<tr>
<td>O₂</td>
<td>[17.95, 18, 18.05]</td>
<td>[13.95, 14, 14.05]</td>
<td>[9.95, 10, 10.05]</td>
<td>[299.95, 300, 300.05]</td>
<td>[3.9, 4, 4.1]</td>
</tr>
<tr>
<td>O₃</td>
<td>[23.95, 24, 24.05]</td>
<td>[12.95, 13, 13.05]</td>
<td>[9.95, 10, 10.05]</td>
<td>[399.95, 400, 400.05]</td>
<td>[2.9, 3, 3.1]</td>
</tr>
<tr>
<td>Demand</td>
<td>[224.95, 225, 225.05]</td>
<td>[274.95, 275, 275.05]</td>
<td>[249.999, 250, 250.001]</td>
<td>[949.999, 950, 950.001]</td>
<td></td>
</tr>
<tr>
<td>P₄</td>
<td>[4.9, 5, 5.1]</td>
<td>[9, 1, 1.1]</td>
<td>[-1, 0, 0.1]</td>
<td>[-1, 0, 0.1]</td>
<td></td>
</tr>
</tbody>
</table>
second allocation and column penalty $P_{II}$ remains the same. In this second allocation maximum penalty is $[4.9, 5, 5.1]$. The cell having fuzzy least cost is again (1, 1). We allocate to this cell with the remaining magnitude $[49.9, 50, 50.1]$. This exhausts the first row by fuzzy zero $[-.2, 0, .2]$. So we delete the first row. The demand remained is $([224.95, 225, 225.05]) - [9.9, 50, 50.1] = [174.85, 175, 175.15]$. There is a change in column penalty $P_{III}$ in the third allocation and row penalty $P_{III}$ remains the same.

### III allocation

<table>
<thead>
<tr>
<th>O2</th>
<th>D2</th>
<th>D3</th>
<th>D4</th>
<th>Supply</th>
<th>P_{III}</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[17.95, 18,</td>
<td>[13.95, 14,</td>
<td>[9.95, 10,</td>
<td>[299.95,300,300.05]</td>
<td>[3.9, 4,</td>
</tr>
<tr>
<td></td>
<td>18.05]</td>
<td>14.05]</td>
<td>10.05]</td>
<td>[124.80,125,125.20]</td>
<td>4.1]</td>
</tr>
<tr>
<td>O3</td>
<td>[23.95, 24,</td>
<td>[12.95, 13,</td>
<td>[9.95, 10,</td>
<td>[399.95,400,400.05]</td>
<td>[2.9, 3,</td>
</tr>
<tr>
<td></td>
<td>24.05]</td>
<td>13.05]</td>
<td>10.05]</td>
<td>[124.80,125,125.20]</td>
<td>3.1]</td>
</tr>
<tr>
<td>Demand</td>
<td>[174.85, 175,</td>
<td>[274.95,275,</td>
<td>[249.95,250,</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>175.15]</td>
<td>275.05]</td>
<td>250.05]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>P_{III}</td>
<td>[5.9, 6, 6.1]</td>
<td>[9, 1, 1.1]</td>
<td>[-.1, 0,.1]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In this third allocation the maximum penalty is $[5.9, 6, 6.1]$. The cell having fuzzy least cost is (1, 1). Allocate to this cell the magnitude $[174.85, 175, 175.15]$. This exhausts the second IV allocation

