CHAPTER - 3*

SOLUTION OF A LINEAR PROGRAMMING PROBLEM WITH FUZZY DATA

3.1 Introduction

Programming problems are concerned with the efficient use or allocation of limited resources to meet desired objectives. These problems are characterized by a large number of solutions that satisfy the basic conditions of each problem. The selection of a particular solution as the best solution to a problem depends on some aim or over-all objective that is implied in the statement of the problem. A solution that satisfies both the conditions of the problem and the given objective, is termed as optimal solution. A typical example is that of the manufacturing company that must determine what combination of available resources will enable it to manufacture products in a way which not only satisfies its production schedule, but also maximizes its profit. This problem has as its basic conditions the limitations of the available resources and the requirements of the production schedule, and as its objective the desire of the company to maximize its gain.

We shall mainly consider only a very special subclass of programming problems called fuzzy linear-programming problem. A fuzzy linear-programming problem differs from the general variety in that a mathematical model or description of the problem can be stated using relationships which are called fuzzy straight line or fuzzy linear. Mathematically, these relations are of the form

\[ \begin{bmatrix} a_1^{(1)} & a_1^{(2)} & a_1^{(3)} \\ a_2^{(1)} & a_2^{(2)} & a_2^{(3)} \\ a_3^{(1)} & a_3^{(2)} & a_3^{(3)} \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_1^{(2)} \\ x_1^{(3)} \end{bmatrix} + \begin{bmatrix} a_4^{(1)} \\ a_4^{(2)} \\ a_4^{(3)} \end{bmatrix} \begin{bmatrix} x_2^{(1)} \\ x_2^{(2)} \\ x_2^{(3)} \end{bmatrix} = \begin{bmatrix} b^{(1)} \\ b^{(2)} \\ b^{(3)} \end{bmatrix} \]

where the \( [a_i^{(1)}, a_i^{(2)}, a_i^{(3)}] \)'s and \( [b^{(1)}, b^{(2)}, b^{(3)}] \) are known fuzzy coefficients and the

*The paper entitled “Solution of a Linear Programming Problem Using Fuzzy Data” has been published in “The Journal of Fuzzy Mathematics (Vol. 12, No. 4, p. 793-811, December 2004)”, International Fuzzy Mathematics Institute, Los Angeles California, USA.
\([x_j^{(1)}, x_j^{(2)}, x_j^{(3)}]\)'s are unknown fuzzy variables. The complete mathematical statement of a fuzzy linear-programming problem includes a set of linear equations, which represent the conditions of the problem and a linear function which expresses the objective of the problem.

An n-dimensional Euclidean space \(E_n\) is a set of objects called vectors with the property that there exist \(n\) linearly independent vectors while every set of \(n+1\) vectors is linearly dependent.

One linear condition in \(E_2\) (such as \(a_1 u_1 + a_2 u_2 = b\)) defines a line, in \(E_3\) defines a plane and one linear condition in \(E_n\) is called a hyperplane.

**Convex Sets:** Convex combination of the points \(U_i = \sum_{i=1}^{k} [u_i^{(1)}, u_i^{(2)}, u_i^{(3)}]\) is a point \(U = \alpha_1 U_1(+)\alpha_2 U_2(+)---(+)\alpha_k U_k\), where the \(\alpha_i\) are scalars, \(\alpha_i \geq 0\) and \(\sum \alpha_i = 1\).

A fuzzy subset \(C\) of \(E_n\) is convex if for all pairs of fuzzy points \(U_1\) and \(U_2\) in \(C\), any fuzzy convex combination \(U = \alpha_1 U_1(+)\alpha_2 U_2\) is also in \(C\). Examples of convex sets are the whole space \(E_n\), a circle and a cube. The set of points that form the boundary of a circle is not a convex set. A point \(U\) in a convex set \(C\) is called an extreme point, if \(U\) can't be expressed as a convex combination of any other two distinct points in \(C\). Every point on the boundary of a circle is an extreme point of the convex set. The extreme points of a triangle are its vertices.

Just as the general programming problem has some objective that guides the selection of the solution to be used, the fuzzy linear-programming problem has a linear function of the fuzzy variables to aid in choosing a solution of the problem. This linear combination of the fuzzy variables, called the objective function, must be optimized by the selected solution. Since the maximum of a linear function is equal to minus the minimum of the negative of the linear function, we loss no generality by considering only the minimization problem (see e.g. exercise we have taken for consideration).

Previous workers [see e.g. Cao (1996), Dutta et al (1992)] have studied problems of fuzzy linear programming for a fuzzy matrix \(A\), a fuzzy vector \(B\) and a fuzzy cost vector \(C\) to optimize,

\[Z = CX\]

while \(AX = B\)

and \(X \geq 0\).

Here we have taken \(X\) also as a fuzzy vector in addition to fuzzy \(A\), \(B\) and \(C\). In this chapter, using triangular fuzzy numbers [as in Goswami et al (1997)] we have provided a fuzzy approach to linear programming problems, using the simplex procedure (see Gass (1984), p.32).
3.2 The Linear Programming Problem in Fuzzified Form

The general linear programming problem is to find a fuzzy vector

\[
([x_1^{(1)}, x_1^{(2)}, x_1^{(3)}], [x_2^{(1)}, x_2^{(2)}, x_2^{(3)}], \ldots, [x_j^{(1)}, x_j^{(2)}, x_j^{(3)}], \ldots, [x_n^{(1)}, x_n^{(2)}, x_n^{(3)}])
\]

which minimizes the objective function

\[
[(c_1 x_1)^{(1)}(c_1 x_1)^{(2)}(c_1 x_1)^{(3)}] + [(c_2 x_2)^{(1)}(c_2 x_2)^{(2)}(c_2 x_2)^{(3)}] + \ldots + [(c_j x_j)^{(1)}(c_j x_j)^{(2)}(c_j x_j)^{(3)}] + \ldots + [(c_n x_n)^{(1)}(c_n x_n)^{(2)}(c_n x_n)^{(3)}]
\]

subject to the linear constraints

\[
[(a_{ij} x_1)^{(1)}(a_{ij} x_1)^{(2)}(a_{ij} x_1)^{(3)}] + [(a_{ij} x_2)^{(1)}(a_{ij} x_2)^{(2)}(a_{ij} x_2)^{(3)}] + \ldots + [(a_{ij} x_j)^{(1)}(a_{ij} x_j)^{(2)}(a_{ij} x_j)^{(3)}] = [b_1^{(1)}, b_1^{(2)}, b_1^{(3)}] i = 1, 2, \ldots, m \tag{3.2.2}
\]

and

\[
[x_j^{(1)}, x_j^{(2)}, x_j^{(3)}] \geq 0 \quad j = 1, 2, \ldots, n \tag{3.2.3}
\]

where the \([a_{ij}^{(1)}, a_{ij}^{(2)}, a_{ij}^{(3)}], [b_1^{(1)}, b_1^{(2)}, b_1^{(3)}] \text{ and } [c_j^{(1)}, c_j^{(2)}, c_j^{(3)}]\) are given fuzzy constants and \(O = [-\delta, 0, \delta], \delta\) is a small positive number. Here \(m < n\).

We shall always assume that equations (3.2.2) have been multiplied by \([-1-\delta, 1, 1+\delta]\) where necessary to make all \([b_1^{(1)}, b_1^{(2)}, b_1^{(3)}] \geq O\).

3.3 Properties of a Solution to the Fuzzy Linear Programming Problem

In this section we shall state a number of standard definitions and describe the most important characteristics of a solution to the fuzzy general linear-programming problem.

**Definition 1:** A feasible solution to the fuzzy linear programming problem is a fuzzy vector

\[
[X^{(1)}, X^{(2)}, X^{(3)}] = ([x_1^{(1)}, x_1^{(2)}, x_1^{(3)}], [x_2^{(1)}, x_2^{(2)}, x_2^{(3)}], \ldots, [x_n^{(1)}, x_n^{(2)}, x_n^{(3)}])
\]

which satisfies conditions (3.2.2) and (3.2.3).

**Definition 2a:** A fuzzy basic solution to (3.2.2) is a solution obtained by setting \(n-m\) fuzzy variables equal to fuzzy zero and solving for the remaining \(m\) fuzzy variables, provided that the determinant of the coefficients of these \(m\) fuzzy variables is fuzzy nonzero. The \(m\) fuzzy variables are called fuzzy basic variables.

**Definition 2b:** A fuzzy basic feasible solution is a fuzzy basic solution which also satisfies (3.2.3); that is, all fuzzy basic variables are fuzzy nonnegative.

**Definition 3:** A fuzzy nondegenerate basic feasible solution is a fuzzy basic feasible solution...
with exactly \( m \) fuzzy positive \([x_i^{(1)}, x_i^{(2)}, x_i^{(3)}]\); that is, all fuzzy basic variables are fuzzy positive.

**Definition 4:** A fuzzy minimum feasible solution is a fuzzy feasible solution which also minimizes (3.2.2).

**Definition 5:** A fuzzy optimal basic feasible solution is a fuzzy basic solution that satisfies conditions (3.2.1), (3.2.2), and (3.2.3).

Unless otherwise stated, when we refer to a fuzzy solution, we shall mean any fuzzy feasible solution.

**Definition 6:** A fuzzy linear functional \( f(X) \) is a real-valued function defined on an \( n \)-dimensional vector space such that, for every fuzzy vector \( X = aU (+) \beta V \), \( f(X) = f(aU (+) \beta V) = a \, f(U) (+) \beta \, f(V) \) for all \( n \)-dimensional fuzzy vectors \( U \) and \( V \) and all scalars \( a \) and \( \beta \).

**Theorem 1.** The set of all fuzzy feasible solutions to the fuzzy linear programming problem is a fuzzy convex set.

**Proof:** We need to show that every fuzzy convex combination of any two fuzzy feasible solutions is also a fuzzy feasible solution. Assume that there are at least two fuzzy solutions

\[ X_1 = [x_1^{(1)}, x_1^{(2)}, x_1^{(3)}] \quad \text{and} \quad X_2 = [x_2^{(1)}, x_2^{(2)}, x_2^{(3)}]. \]

We have \( A(X_1) = B \) for \( X_1 \geq O \), where \( A = [a^{(1)}, a^{(2)}, a^{(3)}] \) and \( B = [b^{(1)}, b^{(2)}, b^{(3)}] \),

\[ A(X_2) = B \quad \text{for} \quad X_2 \geq O. \]

For \( 0 \leq \alpha \leq 1 \), let \( X = \alpha X_1 (+) (1- \alpha)X_2 \) be any fuzzy convex combination of \( X_1 \) and \( X_2 \). We note all the elements of \( X \) are fuzzy nonnegative; that is, \( X \geq O \). We then see that \( X \) is a fuzzy feasible solution, for we have

\[ A(X) = A(\alpha X_1 (+) (1- \alpha)X_2) \]

\[ = \alpha (A(X_1) (-) A(X_2)) (+) A(X_2) \]

\[ = \alpha (B(-)B) (+) B \]

\[ = \alpha [b^{(1)} - b^{(3)}, 0, b^{(3)} - b^{(1)}] (+) [b^{(1)}, b^{(2)}, b^{(3)}], \]

which is fuzzy convex.

In a similar manner, one can prove that the fuzzy sets of solutions to the inequalities \( A(X) \geq B \) and the equalities \( A(X) = B \) are fuzzy convex sets.

46
We shall denote the fuzzy convex set of solutions to the linear programming problem by $F-K$. Since $F-K$ is determined by the intersection of the finite set of linear constraints (3.2.2) and (3.2.3), the boundary of $F-K$ will consist of sections of some of the corresponding hyperplanes. For ease in discussion, we can assume that all our problems have a $F-K$ that is a bounded convex polyhedron. In general, a set of points defined by the intersection of a finite number of linear equations and/or inequalities is termed as a convex polyhedron. With the assumption that $F-K$ is a fuzzy convex polyhedron, we need only to look at the extreme points of the polyhedron in order to determine the fuzzy minimum feasible solution. We prove this with the following theorem:

**Theorem 2.** The objective function (3.2.1) assumes its minimum at an extreme point of the fuzzy Convex set $F-K$ generated by the set of fuzzy feasible solutions to the fuzzy linear-programming problem. If it assumes its minimum at more than one extreme point, then it takes on the same value for every fuzzy convex combination of those particular points.

**Proof:** Since we have assumed $F-K$ to be a bounded fuzzy convex polyhedron, $F-K$ has a finite number of extreme points. In two dimensions let us denote the objective function by $f ([X^{(1)}, X^{(2)}, X^{(3)}])$, the extreme points by $[X_i^{(1)}, X_i^{(2)}, X_i^{(3)}]$ for $i=1,2,\ldots,p$; and the minimum feasible solution by $[X_0^{(1)}, X_0^{(2)}, X_0^{(3)}]$. This means that $f ([X_0^{(1)}, X_0^{(2)}, X_0^{(3)}]) \leq f ([X^{(1)}, X^{(2)}, X^{(3)}])$ for all $[X^{(1)}, X^{(2)}, X^{(3)}]$ in $F-K$. If $[X_0^{(1)}, X_0^{(2)}, X_0^{(3)}]$ is an extreme point, the first part of the theorem is true. Suppose $[X_0^{(1)}, X_0^{(2)}, X_0^{(3)}]$ is not an extreme point. We can then write $[X_0^{(1)}, X_0^{(2)}, X_0^{(3)}]$ as a convex combination of the extreme points of $F-K$, i.e.

$$[X_0^{(1)}, X_0^{(2)}, X_0^{(3)}] = \sum_{i=1}^{p} \alpha_i [X_i^{(1)}, X_i^{(2)}, X_i^{(3)}]$$

for $\alpha_i \geq 0$ and $\sum \alpha_i = 1$.

Then since $f ([X^{(1)}, X^{(2)}, X^{(3)}])$ is a linear functional, we have

$$f ([X_0^{(1)}, X_0^{(2)}, X_0^{(3)}]) = f (\sum_{i=1}^{p} \alpha_i [X_i^{(1)}, X_i^{(2)}, X_i^{(3)}])$$

$$= \alpha_1 f ([X_1^{(1)}, X_1^{(2)}, X_1^{(3)}]) + \alpha_2 f ([X_2^{(1)}, X_2^{(2)}, X_2^{(3)}]) + \ldots$$

$$= \sum_{i=1}^{p} \alpha_i f ([X_i^{(1)}, X_i^{(2)}, X_i^{(3)}])$$

$$= [m^{(1)}, m^{(2)}, m^{(3)}]$$

where $[m^{(1)}, m^{(2)}, m^{(3)}]$ is the minimum of $f ([X^{(1)}, X^{(2)}, X^{(3)}])$ for all $[X^{(1)}, X^{(2)}, X^{(3)}]$ in $F-K$. Since all $\alpha_i \geq 0$, we do not increase the sum (3.3.1) if we substitute for each $f ([X_i^{(1)}, X_i^{(2)}, X_i^{(3)}])$ the minimum of the values $f ([X_i^{(1)}, X_i^{(2)}, X_i^{(3)}])$. 

47
Let \( f([X_1^{(1)}, X_2^{(1)}, X_3^{(1)}]) = \min f([X_1^{(2)}, X_2^{(2)}, X_3^{(2)}]). \) Substituting in (3.3.1) we have, since \( \sum \alpha_i = 1 \)

\[
f([X_0^{(1)}, X_0^{(2)}, X_0^{(3)}]) \geq \alpha_1 f([X_m^{(1)}, X_m^{(2)}, X_m^{(3)}]) \geq \alpha_2 f([X_m^{(1)}, X_m^{(2)}, X_m^{(3)}]) \geq \alpha_3 f([X_m^{(1)}, X_m^{(2)}, X_m^{(3)}]) = f([X_m^{(1)}, X_m^{(2)}, X_m^{(3)}])
\]

Since we assumed \( f([X_0^{(1)}, X_0^{(2)}, X_0^{(3)}]) \leq f([X^{(1)}, X^{(2)}, X^{(3)}]) \) for all \([X^{(1)}, X^{(2)}, X^{(3)}] \) in F-K, we must have

\[
f([X_0^{(1)}, X_0^{(2)}, X_0^{(3)}]) = f([X_m^{(1)}, X_m^{(2)}, X_m^{(3)}]) = [m^{(1)}, m^{(2)}, m^{(3)}].
\]

Therefore there is an extreme point \([X_m^{(1)}, X_m^{(2)}, X_m^{(3)}] \), at which the objective function assumes its minimum value.

To prove the second part of the theorem, let \( f([X^{(1)}, X^{(2)}, X^{(3)}]) \) assume its minimum at more than one extreme point, say at \([X_1^{(1)}, X_1^{(2)}, X_1^{(3)}], [X_2^{(1)}, X_2^{(2)}, X_2^{(3)}], \ldots, [X_q^{(1)}, X_q^{(2)}, X_q^{(3)}] \).

Here we have \( f([X_1^{(1)}, X_1^{(2)}, X_1^{(3)}]) = f([X_2^{(1)}, X_2^{(2)}, X_2^{(3)}]) = \ldots = f([X_q^{(1)}, X_q^{(2)}, X_q^{(3)}]) = [m^{(1)}, m^{(2)}, m^{(3)}] \).

If \([X^{(1)}, X^{(2)}, X^{(3)}] \) is any fuzzy convex combination of the above \([X_i^{(1)}, X_i^{(2)}, X_i^{(3)}] \), say 

\[
[X^{(1)}, X^{(2)}, X^{(3)}] = \sum_{i=1}^q \alpha_i [X_i^{(1)}, X_i^{(2)}, X_i^{(3)}]
\]

for \( \alpha_i \geq 0 \) and \( \sum \alpha_i = 1 \), then

\[
f([X^{(1)}, X^{(2)}, X^{(3)}]) = f(\sum_{i=1}^q \alpha_i [X_i^{(1)}, X_i^{(2)}, X_i^{(3)}])
\]

\[
= \sum_{i=1}^q \alpha_i f([X_i^{(1)}, X_i^{(2)}, X_i^{(3)}])
\]

\[
= \sum \alpha_i [m^{(1)}, m^{(2)}, m^{(3)}]
\]

\[
= [m^{(1)}, m^{(2)}, m^{(3)}].
\]

The proof is now complete. By making the obvious changes, the theorem can be proved for the case where (3.2.1) is to be maximized. By Theorem 2, we need only to consider the extreme points of F-K in our search for a minimum fuzzy feasible solution to the fuzzy linear-programming problem.

A fuzzy feasible solution is a fuzzy vector

\[
[X^{(1)}, X^{(2)}, X^{(3)}] = ([x_1^{(1)}, x_1^{(2)}, x_1^{(3)}], [x_2^{(1)}, x_2^{(2)}, x_2^{(3)}], \ldots, [x_n^{(1)}, x_n^{(2)}, x_n^{(3)}]), \text{ with all } [x_i^{(1)}, x_i^{(2)}, x_i^{(3)}] \geq 0, \text{ such that }
\]

\[
[x_1^{(1)}, x_1^{(2)}, x_1^{(3)}] + [P_1^{(1)}, P_1^{(2)}, P_1^{(3)}] + [x_2^{(1)}, x_2^{(2)}, x_2^{(3)}] + [P_2^{(1)}, P_2^{(2)}, P_2^{(3)}] + \ldots + [x_n^{(1)}, x_n^{(2)}, x_n^{(3)}] + [P_n^{(1)}, P_n^{(2)}, P_n^{(3)}] = [0^{(1)}, 0^{(2)}, 0^{(3)}]
\]

48
Assume we have found a set of \( k \) fuzzy vectors that is linearly independent and that there exists a fuzzy nonnegative combination of these fuzzy vectors that is equal to \([P_0^{(1)}, P_0^{(2)}, P_0^{(3)}]\). Let this set of fuzzy vectors be \([P_1^{(1)}, P_1^{(2)}, P_1^{(3)}], [P_2^{(1)}, P_2^{(2)}, P_2^{(3)}], \ldots, [P_k^{(1)}, P_k^{(2)}, P_k^{(3)}]\). We then have the following theorem:

**Theorem 3.** If a set of \( k \leq m \) fuzzy vectors \([P_1^{(1)}, P_1^{(2)}, P_1^{(3)}], [P_2^{(1)}, P_2^{(2)}, P_2^{(3)}], \ldots, [P_k^{(1)}, P_k^{(2)}, P_k^{(3)}]\) can be found that is linearly independent and such that

\[
\begin{align*}
[x_1^{(1)}, x_1^{(2)}, x_1^{(3)}] & [P_1^{(1)}, P_1^{(2)}, P_1^{(3)}] + [x_2^{(1)}, x_2^{(2)}, x_2^{(3)}] [P_2^{(1)}, P_2^{(2)}, P_2^{(3)}] + \cdots + [x_k^{(1)}, x_k^{(2)}, x_k^{(3)}] [P_k^{(1)}, P_k^{(2)}, P_k^{(3)}] = [P_0^{(1)}, P_0^{(2)}, P_0^{(3)}]
\end{align*}
\]

and all \([x_i^{(1)}, x_i^{(2)}, x_i^{(3)}] \geq 0\), then the fuzzy point

\[
[X^{(1)}, X^{(2)}, X^{(3)}] = ([x_1^{(1)}, x_1^{(2)}, x_1^{(3)}], [x_2^{(1)}, x_2^{(2)}, x_2^{(3)}], \ldots, [x_k^{(1)}, x_k^{(2)}, x_k^{(3)}], [-\delta, 0, \delta], \ldots, [-\delta, 0, \delta])
\]

is an extreme point of the fuzzy convex set of fuzzy feasible solutions. Here

\[
[X^{(1)}, X^{(2)}, X^{(3)}]
\]

is an \( n \)-dimensional fuzzy vector whose last \( n-k \) elements are fuzzy zero.

**Proof:** Suppose \([X^{(1)}, X^{(2)}, X^{(3)}]\) is not an extreme point. Then, since \([X^{(1)}, X^{(2)}, X^{(3)}]\) is a feasible solution, it can be written as a fuzzy convex combination of two other points \([X_1^{(1)}, X_1^{(2)}, X_1^{(3)}]\) and \([X_2^{(1)}, X_2^{(2)}, X_2^{(3)}]\) in F-K. We have

\[
[X^{(1)}, X^{(2)}, X^{(3)}] = \alpha [X_1^{(1)}, X_1^{(2)}, X_1^{(3)}] + (1-\alpha) [X_2^{(1)}, X_2^{(2)}, X_2^{(3)}]
\]

for \( 0 < \alpha < 1 \). Since all the elements \([x_i^{(1)}, x_i^{(2)}, x_i^{(3)}]\) of \([X^{(1)}, X^{(2)}, X^{(3)}]\) are fuzzy nonnegative and since \( 0 < \alpha < 1 \), the last \( n-k \) elements of \([X_1^{(1)}, X_1^{(2)}, X_1^{(3)}]\) and \([X_2^{(1)}, X_2^{(2)}, X_2^{(3)}]\) must also equal fuzzy zero; that is,

\[
[X_1^{(1)}, X_1^{(2)}, X_1^{(3)}] = (\sum_{i=1}^{k} [x_i^{(1)}, x_i^{(2)}, x_i^{(3)}], [-\delta, 0, \delta], \ldots, [-\delta, 0, \delta])
\]

\[
[X_2^{(1)}, X_2^{(2)}, X_2^{(3)}] = (\sum_{i=1}^{k} [x_i^{(1)}, x_i^{(2)}, x_i^{(3)}], [-\delta, 0, \delta], \ldots, [-\delta, 0, \delta])
\]

Since \([X_1^{(1)}, X_1^{(2)}, X_1^{(3)}]\) and \([X_2^{(1)}, X_2^{(2)}, X_2^{(3)}]\) are fuzzy feasible solutions, we have

\[
[A^{(1)}, A^{(2)}, A^{(3)}] [X_1^{(1)}, X_1^{(2)}, X_1^{(3)}] = [b^{(1)}, b^{(2)}, b^{(3)}]
\]

And

\[
[A^{(1)}, A^{(2)}, A^{(3)}] [X_2^{(1)}, X_2^{(2)}, X_2^{(3)}] = [b^{(1)}, b^{(2)}, b^{(3)}]
\]

Rewriting these equations in fuzzy vector notation, we have

\[
\Sigma_i^{k} [x_i^{(1)}, x_i^{(2)}, x_i^{(3)}] [P_i^{(1)}, P_i^{(2)}, P_i^{(3)}] = [P_0^{(1)}, P_0^{(2)}, P_0^{(3)}]
\]

And

\[
\Sigma_i^{k} [x_i^{(1)}, x_i^{(2)}, x_i^{(3)}] [P_i^{(1)}, P_i^{(2)}, P_i^{(3)}] = [P_0^{(1)}, P_0^{(2)}, P_0^{(3)}]
\]
But \([P_1(1), P_1(2), P_1(3)], [P_2(1), P_2(2), P_2(3)], \ldots, [P_k(1), P_k(2), P_k(3)]\) is a linearly independent set, and hence \([P_0(1), P_0(2), P_0(3)]\) can be expressed as a unique linear combination in terms of \([P_1(1), P_1(2), P_1(3)], [P_2(1), P_2(2), P_2(3)], \ldots, [P_k(1), P_k(2), P_k(3)]\). This implies that \([x_1(1), x_1(2), x_1(3)] = [x_i(1), x_i(2), x_i(3)] = [P_k(1), P_k(2), P_k(3)]. Therefore, \([X(1), X(2), X(3)]\) cannot be expressed as a fuzzy convex combination of two distinct points in F-K and must be an extreme point of F-K.

**Theorem 4:** If \([X(1), X(2), X(3)] = ([x_1(1), x_1(2), x_1(3)], [x_2(1), x_2(2), x_2(3)], \ldots, [x_n(1), x_n(2), x_n(3)])\) is an extreme point of F-K, then the fuzzy vectors associated with positive \([x_i(1), x_i(2), x_i(3)]\) form a linearly independent set. From this it follows that, at most, \(m\) of the \([x_i(1), x_i(2), x_i(3)]\) are positive.

**Proof:** Let the fuzzy nonzero coefficients be the first \(k\) coefficients, so that

\[
\Sigma_{i=1}^{k} [x_i(1), x_i(2), x_i(3)] [P_i(1), P_i(2), P_i(3)] = [P_0(1), P_0(2), P_0(3)].
\]

We prove the main part of the theorem by contradiction. Assume that \([P_1(1), P_1(2), P_1(3)], [P_2(1), P_2(2), P_2(3)], \ldots, [P_k(1), P_k(2), P_k(3)]\) is a linearly dependent set. Then there exists a linear combination of these vectors which equals the fuzzy zero vector,

\[
\Sigma_{i=1}^{k} [d_i(1), d_i(2), d_i(3)] [P_i(1), P_i(2), P_i(3)] = O. \tag{3.3.2}
\]

with at least one \([d_i(1), d_i(2), d_i(3)] \neq O\). From the hypothesis of the theorem, we have

\[
\Sigma_{i=1}^{k} [x_i(1), x_i(2), x_i(3)] [P_i(1), P_i(2), P_i(3)] = [P_0(1), P_0(2), P_0(3)]. \tag{3.3.3}
\]

For some \([d(1), d(2), d(3)] > O\), we multiply (3.3.2) by \([d(1), d(2), d(3)]\) and add and subtract the result from (3.3.3) to obtain the two equations

\[
\Sigma_{i=1}^{k} [x_i(1), x_i(2), x_i(3)] [P_i(1), P_i(2), P_i(3)] (+) [d(1), d(2), d(3)] \Sigma_{i=1}^{k} [d_i(1), d_i(2), d_i(3)] [P_i(1), P_i(2), P_i(3)] = [P_0(1), P_0(2), P_0(3)]
\]

\[
\Sigma_{i=1}^{k} [x_i(1), x_i(2), x_i(3)] [P_i(1), P_i(2), P_i(3)] (-) [d(1), d(2), d(3)] \Sigma_{i=1}^{k} [d_i(1), d_i(2), d_i(3)] [P_i(1), P_i(2), P_i(3)] = [P_0(1), P_0(2), P_0(3)].
\]

We then have two solutions to (3.2.2): 

\[
[X_1(1), X_1(2), X_1(3)] = (\Sigma_{i=1}^{k} [x_i(1), x_i(2), x_i(3)] (+) [d(1), d(2), d(3)] \Sigma_{i=1}^{k} [d_i(1), d_i(2), d_i(3)], [-\delta', 0, \delta'], \ldots, [-\delta', 0, \delta'])
\]

and

\[
[X_2(1), X_2(2), X_2(3)] = (\Sigma_{i=1}^{k} [x_i(1), x_i(2), x_i(3)] (-) [d(1), d(2), d(3)] \Sigma_{i=1}^{k} [d_i(1), d_i(2), d_i(3)], [-\delta'', 0, \delta''], \ldots, [-\delta'', 0, \delta''])
\]

Since all \([x_i(1), x_i(2), x_i(3)] > O\), we can let \([d(1), d(2), d(3)]\) be as small as necessary, but still positive,
to make the first $k$ components of both $[X_1(1), X_1(2), X_1(3)]$ and $[X_2(1), X_2(2), X_2(3)]$ positive. Then $[X_1(1), X_1(2), X_1(3)]$ and $[X_2(1), X_2(2), X_2(3)]$ are feasible solutions. But $[X(1), X(2), X(3)] = 1/2[X_1(1), X_1(2), X_1(3)] + 1/2[X_2(1), X_2(2), X_2(3)]$ with some fuzzy zero, which contradicts the hypothesis that $[X(1), X(2), X(3)]$ is an extreme point. The assumption of linear dependence for the vectors $[P_1(1), P_1(2), P_1(3)], [P_2(1), P_2(2), P_2(3)], \ldots, [P_k(1), P_k(2), P_k(3)]$ has thus led to a contradiction and hence must be false; i.e., the set of vectors $[P_1(1), P_1(2), P_1(3)], [P_2(1), P_2(2), P_2(3)], \ldots, [P_k(1), P_k(2), P_k(3)]$ is linearly independent.

Since every set of $m+1$ vectors in $m$-dimensional space is necessarily linearly dependent, we cannot have more than $m$ positive $[x_1(1), x_1(2), x_1(3)]$. For assume that we did. Then the above proof of the main part of the theorem would imply that there exist vectors $[P_1(1), P_1(2), P_1(3)], \ldots, [P_m(1), P_m(2), P_m(3)]$ that are linearly independent.