<table>
<thead>
<tr>
<th>O2</th>
<th>D3</th>
<th>D4</th>
<th>Supply</th>
<th>P_{IV}</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[13.95, 14,</td>
<td>[9.95, 10,</td>
<td>[124.80,125,125.20]</td>
<td>[3.9, 4, 4.1]</td>
</tr>
<tr>
<td></td>
<td>14.05]</td>
<td>10.05]</td>
<td>[124.80,125,125.20]</td>
<td></td>
</tr>
<tr>
<td>O3</td>
<td>[12.95, 13,</td>
<td>[9.95, 10,</td>
<td>[399.95,400,400.05]</td>
<td>[2.9, 3, 3.1]</td>
</tr>
<tr>
<td></td>
<td>13.05]</td>
<td>10.05]</td>
<td>[124.80,125,125.20]</td>
<td></td>
</tr>
<tr>
<td>Demand</td>
<td>[274.95,275,</td>
<td>[249.95,250,</td>
<td>[124.75, 125, 125.25]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>275.05]</td>
<td>250.05]</td>
<td>[124.75, 125, 125.25]</td>
<td></td>
</tr>
<tr>
<td>P_{IV}</td>
<td>[9, 1, 1.1]</td>
<td>[9, 1, 1.1]</td>
<td>[-.1, 0,.1]</td>
<td></td>
</tr>
</tbody>
</table>
column \( D_2 \) by fuzzy zero \([-0.3, 0, 0.3]\) and supply remained is \([299.95, 300, 300.05]\)\([\rightarrow]\)[174.85, 175, 175.15] = \([124.80, 125, 125.20]\). There is no change in penalties \( P_{IV} \). In this fourth allocation maximum penalty is the row penalty \([3.9, 4, 4.1]\). The cell having fuzzy least cost is \((1, 2)\). Allocate to this cell with minimum magnitude (i.e., \([124.80, 125, 125.20]\), \([249.95, 250, 250.05]\)) = \([124.80, 125, 125.20]\)). This exhausts the second row \( O_2 \) by fuzzy zero \([-0.4, 0, 0.4]\) and the remaining demand is \([249.95, 250, 250.05]\)\([\rightarrow]\)[124.80, 125, 125.20] = \([124.75, 125, 125.25]\). There is change now in column penalty \( P_y \).

\( V \) allocation

<table>
<thead>
<tr>
<th></th>
<th>( D_3 )</th>
<th>( D_4 )</th>
<th>Supply</th>
<th>( P_y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( O_3 )</td>
<td>[12.95, 13, 13.05]</td>
<td>[9.95, 10, 10.05]</td>
<td>[399.95, 400, 400.05]</td>
<td>[2.998, 3.002]</td>
</tr>
<tr>
<td>Demand</td>
<td>[274.95, 275, 275.05]</td>
<td>[124.75, 125, 125.25]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( P_y )</td>
<td>[12.95, 13, 13.05]</td>
<td>[9.95, 10, 10.05]</td>
<td>[9.95, 10, 10.05]</td>
<td></td>
</tr>
</tbody>
</table>

In this fifth allocation the maximum penalty is \([12.95, 13, 13.05]\). The remaining demand is allocated to the cell \((1, 1)\). Then the column \( D_3 \) is exhausted by fuzzy zero \([-0.1, 0, 0.1]\). The remaining supply is \([399.95, 400, 400.05]\)\([\rightarrow]\)[274.95, 275, 275.05] = \([124.9, 125, 125.1]\). There is a change now in row penalty \( P_{VI} \).

In the sixth allocation we allocate the last remaining supply \([124.9, 125, 125.1]\) to column \( D_4 \).

\( VI \) allocation

<table>
<thead>
<tr>
<th></th>
<th>( D_4 )</th>
<th>Supply</th>
<th>( P_{VI} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( O_3 )</td>
<td>[9.999, 10, 10.001]</td>
<td>[124.9, 125, 125.1]</td>
<td>[9.95, 10, 10.05]</td>
</tr>
<tr>
<td>Demand</td>
<td>[124.75, 125, 125.25]</td>
<td>[-0.2, 0, 0.2]</td>
<td>[9.95, 10, 10.05]</td>
</tr>
<tr>
<td>( P_{VI} )</td>
<td>[9.95, 10, 10.05]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**NB:** we have used here a cut symbol to mean that the particular amount in that row or column is change by the amount below.
We have seen that all the requirements are fulfilled and in this sixth allocation demand and supply are differed only by fuzzy zero \([-0.35, 0, 0.35]\)\([-0.2, 0, 0.2]=\) \([-0.55, 0, 0.55]\). Finally we arrive at the initial basic feasible solution which is shown in the table below:

**Table-2**

<table>
<thead>
<tr>
<th></th>
<th>(D_1)</th>
<th>(D_2)</th>
<th>(D_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(O_1)</td>
<td>([10.95, 11, 11.05]) ([199.95, 200, 200.05])</td>
<td>([12.95, 13, 13.05]) ([49.9, 50, 50.1])</td>
<td>([16.95, 17, 17.05])</td>
</tr>
<tr>
<td>(O_2)</td>
<td>([15.95, 16, 16.05]) ([174.85, 175, 175.15])</td>
<td>([17.95, 18, 18.05]) ([13.95, 14, 14.05])</td>
<td>()</td>
</tr>
<tr>
<td>(O_3)</td>
<td>([20.95, 21, 21.05]) ([274.95, 275, 275.05])</td>
<td>([23.95, 24, 24.05]) ([12.95, 13, 13.05])</td>
<td>()</td>
</tr>
<tr>
<td>Demand</td>
<td>([199.95, 200, 200.05]) ([224.95, 225, 225.05])</td>
<td>([274.95, 275, 275.05]) ([)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(D_4)</th>
<th>Supply</th>
</tr>
</thead>
<tbody>
<tr>
<td>([13.95, 14, 14.05]) ([124.80, 125, 125.20])</td>
<td>([249.95, 250, 50.05])</td>
</tr>
<tr>
<td>([9.95, 10, 10.05]) ([124.75, 125, 125.25])</td>
<td>([299.95, 300, 300.05])</td>
</tr>
<tr>
<td>([9.95, 10, 10.05]) ([124.75, 125, 125.25])</td>
<td>([399.95, 400, 400.05])</td>
</tr>
<tr>
<td>([249.95, 250, 250.05])</td>
<td>()</td>
</tr>
</tbody>
</table>

There are 6 positive independent allocations which equals to \(m + n - 1 = 3 + 4 - 1\). This ensures that the solution is a fuzzy non-degenerate basic feasible solution.

**The transportation cost** = \(C_{i1}D_{11} + C_{12}D_{12} + C_{22}D_{22} + C_{33}D_{33} + C_{24}D_{24} + C_{34}D_{34}\)

\[
= [10.95, 11, 11.05]\)[199.95, 200, 200.05][12.95, 13, 13.05][49.9, 50, 50.1][17.95, 18, 18.05][174.85, 175, 175.15][9.95, 10, 10.05][124.8, 125, 125.2][12.95, 13, 13.05][274.95, 275, 275.05][9.95, 10, 10.05][124.75, 125, 125.25]
\]

\(= Rs[12017.84, 12075, 12132.24]\) (4.3.1)

Here \(C_{ij}\) are cost coefficients and \(D_{ij}\) \((i = 1,2,3; j = 1,2,3,4)\) are allocations.
We now attempt to find fuzzy membership function \( (f.m.f) \) of \( C_g \) and \( D_g \) and then of transportation cost as follows:

\[
\mu_{C_1}(x) = \begin{cases} 
(x - 10.95)/.05, & 10.95 \leq x \leq 11 \\
-x + 11.05)/.05, & 11 \leq x \leq 11.05 \\
0, & \text{otherwise}
\end{cases}
\]

Let \( C_{1\alpha} \) be the interval of confidence for the level of presumption \( \alpha \), \( \alpha \in [0, 1] \).

\[
C_{1\alpha} = [c_1^{(\alpha)}, c_2^{(\alpha)}] = [.05\alpha + 10.95, -.05\alpha + 11.05]
\]

\[
\mu_{D_1}(x) = \begin{cases} 
(x - 199.95)/.05, & 199.95 \leq x \leq 200 \\
-x + 200.05)/.05, & 200 \leq x \leq 200.05 \\
0, & \text{otherwise}
\end{cases}
\]

Let \( D_{1\alpha} \) be the interval of confidence for the level of presumption \( \alpha \), \( \alpha \in [0, 1] \).

\[
D_{1\alpha} = [d_1^{(\alpha)}, d_2^{(\alpha)}] = [.05\alpha + 199.95, -.05\alpha + 200.05]
\]

\[
C_{1\alpha} (.) D_{1\alpha} = [(-.05\alpha + 12.95)(.05\alpha + 49.9), (-.05\alpha + 13.05)(-.05\alpha + 50.1)]
\]

Similarly we find,

\[
\mu_{C_2}(x) = \begin{cases} 
(x - 12.95)/.05, & 12.95 \leq x \leq 13 \\
-x + 13.05)/.05, & 13 \leq x \leq 13.05 \\
0, & \text{otherwise}
\end{cases}
\]

\[
C_{2\alpha} = [.05\alpha + 12.95, -.05\alpha + 13.05]
\]

\[
\mu_{D_2}(x) = \begin{cases} 
(x - 49.9)/.1, & 49.9 \leq x \leq 50 \\
-x + 50.1)/.1, & 50 \leq x \leq 50.1 \\
0, & \text{otherwise}
\end{cases}
\]

\[
D_{2\alpha} = [.1\alpha + 49.9, -.1\alpha + 50.1]
\]

\[
C_{2\alpha} (.) D_{2\alpha} = [(.05\alpha + 12.95)(.1\alpha + 49.9), (-.05\alpha + 13.05)(-.1\alpha + 50.1)]
\]

\[
\mu_{C_{22}}(x) = \begin{cases} 
(x - 17.95)/.05, & 17.95 \leq x \leq 18 \\
-x + 18.05)/.05, & 18 \leq x \leq 18.05 \\
0, & \text{otherwise}
\end{cases}
\]

\[
C_{22\alpha} = [.05\alpha + 17.95, -.05\alpha + 18.05]
\]

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\[
\mu_{D_{22}}(x) = \begin{cases} 
(x - 174.85)/.15, & 174.85 \leq x \leq 175 \\
(-x + 175.15)/.15, & 175 \leq x \leq 175.15 \\
0, & \text{otherwise}
\end{cases}
\]

\[
D_{22a} = [.15\alpha + 174.85, -.15\alpha + 175.15] \tag{viii}
\]

\[
C_{22a} \cdot D_{22a} = [(.05\alpha + 17.95)(.15\alpha + 174.85), (-.05\alpha + 18.05)(-.15\alpha + 175.15)]
\]

\[
= [.0075\alpha^2 + 11.435\alpha + 3138.5575, .0075\alpha^2 - 11.465\alpha + 3161.4575] \tag{ix}
\]

\[
\mu_{C_{24}}(x) = \begin{cases} 
(x - 9.95)/.05, & 9.95 \leq x \leq 10 \\
(-x + 10.05)/.05, & 10 \leq x \leq 10.05 \\
0, & \text{otherwise}
\end{cases}
\]

\[
C_{24a} = [.05\alpha + 9.95, -.05\alpha + 10.05] \tag{x}
\]

\[
\mu_{D_{24}}(x) = \begin{cases} 
(x - 124.8)/.2, & 124.8 \leq x \leq 125 \\
(-x + 125.2)/.2, & 125 \leq x \leq 125.2 \\
0, & \text{otherwise}
\end{cases}
\]

\[
D_{24a} = [.2\alpha + 124.8, -.2\alpha + 125.2] \tag{xii}
\]

\[
C_{24a} \cdot D_{24a} = [(.05\alpha + 9.95)(.2\alpha + 124.8), (-.05\alpha + 10.05)(-.2\alpha + 125.2)]
\]

\[
= [.01\alpha^2 + 8.23\alpha + 1241.76, .01\alpha^2 - 8.27\alpha + 1258.26] \tag{xii}
\]

\[
\mu_{C_{33}}(x) = \begin{cases} 
(x - 12.95)/.05, & 12.95 \leq x \leq 13 \\
(-x + 13.05)/.05, & 13 \leq x \leq 13.05 \\
0, & \text{otherwise}
\end{cases}
\]

\[
C_{33a} = [.05\alpha + 12.95, -.05\alpha + 13.05] \tag{xi}
\]

\[
\mu_{D_{33}}(x) = \begin{cases} 
(x - 274.95)/.05, & 274.95 \leq x \leq 275 \\
(-x + 275.05)/.05, & 275 \leq x \leq 275.05 \\
0, & \text{otherwise}
\end{cases}
\]