We can, without any loss of generality, assume that the set of vectors $[P_1(1), P_1(2), P_1(3)], [P_2(1), P_2(2), P_2(3)], \ldots, [P_n(1), P_n(2), P_n(3)]$ of the linear programming problem always contains a set of $m$ linearly independent vectors.

**Corollary 1.** Associated with every extreme point of $F-K$ is a set of $m$ linearly independent fuzzy vectors from the given set $[P_1(1), P_1(2), P_1(3)], [P_2(1), P_2(2), P_2(3)], \ldots, [P_n(1), P_n(2), P_n(3)]$.

**Proof:** Theorem 4 has shown that there are $k \leq m$ such vectors. For $k = m$, the corollary is proved. Assume that $k < m$ and that we can find only additional fuzzy vectors $[P_{k+1}(1), P_{k+1}(2), P_{k+1}(3)], \ldots, [P_r(1), P_r(2), P_r(3)]$ such that the set $[P_1(1), P_1(2), P_1(3)], \ldots, [P_k(1), P_k(2), P_k(3)], [P_{k+1}(1), P_{k+1}(2), P_{k+1}(3)], \ldots, [P_r(1), P_r(2), P_r(3)]$ for $r < m$ is linearly independent. Then this implies that the remaining $n-r$ vectors are dependent on $[P_1(1), P_1(2), P_1(3)], \ldots, [P_r(1), P_r(2), P_r(3)]$. But this contradicts the assumption that we always have a set of $m$ linearly independent vectors in the given set of $[P_1(1), P_1(2), P_1(3)], \ldots, [P_n(1), P_n(2), P_n(3)]$. Therefore, there must be $m$ linearly independent fuzzy vectors $[P_1(1), P_1(2), P_1(3)], \ldots, [P_m(1), P_m(2), P_m(3)]$ associated with every extreme point, such that

\[
\sum_{i=1}^{k} [x_i(1), x_i(2), x_i(3)] [P_i(1), P_i(2), P_i(3)] (+) \sum_{i=k+1}^{m} [-\delta, 0, \delta] [P_i(1), P_i(2), P_i(3)] = [P_0(1), P_0(2), P_0(3)].
\]

We can sum up the preceding theorems by the following:

**Theorem 5:** $[X(1), X(2), X(3)] = ([x_1(1), x_1(2), x_1(3)], [x_2(1), x_2(2), x_2(3)], \ldots, [x_n(1), x_n(2), x_n(3)])$ is an
extreme point of F-K iff the positive \([x_j^{(1)}, x_j^{(2)}, x_j^{(3)}]\) are coefficients of linearly independent fuzzy vectors \([P_j^{(1)}, P_j^{(2)}, P_j^{(3)}]\) in \(\sum_{j=1}^{n} [x_j^{(1)}, x_j^{(2)}, x_j^{(3)}][P_j^{(1)}, P_j^{(2)}, P_j^{(3)}] = [P_0^{(1)}, P_0^{(2)}, P_0^{(3)}]\).

As a result of the assumptions and theorems of this section, we have:

1. There is an extreme point of F-K at which the objective function takes on its minimum.
2. Every fuzzy basic feasible solution corresponds to an extreme point of F-K.
3. Every extreme point of F-K has \(m\) linearly independent fuzzy vectors of the given set of \(n\) associated with it (see Gass p.67-76).

3.4 The Fuzzy Simplex Computational Procedure

From the above we can conclude that we need only investigate fuzzy extreme-point solutions and hence only those feasible solutions generated by \(m\) linearly independent fuzzy vectors. For large \(n\) and \(m\) it would be an impossible task to evaluate all the possible solutions and select one that minimizes the objective function. What is required is a computational scheme that selects, in an orderly fashion, a small subset of the possible solutions which converges to a minimum solution. The simplex procedure, devised by G. B. Dantzig, is such a scheme.\(^1\) This procedure finds an extreme point and determines whether it is the minimum. If it is not, the procedure finds an adjacent extreme point\(^2\) whose corresponding value of the objective function is less than or equal to the preceding value. In a finite number of such steps (usually between \(m\) and \(3m\)), a minimum feasible solution is found. The simplex method makes it possible to discover whether the problem has no finite minimum solutions or no fuzzy feasible solutions. It is a powerful scheme for solving any linear-programming problem.

We shall now discuss the validity of the basic elements of the simplex procedure and related computational algorithms. By simplex procedure, we can obtain a minimum feasible solution in a finite number of steps. These steps, or iterations, consist in finding a new fuzzy feasible solution whose corresponding value of the objective function is less than the value of the objective function for the preceding solution. This process is continued until a minimum solution has been reached.

3.5 Development of a Minimum Feasible Solution

We assume that the fuzzy linear programming problem is feasible, that every fuzzy basic feasible solution is fuzzy nondegenerate, and that we are given a fuzzy basic feasible solution.
These assumptions are made without any loss in generality. Let the given solution be 
\[X_0(1), X_0(2), X_0(3)] = \{(X_{01}(1), X_{02}(2), X_{03}(3)), \ldots, (X_{m1}(1), X_{m2}(2), X_{m3}(3))\} \text{ and the associated set of linearly independent vectors be } [P_1(1), P_1(2), P_1(3)], [P_2(1), P_2(2), P_2(3)], \ldots, [P_m(1), P_m(2), P_m(3)]. \text{ We then have}

\[
\begin{align*}
\left[(X_{01}P_1)^{(1)}, (X_{01}P_1)^{(2)}, (X_{01}P_1)^{(3)}\right] + & \left[(X_{02}P_2)^{(1)}, (X_{02}P_2)^{(2)}, (X_{02}P_2)^{(3)}\right] + & \ldots & + \left[(X_{0m}P_m)^{(1)}, (X_{0m}P_m)^{(2)}, (X_{0m}P_m)^{(3)}\right] = [P_0^{(1)}, P_0^{(2)}, P_0^{(3)}] \\
\left[(X_{m1}C_1)^{(1)}, (X_{m1}C_1)^{(2)}, (X_{m1}C_1)^{(3)}\right] + & \left[(X_{m2}C_2)^{(1)}, (X_{m2}C_2)^{(2)}, (X_{m2}C_2)^{(3)}\right] + & \ldots & + \left[(X_{mj}C_j)^{(1)}, (X_{mj}C_j)^{(2)}, (X_{mj}C_j)^{(3)}\right] + & \ldots & + \left[(X_{m1}C_m)^{(1)}, (X_{m1}C_m)^{(2)}, (X_{m1}C_m)^{(3)}\right] = [Z_1^{(1)}, Z_1^{(2)}, Z_1^{(3)}] \quad j = 1, \ldots, n
\end{align*}
\]

(3.5.1)

(3.5.2)

where all \([x_{01}(1), x_{01}(2), x_{01}(3)] > 0\), the \([c_1^{(1)}, c_1^{(2)}, c_1^{(3)}]\) are the cost coefficients of the fuzzy objective function, and \([z_0^{(1)}, z_0^{(2)}, z_0^{(3)}]\) is the corresponding fuzzy value of the fuzzy objective function for the given solution.

Since the set \([P_1(1), P_1(2), P_1(3)], [P_2(1), P_2(2), P_2(3)], \ldots, [P_m(1), P_m(2), P_m(3)]\) is linearly independent and thus forms a fuzzy basis, we can express any fuzzy vector from the set \([P_1(1), P_1(2), P_1(3)], [P_2(1), P_2(2), P_2(3)], \ldots, [P_m(1), P_m(2), P_m(3)]\) in terms of \([P_1(1), P_1(2), P_1(3)], [P_2(1), P_2(2), P_2(3)], \ldots, [P_m(1), P_m(2), P_m(3)]\) by

\[
\begin{align*}
\left[(X_{01}P_1)^{(1)}, (X_{01}P_1)^{(2)}, (X_{01}P_1)^{(3)}\right] + & \left[(X_{02}P_2)^{(1)}, (X_{02}P_2)^{(2)}, (X_{02}P_2)^{(3)}\right] + & \ldots & + \left[(X_{0m}P_m)^{(1)}, (X_{0m}P_m)^{(2)}, (X_{0m}P_m)^{(3)}\right] = [P_j^{(1)}, P_j^{(2)}, P_j^{(3)}] \quad j = 1, \ldots, n
\end{align*}
\]

(3.5.3)

and define

\[
\begin{align*}
\left[(X_{m1}C_1)^{(1)}, (X_{m1}C_1)^{(2)}, (X_{m1}C_1)^{(3)}\right] + & \left[(X_{m2}C_2)^{(1)}, (X_{m2}C_2)^{(2)}, (X_{m2}C_2)^{(3)}\right] + & \ldots & + \left[(X_{mj}C_j)^{(1)}, (X_{mj}C_j)^{(2)}, (X_{mj}C_j)^{(3)}\right] + & \ldots & + \left[(X_{m1}C_m)^{(1)}, (X_{m1}C_m)^{(2)}, (X_{m1}C_m)^{(3)}\right] = [z_j^{(1)}, z_j^{(2)}, z_j^{(3)}] \quad j = 1, \ldots, n
\end{align*}
\]

(3.5.4)

where the \([c_1^{(1)}, c_1^{(2)}, c_1^{(3)}]\) are the cost coefficients corresponding to the \([P_1(1), P_1(2), P_1(3)]\).

**Theorem 1:** If for any fixed \(j\), the condition \([z_j^{(1)}, z_j^{(2)}, z_j^{(3)}] > 0\) holds, then a set of fuzzy feasible solutions can be constructed s.t. \([z_1^{(1)}, z_2^{(2)}, z_3^{(3)}] < [z_0^{(1)}, z_0^{(2)}, z_0^{(3)}]\) for any member of the set, where the lower bound of \([z_1^{(1)}, z_2^{(2)}, z_3^{(3)}]\) is either finite or infinite.

\(^1\) The name simplex method is due to the use of the equation \(\sum X_j = 1\) as a constraint in a geometric interpretation of this procedure, as described in Dantzig (1951).

\(^2\) Two extreme points are said to be adjacent if they are joined by a boundary of the convex polyhedron. We define an edge of a fuzzy convex polyhedron F-K as the line segment joining two extreme points such that no point on the segment is the midpoint of the two other points in F-K not on the segment. The extreme points are called adjacent.
$([z^{(1)}, z^{(2)}, z^{(3)}]$ is the value of the objective function for a particular member of the set of fuzzy feasible solutions.

**Case 1.** If the lower bound is finite, a new fuzzy feasible solution consisting of exactly $m$ positive fuzzy variables can be constructed whose value of the objective function is less than the value for the preceding solution.

**Case 11.** If the lower bound is infinite, a new fuzzy feasible solution consisting of exactly $m+1$ fuzzy positive variables can be constructed whose value of the objective function can be made arbitrarily small.

The following analysis applies to the proof of both the cases:

Multiplying (3.5.3) by some fuzzy number $[\theta^{(1)}, \theta^{(2)}, \theta^{(3)}]$ and subtracting from (3.5.1) and similarly multiplying (3.5.4) by the same $[\theta^{(1)}, \theta^{(2)}, \theta^{(3)}]$ and subtracting from (3.5.2) for $j=1, 2, \ldots$, we get,

$$
\Sigma_{i=1}^{m} \left( [x_{i0}^{(1)}, x_{i0}^{(2)}, x_{i0}^{(3)}] - [\theta^{(1)}, \theta^{(2)}, \theta^{(3)}] [x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}] \right) [P_{i}^{(1)}, P_{i}^{(2)}, P_{i}^{(3)}] = [P_{0}^{(1)}, P_{0}^{(2)}, P_{0}^{(3)}] (3.5.5)
$$

$$
\Sigma_{i=1}^{m} \left( [x_{i0}^{(1)}, x_{i0}^{(2)}, x_{i0}^{(3)}] - [\theta^{(1)}, \theta^{(2)}, \theta^{(3)}] [x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}] [c_{j}^{(1)}, c_{j}^{(2)}, c_{j}^{(3)}] \right) = [z_{0}^{(1)}, z_{0}^{(2)}, z_{0}^{(3)}] (-) [\theta^{(1)}, \theta^{(2)}, \theta^{(3)}] [z_{j}^{(1)}, z_{j}^{(2)}, z_{j}^{(3)}] (-) [c_{j}^{(1)}, c_{j}^{(2)}, c_{j}^{(3)}] (3.5.6)
$$

where $[\theta^{(1)}, \theta^{(2)}, \theta^{(3)}] [c_{j}^{(1)}, c_{j}^{(2)}, c_{j}^{(3)}]$ has been added to both sides of (3.5.6). If all the coefficients of the vectors $[P_{1}^{(1)}, P_{1}^{(2)}, P_{1}^{(3)}], \ldots, [P_{m}^{(1)}, P_{m}^{(2)}, P_{m}^{(3)}], [P_{j}^{(1)}, P_{j}^{(2)}, P_{j}^{(3)}]$ in (3.5.5) are fuzzy nonnegative, then we have determined a new fuzzy feasible solution whose value of the objective function is by (3.5.6), $[z^{(1)}, z^{(2)}, z^{(3)}] = [z_{0}^{(1)}, z_{0}^{(2)}, z_{0}^{(3)}] (-) [\theta^{(1)}, \theta^{(2)}, \theta^{(3)}] [z_{j}^{(1)}, z_{j}^{(2)}, z_{j}^{(3)}] (-) [c_{j}^{(1)}, c_{j}^{(2)}, c_{j}^{(3)}]$. Since the variables $\Sigma_{i=1}^{m} ([x_{i0}^{(1)}, x_{i0}^{(2)}, x_{i0}^{(3)}])$ in (3.5.5) are all fuzzy positive, it is clear from our discussion that there is a value of $[\theta^{(1)}, \theta^{(2)}, \theta^{(3)}] > 0$ (either finite or infinite) for which the coefficients of the fuzzy vectors in (3.5.5) remain fuzzy positive.

From the assumption that, for a fixed $j$, $([z_{j}^{(1)}, z_{j}^{(2)}, z_{j}^{(3)}] (-) [c_{j}^{(1)}, c_{j}^{(2)}, c_{j}^{(3)}]) > 0$, we have $[z^{(1)}, z^{(2)}, z^{(3)}] = [z_{0}^{(1)}, z_{0}^{(2)}, z_{0}^{(3)}] (-) [\theta^{(1)}, \theta^{(2)}, \theta^{(3)}] [z_{j}^{(1)}, z_{j}^{(2)}, z_{j}^{(3)}] (-) [c_{j}^{(1)}, c_{j}^{(2)}, c_{j}^{(3)}] < [z_{0}^{(1)}, z_{0}^{(2)}, z_{0}^{(3)}]$ for $[\theta^{(1)}, \theta^{(2)}, \theta^{(3)}] > 0$.

We see that in either event a new fuzzy feasible solution can be obtained whose corresponding value of the objective function is less than the value for the preceding solution.
The proof of Case 1 follows:

If, for the fixed $j$, at least one $[x_{ij}(1), x_{ij}(2), x_{ij}(3)] > 0$ in (3.5.3) for $i=1, \ldots, m$, the largest value of $[\theta^{(1)}, \theta^{(2)}, \theta^{(3)}]$ for which all coefficients of (3.5.5) remain fuzzy nonnegative is given by

$$[\theta^{(1)}, \theta^{(2)}, \theta^{(3)}] = \min_i ([x_{i0}(1), x_{i0}(2), x_{i0}(3)](+) [x_{ij}(1), x_{ij}(2), x_{ij}(3)]) > 0 \quad (3.5.7)$$

for $[x_{ij}(1), x_{ij}(2), x_{ij}(3)] > 0$.

Since we assumed that the problem is fuzzy nondegenerate, i.e., that all fuzzy basic feasible solutions have fuzzy positive elements, the minimum in (3.5.7) will be obtained for a unique $i$. If $[\theta^{(1)}, \theta^{(2)}, \theta^{(3)}]$ is substituted for $[\theta^{(1)}$, $\theta^{(2)}$, $\theta^{(3)}]$ in (3.5.5) and (3.5.6), the coefficient corresponding to this unique $i$ will be fuzzy zero. We have then constructed a new fuzzy basic feasible solution consisting of $[P_{1}(1), P_{2}(2), P_{3}(3)]$ and $(m-1)$ fuzzy vectors of the original fuzzy basis. This new fuzzy basis can be used as the previous one. If a new $([z_{j}(1), z_{j}(2), z_{j}(3)](-) [c_{j}(1), c_{j}(2), c_{j}(3)]) > 0$ and a corresponding $[x_{ij}(1), x_{ij}(2), x_{ij}(3)] > 0$, another solution can be obtained which has a smaller value of the objective function. This process will continue either until all $([z_{j}(1), z_{j}(2), z_{j}(3)](-) [c_{j}(1), c_{j}(2), c_{j}(3)]) \leq 0$, or until for some $([z_{j}(1), z_{j}(2), z_{j}(3)](-) [c_{j}(1), c_{j}(2), c_{j}(3)]) > 0$, all $[x_{ij}(1), x_{ij}(2), x_{ij}(3)] \leq 0$. If all $([z_{j}(1), z_{j}(2), z_{j}(3)](-) [c_{j}(1), c_{j}(2), c_{j}(3)]) \leq 0$, the process terminates.

For Case 11:

If at any stage we have, for some $j$, $([z_{j}(1), z_{j}(2), z_{j}(3)](-) [c_{j}(1), c_{j}(2), c_{j}(3)]) > 0$ and all $[x_{ij}(1), x_{ij}(2), x_{ij}(3)] \leq 0$, then there is no upper bound to $[\theta^{(1)}, \theta^{(2)}, \theta^{(3)}]$ and the objective function has a lower bound of $-\infty$. We see for this case that, for any $[\theta^{(1)}, \theta^{(2)}, \theta^{(3)}] > 0$, all the coefficients of (3.5.5) are positive. We then have a fuzzy feasible solution consisting of $m+1$ fuzzy positive elements. Hence by taking $[\theta^{(1)}, \theta^{(2)}, \theta^{(3)}]$ large enough, the corresponding value of the fuzzy objective function given by the right hand side of (3.5.6) can be made arbitrarily small.

**Theorem 2:** If for any fuzzy basic feasible solution $[X^{(1)}, X^{(2)}, X^{(3)}]=[x_{10}(1), x_{10}(2), x_{10}(3)], [x_{20}(1), x_{20}(2), x_{20}(3)], \ldots, [x_{m0}(1), x_{m0}(2), x_{m0}(3)]$ the conditions $([z_{j}(1), z_{j}(2), z_{j}(3)](-) [c_{j}(1), c_{j}(2), c_{j}(3)]) \leq 0$ hold for all $j=1,2,\ldots,n$ then (3.5.1) and (3.5.2) constitute a fuzzy minimum feasible solution.