\[
D_{33a} = [.05\alpha + 274.95, -.05\alpha + 275.05] \tag{xiv}
\]

\[
C_{33a} \cdot D_{33a} = [(.05\alpha + 12.95)(.05\alpha + 274.95), (-.05\alpha + 13.05)(-.05\alpha + 275.05)]
\]

\[
= [.0025\alpha^2 + 14.395\alpha + 3560.6025, .0025\alpha^2 - 14.405\alpha + 3589.4025] \tag{xv}
\]

\[
\mu_{C_{36}}(x) = \mu_{C_{26}}(x)
\]

Therefore \( C_{34a} = C_{24a} \)

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\[
\mu_{D_36}(x) = \begin{cases} 
(x - 124.75)/.25, & 124.75 \leq x \leq 125 \\
(-x + 125.25)/.25, & 125 \leq x \leq 125.25 \\
0, & \text{otherwise}
\end{cases}
\]

\[
D_{34a} = [.25\alpha + 124.75, -.25\alpha + 125.25]
\]

\[
C_{34a}(\cdot) D_{34a} = [(0.05\alpha + 9.95)(0.25\alpha + 124.75), (-0.05\alpha + 10.05)(-0.25\alpha + 125.25)]
\]

\[
= [.0125\alpha^2 + 8.725\alpha + 1241.2625, .0125\alpha^2 - 8.775\alpha + 1258.7625]
\]

We can now write
\[
\text{Cost}_a = C_{11a}(\cdot) D_{11a} (\cdot) + C_{12a}(\cdot) D_{12a} (\cdot) + C_{22a}(\cdot) D_{22a} (\cdot) + C_{24a}(\cdot) D_{24a} (\cdot) + C_{33a}(\cdot) D_{33a} (\cdot)
\]

\[
\text{Cost}_a = [.04\alpha^2 + 57.12\alpha + 12017.84, .04\alpha^2 - 57.28\alpha + 12132.24]
\]

The equations to be solved are:
\[
.04\alpha^2 + 57.12\alpha + 12017.84 - x_1 = 0 \quad (I)
\]
\[
.04\alpha^2 - 57.28\alpha + 12132.24 - x_2 = 0 \quad (II)
\]

We are to retain only two roots in [0, 1].

From (I) we get,
\[
\alpha = \frac{-57.12 + \sqrt{((57.12)^2 - 4 \times 0.04 (12017.84 - x_1))}}{2 \times 0.04}
\]

and from (II) we get,
\[
\alpha = \frac{57.28 - \sqrt{((-57.28)^2 - 4 \times 0.04 (12132.24 - x_2))}}{2 \times 0.04}
\]

\[
\therefore \quad \mu_{\text{Cost}}(x) = \begin{cases} 
(-57.12 + \sqrt{((57.12)^2 - 4 \times 0.04 (12017.84 - x))})/ (2 \times 0.04), & 12017.84 \leq x \leq 12075 \\
(57.28 - \sqrt{((-57.28)^2 - 4 \times 0.04 (12132.24 - x))})/ (2 \times 0.04), & 12075 \leq x \leq 12132.24 \\
0, & \text{otherwise}
\end{cases}
\]

which is the required membership function of fuzzy transportation cost (using (9)).
4.3(b) Optimality Test

Once the fuzzy initial basic feasible solution has been completed, the next step in the problem is to determine whether the solution obtained is fuzzy optimum or not.

Optimality test can be conducted to any initial basic feasible solution of a transportation problem provided such allocations have exactly $m + n - 1$ non-negative allocations, where $m$ is the origin and $n$ is the number of destinations. Also these allocations must be in independent positions.

To perform fuzzy optimality test, we are going to discuss the Modified Distribution Method (MODI) (see e.g., Kalavathy p. 136-137) in fuzzified form. The various steps involved in MODI method for performing fuzzy optimality test are given below.

4.3(b)(i) MODI Method

**Step 1** Find the fuzzy initial basic feasible solution of a fuzzy transportation problem by using any one of the methods for initial solution.