The results of theorem 1 and theorem 2 enable us to start with a fuzzy basic feasible solution and generate a set of new fuzzy basic feasible solutions that converge to the minimum solution.
We can summarize the above discussion of the optimality of a fuzzy basic feasible solution in the following manner:

Define the linear programming problem by: Minimize
\[
[z(1), z(2), z(3)] = [c_1(1), c_1(2), c_1(3)] [x_1(1), x_1(2), x_1(3)] (+) \quad \ldots \quad (+) [c_n(1), c_n(2), c_n(3)] [x_n(1), x_n(2), x_n(3)]
\]

(3.5.8)

Subject to
\[
[a_{11}(1), a_{11}(2), a_{11}(3)] [x_1(1), x_1(2), x_1(3)] (+) \quad \ldots \quad (+) [a_{1n}(1), a_{1n}(2), a_{1n}(3)] [x_n(1), x_n(2), x_n(3)] = [b_1(1), b_1(2), b_1(3)]
\]

\vdots

\[
[a_{m1}(1), a_{m1}(2), a_{m1}(3)] [x_1(1), x_1(2), x_1(3)] (+) \quad \ldots \quad (+) [a_{mn}(1), a_{mn}(2), a_{mn}(3)] [x_n(1), x_n(2), x_n(3)] = [b_m(1), b_m(2), b_m(3)]
\]

(3.5.9)

and
\[
[x_j(1), x_j(2), x_j(3)] \geq 0
\]

(3.5.10)

Note that \([z(1), z(2), z(3)]\) is an unrestricted variable which measures the value of the given objective function. For discussion purposes, assume that the first \(m\) fuzzy variables \(([x_1(1), x_2(1), x_3(1)], [x_2(1), x_2(2), x_2(3)], \ldots, [x_m(1), x_m(2), x_m(3)]\) form a fuzzy basic feasible solution and thus we can solve (3.5.9) for these fuzzy basic variables by the elimination procedure to obtain

\[
[x_1(1), x_1(2), x_1(3)] = [x_{10}(1), x_{10}(2), x_{10}(3)] (-) [x_{1,m+1}(1), x_{1,m+1}(2), x_{1,m+1}(3)] [x_{m+1}(1), x_{m+1}(2), x_{m+1}(3)] (-) \ldots
\]

\[
(-) [x_{m+1}(1), x_{m+1}(2), x_{m+1}(3)] [x_n(1), x_n(2), x_n(3)]
\]

(3.5.11)

Equations (3.5.11) are solved for the fuzzy basic variables in terms of the solution \(([x_{10}(1), x_{10}(2), x_{10}(3)], [x_{20}(1), x_{20}(2), x_{20}(3)], \ldots, [x_{0(1), x_{0}(2), x_{0}(3)}])\) and the fuzzy nonbasic variables \(([x_{m+1}(1), x_{m+1}(2), x_{m+1}(3)], \ldots, [x_n(1), x_n(2), x_n(3)]\) of (3.5.3). By letting all the fuzzy nonbasic variables equal fuzzy zero we obtain the usual solution for the fuzzy basic variables \([X_0(1), X_0(2), X_0(3)] = ([x_{0(1), x_{10}(2), x_{10}(3)]}, [x_{20}(1), x_{20}(2), x_{20}(3)], \ldots, [x_{m0}(1), x_{m0}(2), x_{m0}(3)])\). We next substitute for \(([x_1(1), x_2(1), x_3(1)], [x_2(1), x_2(2), x_2(3)], \ldots, [x_m(1), x_m(2), x_m(3)])\) the corresponding right-hand sides of (3.5.11)
i.e., \[ [X_1(1), X_1(2), X_1(3)] = [X_0(1), X_0(2), X_0(3)] \]
\[ \cdot [X_{inv1}(1), X_{inv1}(2), X_{inv1}(3)] \cdot [X_{inv1}(1), X_{inv1}(2), X_{inv1}(3)] \cdot (\cdot) \]
\[ [X_i(1), X_i(2), X_i(3)] = [X_i(1), X_i(2), X_i(3)] \cdot (\cdot) \]
\[ [X_m(1), X_m(2), X_m(3)] \cdot [X_m(1), X_m(2), X_m(3)] \cdot (\cdot) \]

in the objective function (3.5.8) to obtain
\[ [z(1), z(2), z(3)] = [z_0(1), z_0(2), z_0(3)] \cdot (\cdot) \]
\[ \sum_{j=m+1}^n \left( [z_j(1), z_j(2), z_j(3)] \cdot (\cdot) \right) \cdot [c_j(1), c_j(2), c_j(3)] \cdot [x_j(1), x_j(2), x_j(3)] \]

(3.5.12)

When all the fuzzy nonbasic variables \([x_j(1), x_j(2), x_j(3)]\) are set equal to fuzzy zero, we have the
\[ [z(1), z(2), z(3)] = [z_0(1), z_0(2), z_0(3)] \]

of (3.5.2).

Since all \([x_j(1), x_j(2), x_j(3)] \geq 0\), by (3.5.12) we see that, if for the given fuzzy basic feasible solution all \(( [z_j(1), z_j(2), z_j(3)] \cdot (\cdot) \cdot [c_j(1), c_j(2), c_j(3)] < 0)\) would increase \([z(1), z(2), z(3)]\); while if some \(( [z_j(1), z_j(2), z_j(3)] \cdot (\cdot) \cdot [c_j(1), c_j(2), c_j(3)] > 0)\), an increase of the corresponding fuzzy nonbasic variable would decrease \([z(1), z(2), z(3)]\). Thus, in the former situation (all \([z_j(1), z_j(2), z_j(3)] \cdot (\cdot) \cdot [c_j(1), c_j(2), c_j(3)] < 0)\) we have an optimal solution, and in the latter we can improve the solution.

The nondegeneracy assumption was in voked to ensure the convergence to the minimum solution. Dantzig, Orden, and Wolfe (1954), Wolfe (1963), Charnes, Cooper, and Henderson (1953), and Bland (1977) have resolved degeneracy from both the theoretical and computational points of view (see Gass, p.88-99).

3.6 Computational Procedure

We assume that the given set of fuzzy vectors \([P_1(1), P_1(2), P_1(3)], [P_2(1), P_2(2), P_2(3)], \ldots, [P_n(1), P_n(2), P_n(3)]\) contains m fuzzy unit vectors that can be grouped together to form an \(m \times m\) fuzzy unit matrix. We let these vectors be \([P_1(1), P_1(2), P_1(3)], [P_2(1), P_2(2), P_2(3)], \ldots, [P_m(1), P_m(2), P_m(3)]\)

and take as our admissible basis
\[ [B^1, B^2, B^3] = ([P_1(1), P_1(2), P_1(3)], [P_2(1), P_2(2), P_2(3)], \ldots, [P_m(1), P_m(2), P_m(3)]) = [I_m(1), I_m(2), I_m(3)] \]

Since all the elements of \([P_0(1), P_0(2), P_0(3)]\) were originally assumed to be fuzzy nonnegative, we have the initial extreme-point solution
\[ [X_0(1), X_0(2), X_0(3)] = [P_0(1), P_0(2), P_0(3)] \]
and
\[ [X_j(1), X_j(2), X_j(3)] = [P_j(1), P_j(2), P_j(3)] \]

where
\[ [X_0(1), X_0(2), X_0(3)] = ([X_{10}(1), X_{10}(2), X_{10}(3)], [X_{20}(1), X_{20}(2), X_{20}(3)], \ldots, [X_{m0}(1), X_{m0}(2), X_{m0}(3)]) \geq 0 \]

and
To start the simplex procedure, we arrange the problem matrix as shown in Table-1. (In practice, one does not have to group the unit vectors together or even to keep track of them, but we shall do so for illustrative purposes.) From the original equation of the problem given by $A(.)X = B$, where $B = [b_1(1), b_1(2), b_1(3)]$ we have let $[x_{i0}(1), x_{i0}(2), x_{i0}(3)] = [b_1(1), b_1(2), b_1(3)]$ and $[x_{ij}(1), x_{ij}(2), x_{ij}(3)] = [a_{ij}(1), a_{ij}(2), a_{ij}(3)]$.

$[z_j(1), z_j(2), z_j(3)]$ for $j = 0, 1, \ldots, n$ is obtained by taking the inner product of the $j$th vector with the column vector labeled $[c_1(1), c_1(2), c_1(3)]$, that is,

$[z_0(1), z_0(2), z_0(3)] = \sum_{i=1}^{m} [c_i(1), c_i(2), c_i(3)] \cdot [x_{i0}(1), x_{i0}(2), x_{i0}(3)]$

$[z_j(1), z_j(2), z_j(3)] = \sum_{i=1}^{m} [c_i(1), c_i(2), c_i(3)] \cdot [x_{ij}(1), x_{ij}(2), x_{ij}(3)]$, $j = 1, 2, \ldots, n$

**Table -1  Initial step of Computational procedure**

<table>
<thead>
<tr>
<th>Basis</th>
<th>C</th>
<th>P₀</th>
<th>C₁</th>
<th>C₂</th>
<th>C₃</th>
<th>C₄</th>
<th>C₅</th>
<th>C₆</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>P₁</td>
<td>[c₁(1), c₁(2), c₁(3)]</td>
<td>[x₁₀(1), x₁₀(2), x₁₀(3)]</td>
<td>[1-δ, 1, 1+δ]</td>
<td>[-δ, 0, δ]</td>
<td>[-δ, 0, δ]</td>
<td>[-δ, 0, δ]</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>P₂</td>
<td>[c₂(1), c₂(2), c₂(3)]</td>
<td>[x₂₀(1), x₂₀(2), x₂₀(3)]</td>
<td>[1-δ, 1, 1+δ]</td>
<td>[-δ, 0, δ]</td>
<td>[-δ, 0, δ]</td>
<td>[-δ, 0, δ]</td>
<td></td>
</tr>
<tr>
<td>r</td>
<td>Pᵣ</td>
<td>[cᵣ(1), cᵣ(2), cᵣ(3)]</td>
<td>[xᵣ₀(1), xᵣ₀(2), xᵣ₀(3)]</td>
<td>[-δ, 0, δ]</td>
<td>[-δ, 0, δ]</td>
<td>[-δ, 0, δ]</td>
<td>[-δ, 0, δ]</td>
<td></td>
</tr>
<tr>
<td>m</td>
<td>Pₘ</td>
<td>[cₘ(1), cₘ(2), cₘ(3)]</td>
<td>[xₘ₀(1), xₘ₀(2), xₘ₀(3)]</td>
<td>[-δ, 0, δ]</td>
<td>[-δ, 0, δ]</td>
<td>[-δ, 0, δ]</td>
<td>[-δ, 0, δ]</td>
<td></td>
</tr>
<tr>
<td>m+1</td>
<td>Pₘ₊₁</td>
<td>[z₀(1), z₀(2), z₀(3)]</td>
<td>[1-δ, 1, 1+δ]</td>
<td>[-δ, 0, δ]</td>
<td>[-δ, 0, δ]</td>
<td>[-δ, 0, δ]</td>
<td>[-δ, 0, δ]</td>
<td></td>
</tr>
</tbody>
</table>
The elements \([z_0(1), z_0(2), z_0(3)]\) and \([z_j(1), z_j(2), z_j(3)]\) \((-\{c_j(1), c_j(2), c_j(3)\}\) for those fuzzy vectors in the basis will always equal fuzzy zero. If all the numbers \([z_j(1), z_j(2), z_j(3)]\) \((-\{c_j(1), c_j(2), c_j(3)\}\) \([-O< j = 1, 2, \ldots, n,\) then the solution \([X_0(1), X_0(2), X_0(3)] = ([x_{01}(1), x_{01}(2), x_{01}(3)], \]
\([x_{02}(1), x_{02}(2), x_{02}(3)], \ldots, [x_{0m}(1), x_{0m}(2), x_{0m}(3)]\) \(= (b_1(1), b_1(2), b_1(3),\]
\(b_2(1), b_2(2), b_2(3),\ldots, b_m(1), b_m(2), b_m(3))\) is a minimum feasible solution, and the corresponding value of the objective function is \([z_0(1), z_0(2), z_0(3)]\). We shall assume at least one \([z_j(1), z_j(2), z_j(3)]\) \((-\{c_j(1), c_j(2), c_j(3)\}\) \(> 0\) and compute a new fuzzy feasible solution whose fuzzy basis contains \((m-1)\) fuzzy vectors of the original basis \([P_1(1), P_1(2), P_1(3)], [P_2(1), P_2(2), P_2(3)], \ldots, [P_m(1), P_m(2), P_m(3)]\). In searching for a new fuzzy vector to enter the basis, we can theoretically select any fuzzy vector whose corresponding \([z_j(1), z_j(2), z_j(3)]\) \((-\{c_j(1), c_j(2), c_j(3)\}\) \(> 0\) As Dantzig (1951) points out, the number of iterations i.e., the number of basis changes, necessary to obtain a minimum solution can, in general, be greatly reduced by not selecting at random any vector \([P_j(1), P_j(2), P_j(3)]\) with its \([z_j(1), z_j(2), z_j(3)]\) \((-\{c_j(1), c_j(2), c_j(3)\}\) \(> 0\), but by selecting the one which gives the greatest immediate decrease in the value of the objective function. The fuzzy vector \([P_j(1), P_j(2), P_j(3)]\) should then be the one which corresponds to the

\[
\max_j[\theta_j(1), \theta_j(2), \theta_j(3)]((z_j(1), z_j(2), z_j(3))(c_j(1), c_j(2), c_j(3))] \tag{3.6.1}
\]

where, for each j

\[
[\theta_j(1), \theta_j(2), \theta_j(3)] = \min_i ([x_{ij}(1), x_{ij}(2), x_{ij}(3)] /[x_{j0}(1), x_{j0}(2), x_{j0}(3)] > 0 \tag{3.6.2}
\]

for \([x_{ij}(1), x_{ij}(2), x_{ij}(3)] > 0\). If there are a number of j for which \([z_j(1), z_j(2), z_j(3)]\) \((-\{c_j(1), c_j(2), c_j(3)\}\) \(> 0\), the above rule is rather complicated to apply. A much simpler criterion for selecting the fuzzy vector to be introduced is to select the one which corresponds to the

\[
\max_j (z_j(1), z_j(2), z_j(3)) \((-\{c_j(1), c_j(2), c_j(3)\}\) \(\geq 0\)) \tag{3.6.3}
\]

In our example, let

\[
\max_j (z_j(1), z_j(2), z_j(3)) \((-\{c_j(1), c_j(2), c_j(3)\}\) \(= z_k(1), z_k(2), z_k(3)) \((-\{c_k(1), c_k(2), c_k(3)\}\) \(> 0\)
\]

Then the vector \([P_k(1), P_k(2), P_k(3)]\) is to be introduced into the basis. We next compute

\[
[\theta_k(1), \theta_k(2), \theta_k(3)] = \min_i ([x_{ik}(1), x_{ik}(2), x_{ik}(3)] [(z_k(1), z_k(2), z_k(3)] \((-\{c_k(1), c_k(2), c_k(3)\}\) \(\leq 0\)) \tag{3.6.4}
\]

for \([x_{ik}(1), x_{ik}(2), x_{ik}(3)] \leq 0\). If all \([x_{ik}(1), x_{ik}(2), x_{ik}(3)] \leq 0\), we can then find a fuzzy feasible solution whose value of the objective function can be made arbitrarily small. Our computation is then complete. Assume however, some \([x_{ik}(1), x_{ik}(2), x_{ik}(3)] > 0\) and
Vector \( [P_r(1), P_r(2), P_r(3)] \) will be the one removed from the frizzy basis. Our new feasible solution will have a new basis consisting of \( [P_1(1), P_1(2), P_1(3)], \ldots, [P_r-1(1), P_r-1(2), P_r-1(3)], [P_k(1), P_k(2), P_k(3)], [P_{r+1}(1), P_{r+1}(2), P_{r+1}(3)], \ldots, [P_m(1), P_m(2), P_m(3)] \). We next wish to compute the new solution explicitly and to express each vector not in the basis, in terms of the new basis.

Since our initial fuzzy basis is \( ([P_1(1), P_1(2), P_1(3)] [P_2(1), P_2(2), P_2(3)], \ldots, [P_m(1), P_m(2), P_m(3)]) = ([I_m(1), I_m(2), I_m(3)]) \), we can readily express all the fuzzy vectors \( [P_j(1), P_j(2), P_j(3)] \) in terms of this basis. We then have

\[
[P_0(1), P_0(2), P_0(3)] = [x_{10}(1), x_{10}(2), x_{10}(3)] [P_1(1), P_1(2), P_1(3)] + \ldots + [x_{0}(1), x_{0}(2), x_{0}(3)] [P_r(1), P_r(2), P_r(3)]
\]

\[
[P_k(1), P_k(2), P_k(3)] = [x_{1k}(1), x_{1k}(2), x_{1k}(3)] [P_1(1), P_1(2), P_1(3)] + \ldots + [x_{k}(1), x_{k}(2), x_{k}(3)] [P_r(1), P_r(2), P_r(3)]
\]

\[
[P_i(1), P_i(2), P_i(3)] = [x_{j}(1), x_{j}(2), x_{j}(3)] [P_1(1), P_1(2), P_1(3)] + \ldots + [x_{j}(1), x_{j}(2), x_{j}(3)] [P_r(1), P_r(2), P_r(3)]
\]

From (3.6.5)

\[
[P_r(1), P_r(2), P_r(3)] = 1/[x_{rk}(1), x_{rk}(2), x_{rk}(3)] \{[P_k(1), P_k(2), P_k(3)] (-)[x_{ik}(1), x_{ik}(2), x_{ik}(3)] [P_1(1), P_1(2), P_1(3)] \ldots \}

\[
(-)[x_{mnk}(1), x_{mnk}(2), x_{mnk}(3)] [P_m(1), P_m(2), P_m(3)]
\]

Substituting the above expression for \( [p_1(1), p_1(2), p_1(3)] \) in (3.6.4) we obtain

\[
[P_0(1), P_0(2), P_0(3)] = [x_{10}(1), x_{10}(2), x_{10}(3)] [P_1(1), P_1(2), P_1(3)] + \ldots + [x_{0}(1), x_{0}(2), x_{0}(3)] [P_r(1), P_r(2), P_r(3)]
\]

or

\[
[P_0(1), P_0(2), P_0(3)] = ([x_{10}(1), x_{10}(2), x_{10}(3)] \ldots [x_{0}(1), x_{0}(2), x_{0}(3)] [x_{rk}(1), x_{rk}(2), x_{rk}(3)] [x_{ik}(1), x_{ik}(2), x_{ik}(3)] [P_k(1), P_k(2), P_k(3)] \ldots)
\]

\[
[P_0(1), P_0(2), P_0(3)] = ([x_{10}(1), x_{10}(2), x_{10}(3)] \ldots [x_{0}(1), x_{0}(2), x_{0}(3)] [x_{rk}(1), x_{rk}(2), x_{rk}(3)] [x_{ik}(1), x_{ik}(2), x_{ik}(3)] [P_k(1), P_k(2), P_k(3)] \ldots)
\]

The new fuzzy feasible solution

\[
[X_0(0), X_0(2), X_0(3)] = ([x_{10}(1), x_{10}(2), x_{10}(3)] \ldots [x_{0}(1), x_{0}(2), x_{0}(3)] [x_{rk}(1), x_{rk}(2), x_{rk}(3)] [x_{ik}(1), x_{ik}(2), x_{ik}(3)] [P_k(1), P_k(2), P_k(3)] \ldots)
\]

\[
[X_0(0), X_0(2), X_0(3)] \geq \text{fuzzy zero}, \text{ is given by}
\]

60
where
\[
\begin{align*}
[x_{i0}^{(1)}, x_{i0}^{(2)}, x_{i0}^{(3)}] &= [x_{i0}^{(1)}, x_{i0}^{(2)}, x_{i0}^{(3)}](-)[x_{i0}^{(1)}, x_{i0}^{(2)}, x_{i0}^{(3)}] (+)\cdots (+) [x_{ik}^{(1)}, x_{ik}^{(2)}, x_{ik}^{(3)}](-) [x_{ik}^{(1)}, x_{ik}^{(2)}, x_{ik}^{(3)}] (+)\cdots (+) [x_{rk}^{(1)}, x_{rk}^{(2)}, x_{rk}^{(3)}], \\
[x_{jk}^{(1)}, x_{jk}^{(2)}, x_{jk}^{(3)}] &= [x_{jk}^{(1)}, x_{jk}^{(2)}, x_{jk}^{(3)}] (+)\cdots (+) [x_{mj}^{(1)}, x_{mj}^{(2)}, x_{mj}^{(3)}] [P^{(1)}, P^{(2)}, P^{(3)}]
\end{align*}
\] (3.6.8)

Similarly, by substituting (3.6.7) into (3.6.6) we can obtain the expression for each \([P^{(1)}, P^{(2)}, P^{(3)}]\)not in the new basis in terms of this basis. This yields
\[
\begin{align*}
[x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}] &= [x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}](-)[x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}] (+)\cdots (+) [x_{kij}^{(1)}, x_{kij}^{(2)}, x_{kij}^{(3)}][P^{(1)}, P^{(2)}, P^{(3)}] [P^{(1)}, P^{(2)}, P^{(3)}] (-)\cdots (-) [x_{kij}^{(1)}, x_{kij}^{(2)}, x_{kij}^{(3)}] [P^{(1)}, P^{(2)}, P^{(3)}]
\end{align*}
\] (3.6.9)