**Step 2** Find out a set of numbers $U_i = [u_i^{(1)}, u_i^{(2)}, u_i^{(3)}]$ and $V_j = [v_j^{(1)}, v_j^{(2)}, v_j^{(3)}]$ for each row and column satisfying $U_i (+) V_j = C_{ij}$ for each occupied cell. To start with we assign a number 'fuzzy zero' (here it is $[-.05, 0, .05]$) to any row or column having maximum number of allocations. If this maximum number of allocations is more than one, we are to choose any one arbitrarily.

**Step 3** For each empty (unoccupied) cell, we find the sum $U_i$ and $V_j$ written in the bottom left corner of that cell.

**Step 4** Find out for each empty cell the net evaluation value $\Delta_{ij} = C_{ij}(-)(U_i(+)+V_j)$ and it is to be written at the bottom right corner of that cell.

This step gives the optimality conclusion:

(i) If all $\Delta_{ij} > [-\delta, 0, \delta]$, the solution is fuzzy optimum and a fuzzy unique solution exists.

(ii) If $\Delta_{ij} \geq [-\delta, 0, \delta]$, then the solution is fuzzy optimum, but an alternate solution exists.

(iii) If at least one $\Delta_{ij} < [-\delta, 0, \delta]$, the solution is not fuzzy optimum. In this case we go to the next step, to improve the total transportation cost.

**Step 5** Select the empty cell having the most negative value of $\Delta_{ij}$. From this cell we draw a closed path by drawing horizontal and vertical lines with the corner cells occupied.
Assign sign + and − alternately and find the minimum allocation from the cell having negative sign.

This allocation should be added to the allocation having positive sign and subtracted from the allocation having negative sign.

**Step 6** The above step yield a better solution by making one (or more) occupied cell as empty and one empty cell as occupied. For this new set of basic feasible allocations repeat from the step (2) till an optimum basic feasible solution is obtained.

4.3(b)(ii) To Find the Optimal Solution of the problem taken into consideration

We apply the MODI method in the example taken above in order to determine the fuzzy optimal solution. We determine a set of numbers $U_i$ and $V_j$ for each row and column of Table-2 with $U_i (+) V_j = C_{ij}$ for each occupied cell. To start with we give $U_1 = [-.05, 0, .05]$ arbitrarily.

From the occupied cells we find,

- $V_1 = C_{11}(-)U_1 = [10.95, 11, 11.05][-0.05, 0, .05] = [10.9, 11, 11.1]$
- $V_2 = C_{12}(-)U_1 = [12.95, 13, 13.05][-0.05, 0, .05] = [12.9, 13, 13.1]$
- $U_2 = C_{22}(-)V_2 = [17.95, 18, 18.05][-0.05, 0, .05] = [17.9, 18, 18.1]$
- $V_4 = C_{24}(-)U_2 = [9.95, 10, 10.05][-0.05, 0, .05] = [9.8, 10, 10.1]$
- $U_3 = C_{34}(-)V_4 = [9.95, 10, 10.05][-0.05, 0, .05] = [9.8, 10, 10.1]$
- $V_3 = C_{33}(-)U_3 = [12.95, 13, 13.05][-0.05, 0, .05] = [12.8, 13, 13.1]$

Next we find the sum $U_i$ and $V_j$ for each empty cell and enter at the bottom left corner of that cell which are as follows:

- $U_1 (+) V_3 = [7.65, 8, 8.35]$
- $U_1 (+) V_4 = [4.75, 5, 5.25]$
- $U_2 (+) V_1 = [15.75, 16, 16.25]$
- $U_2 (+) V_3 = [12.55, 13, 13.45]$
- $U_3 (+) V_1 = [15.65, 16, 16.35]$
- $U_3 (+) V_2 = [17.65, 18, 18.35]$

Next we find the net evaluations $\Delta_{ij} = C_{ij}(-)(U_i(+))V_j$ for each unoccupied cell and enter at the bottom right corner of that cell which are as follows:

- $\Delta_{13} = C_{13}(-)(U_1(+)V_3) = [8.6, 9, 9.4]$
\[ \Delta_{14} = C_{14}(-)(U_1(+)V_4) = [8.7, 9, 9.3] \]
\[ \Delta_{21} = C_{21}(-)(U_2(+)V_1) = [-.3, 0, .3] \]
\[ \Delta_{23} = C_{23}(-)(U_2(+)V_3) = [.5, 1, 1.5] \]
\[ \Delta_{31} = C_{31}(-)(U_3(+)V_1) = [4.6, 5, 5.4] \]
\[ \Delta_{32} = C_{32}(-)(U_3(+)V_2) = [5.6, 6, 6.4] \]

We now write the above values in tabular form as follows:

Table-3

<table>
<thead>
<tr>
<th>( D_1 )</th>
<th>( D_2 )</th>
<th>( D_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( O_1 )</td>
<td>[10.95, 11, 11.05]</td>
<td>[12.95, 13, 13.05]</td>
</tr>
<tr>
<td></td>
<td>([199.95, 200, 200.05])</td>
<td>([49.9, 50, 50.1])</td>
</tr>
<tr>
<td>( O_2 )</td>
<td>[15.95, 16, 16.05]</td>
<td>[17.95, 18, 18.05]</td>
</tr>
<tr>
<td></td>
<td>[15.75, 16, 16.25]</td>
<td>[-.3, 0, .3]</td>
</tr>
<tr>
<td>( O_3 )</td>
<td>[20.95, 21, 21.05]</td>
<td>[23.95, 24, 24.05]</td>
</tr>
<tr>
<td></td>
<td>[15.65, 16, 16.35]</td>
<td>[4.65, 5, 5.4]</td>
</tr>
<tr>
<td>( V_j )</td>
<td>[10.9, 11, 11.1]</td>
<td>[12.9, 13, 13.1]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( D_4 )</th>
<th>( U_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>[13.95, 14, 14.05]</td>
<td>[-.05, 0, .05]</td>
</tr>
<tr>
<td>[4.75, 5, 5.25]</td>
<td>[8.7, 9, 9.3]</td>
</tr>
<tr>
<td>[9.95, 10, 10.05]</td>
<td>[4.85, 5, 5.15]</td>
</tr>
<tr>
<td>([124.8, 125, 125.2])</td>
<td></td>
</tr>
<tr>
<td>[9.95, 10, 10.05]</td>
<td>[4.75, 5, 5.25]</td>
</tr>
<tr>
<td>([124.75, 125, 125.25])</td>
<td></td>
</tr>
<tr>
<td>[4.8, 5, 5.2]</td>
<td></td>
</tr>
</tbody>
</table>
Here since all $\Delta_i \geq [-.3, 0, .3]$ the solution is fuzzy optimal but an alternate solution exists as $\Delta_{21} = [-.3, 0, .3]$, a fuzzy zero. Therefore, in this case, the **fuzzy optimal allocation** is given by

$X_{11} = [199.95, 200, 200.05]$, $X_{12} = [49.9, 50, 50.1]$, $X_{22} = [174.85, 175, 175.15]$, $X_{24} = [124.8, 125, 125.2]$, $X_{33} = [274.95, 275, 275.05]$, $X_{34} = [124.75, 125, 125.25]$.

The optimum transportation cost is

$[10.95, 11, 11.05](+) [199.95, 200, 200.05] (+) [12.95, 13, 13.05] (+) [49.9, 50, 50.1] (+) [17.95, 18, 18.05] (+) [174.85, 175, 175.15] (+) [9.95, 10, 10.05] (+) [124.8, 125, 125.2] (+) [12.95, 13, 13.05] (+) [274.95, 275, 275.05] (+) [9.95, 10, 10.05] (+) [124.75, 125, 125.25] = Rs[12017.84, 12075, 12132.24]$ (from Table-3).

### 4.4 Conclusion

Using exactly the same procedure as in the fuzzy initial basic feasible solution we could find fuzzy membership functions of optimality test also for verification of the results.

Similarly we could also attempt to fuzzify *Degeneracy in Transportation Problem* and *Unbalanced Transportation Problem*, though much numerical computations are needed.

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