Since
\[
\begin{align*}
[z_{1}^{(1)}, z_{2}^{(2)}, z_{3}^{(3)}] (-) [c_{1}^{(1)}, c_{2}^{(2)}, c_{3}^{(3)}] &= ([x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}](-)[c_{1}^{(1)}, c_{2}^{(2)}, c_{3}^{(3)}] (+)\cdots (+) [x_{k_{ij}}^{(1)}, x_{k_{ij}}^{(2)}, x_{k_{ij}}^{(3)}] (+)\cdots (+) [x_{mj}^{(1)}, x_{mj}^{(2)}, x_{mj}^{(3)}] (+)\cdots (+) [c_{m}^{(1)}, c_{m}^{(2)}, c_{m}^{(3)}] (-) [c_{1}^{(1)}, c_{2}^{(2)}, c_{3}^{(3)}] \end{align*}
\] (3.6.10a)

and by substituting the values (3.6.8) for \([x_{i0}^{(1)}, x_{i0}^{(2)}, x_{i0}^{(3)}]\) into
\[
\begin{align*}
[z_{1}^{(1)}, z_{2}^{(2)}, z_{3}^{(3)}] &= [c_{1}^{(1)}, c_{2}^{(2)}, c_{3}^{(3)}] ([x_{10}^{(1)}, x_{10}^{(2)}, x_{10}^{(3)}] (+)\cdots (+) [c_{k^{(1)}, c_{k^{(2)}, c_{k^{(3)}}}] (-) [x_{k0}^{(1)}, x_{k0}^{(2)}, x_{k0}^{(3)}] (+)\cdots (+) [c_{m}^{(1)}, c_{m}^{(2)}, c_{m}^{(3)}] (+) [x_{m0}^{(1)}, x_{m0}^{(2)}, x_{m0}^{(3)}] (+)\cdots (+) [c_{1}^{(1)}, c_{2}^{(2)}, c_{3}^{(3)}] (-) [z_{k}^{(1)}, z_{k}^{(2)}, z_{k}^{(3)}] (-) [c_{k}^{(1)}, c_{k}^{(2)}, c_{k}^{(3)}] \end{align*}
\] (3.6.10b)
We then note that, in order to obtain the new solution \([X_0'(1), X_0'(2), X_0'(3)]\), the new vectors \([X_j'(1), X_j'(2), X_j'(3)]\), and the corresponding \([y_j'(1), y_j'(2), y_j'(3)]\), \((-\), \([c_j'(1), c_j'(2), c_j'(3)]\), all elements in Table-1 for rows \(i = 1, \ldots, m+1\) and columns \(j = 0, 1, \ldots, n\) are transformed by the formulas

\[
\begin{align*}
[x_i'(1), x_i'(2), x_i'(3)] &= [x_i(1), x_i(2), x_i(3)](-)[x_i(1), x_i(2), x_i(3)](+) [x_k(1), x_k(2), x_k(3)] \\
&= (\cdot)[x_k(1), x_k(2), x_k(3)]
\end{align*}
\]

for \(i \neq r\)

\[
[x'_0(1), x'_0(2), x'_0(3)] = [x_0(1), x_0(2), x_0(3)](+) [x_k(1), x_k(2), x_k(3)]
\]

where \([y_0'(1), y_0'(2), y_0'(3)] = [x_{m+1,0}(1), x_{m+1,0}(2), x_{m+1,0}(3)]\)

\([z_j'(1), z_j'(2), z_j'(3)](-)[c_j'(1), c_j'(2), c_j'(3)] = [x_{m+1,j}(1), x_{m+1,j}(2), x_{m+1,j}(3)]\]

Here we are letting the general formulas (3.6.11) apply to all elements of the computational table including the \([P_0'(1), P_0'(2), P_0'(3)]\) column and the \((m+1)st\) row. The transformation defined by (3.6.11) is equivalent to the complete elimination formulas when the pivot element is \([x_k(1), x_k(2), x_k(3)]\).

Once an initial computational table has been constructed, the simplex procedure calls for the successive application (i.e. an iteration) of:

1. The testing of the \([z_j'(1), z_j'(2), z_j'(3)](-)[c_j'(1), c_j'(2), c_j'(3)]\) elements to determine whether a minimum solution has been found, i.e. whether \([z_j'(1), z_j'(2), z_j'(3)](-)[c_j'(1), c_j'(2), c_j'(3)] \leq 0\) for all \(j\).

2. The selection of the vector to be introduced into the basis if some \([z_j'(1), z_j'(2), z_j'(3)](-)[c_j'(1), c_j'(2), c_j'(3)] > 0\), i.e. selection of the vector with maximum \([z_j'(1), z_j'(2), z_j'(3)](-)[c_j'(1), c_j'(2), c_j'(3)]\).

3. The selection of the vector to be removed from the basis to ensure the feasibility of the new solution, i.e. the vector with minimum \(((x_{00}(1), x_{00}(2), x_{00}(3))([x_k(1), x_k(2), x_k(3)]))\) for those \([x_k(1), x_k(2), x_k(3)] > 0\), where \(k\) corresponds to the vector selected in step 2. If all \([x_k(1), x_k(2), x_k(3)] < 0\), then the solution is unbounded.

4. The transformation of the table by the complete elimination procedure to obtain the new solution and associated elements.

Each such iteration produces a new basic feasible solution.

We are now going to write the fuzzy membership function (f.m.f.) of \(X_{ij}\) (\(=\)) \([x_j'(1), x_j'(2), x_j'(3)]\) only as follows: \(X_{ij} = X_{ij}(-)X_{ij}(+)X_{ij}(-)X_{ij}(+)X_{ij}\)

\[
= [x_{ij}(1) - (x_{ij}(3)/x_{ik}(1)) x_{ik}(3), x_{ij}(2) - (x_{ij}(2)/x_{ik}(2)) x_{ik}(2), x_{ij}(3) - (x_{ij}(1)/x_{ik}(3)) x_{ik}(1)]
\]

(3.6.8)
Let us now write fuzzy membership function (f.m.f.)s of $X_{ij}$, $X_{ij}^*$, $X_{jk}$ and $X_{jk}$ one by one.

Since $X_{ij} = [x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}]
\mu_{X_{ij}}(x) = \begin{cases} 
(x - x_{ij}^{(1)})/(x_{ij}^{(2)} - x_{ij}^{(1)}), & x_{ij}^{(1)} \leq x \leq x_{ij}^{(2)} \\
(-x + x_{ij}^{(3)})/(x_{ij}^{(3)} - x_{ij}^{(2)}), & x_{ij}^{(2)} \leq x \leq x_{ij}^{(3)} \\
0, & \text{otherwise}
\end{cases}
(1)

To compute the interval of confidence for each level $\alpha$, the triangular shapes will be described by functions of $\alpha$ in the following manner:

From (1) $\alpha = (x_{ij}^{(a)} - x_{ij}^{(1)})/(x_{ij}^{(2)} - x_{ij}^{(1)})$ and $\alpha = (-x_{ij}^{(a)} + x_{ij}^{(3)})/(x_{ij}^{(3)} - x_{ij}^{(2)})$
Therefore $x_{ij}^{(a)} = [x_1^{(a)}, x_2^{(a)}]$

Or, $x_{ij}^{(a)} = [(x_{ij}^{(2)} - x_{ij}^{(1)})a + x_{ij}^{(1)}, -(x_{ij}^{(3)} - x_{ij}^{(2)})a + x_{ij}^{(3)}]$ \hspace{1cm} (ii)

Exactly in the same way we write

$$
\mu_{x_{ij}}(x) = \begin{cases} 
(x - x_{ij}^{(1)})/(x_{ij}^{(2)} - x_{ij}^{(1)}), & x_{ij}^{(1)} \leq x \leq x_{ij}^{(2)} \\
(-x + x_{ij}^{(3)})/(x_{ij}^{(3)} - x_{ij}^{(2)}), & x_{ij}^{(2)} \leq x \leq x_{ij}^{(3)} \\
0, & \text{otherwise}
\end{cases}
$$ \hspace{1cm} (iii)

and $x_{ij}^{(a)} = [(x_{ij}^{(2)} - x_{ij}^{(1)})a + x_{ij}^{(1)}, -(x_{ij}^{(3)} - x_{ij}^{(2)})a + x_{ij}^{(3)}]$ \hspace{1cm} (iv)

Lastly

$$
\mu_{x_{ik}}(x) = \begin{cases} 
(x - x_{ik}^{(1)})/(x_{ik}^{(2)} - x_{ik}^{(1)}), & x_{ik}^{(1)} \leq x \leq x_{ik}^{(2)} \\
(-x + x_{ik}^{(3)})/(x_{ik}^{(3)} - x_{ik}^{(2)}), & x_{ik}^{(2)} \leq x \leq x_{ik}^{(3)} \\
0, & \text{otherwise}
\end{cases}
$$ \hspace{1cm} (vii)

and $x_{ik}^{(a)} = [(x_{ik}^{(2)} - x_{ik}^{(1)})a + x_{ik}^{(1)}, -(x_{ik}^{(3)} - x_{ik}^{(2)})a + x_{ik}^{(3)}]$ \hspace{1cm} (viii)

The equations to be solved are:

\begin{align*}
\{(x_{ij}^{(2)} - x_{ij}^{(1)})a + x_{ij}^{(1)}\} - \{(x_{ij}^{(3)} - x_{ij}^{(2)})a + x_{ij}^{(3)}\} & - (x_{ij}^{(3)} - x_{ij}^{(2)})a + x_{ij}^{(3)} = 0 \\
\{(x_{ik}^{(2)} - x_{ik}^{(1)})a + x_{ik}^{(1)}\} - \{(x_{ik}^{(3)} - x_{ik}^{(2)})a + x_{ik}^{(3)}\} & - (x_{ik}^{(3)} - x_{ik}^{(2)})a + x_{ik}^{(3)} = 0
\end{align*}

or,

\begin{align*}
\{(x_{ij}^{(2)} - x_{ij}^{(1)})a(x_{ik}^{(2)} - x_{ik}^{(1)}) - (x_{ij}^{(3)} - x_{ij}^{(2)})(x_{ik}^{(3)} - x_{ik}^{(2)})\}a^2 + \{x_{ik}^{(1)}(x_{ij}^{(2)} - x_{ij}^{(1)}) + x_{ij}^{(1)}(x_{ik}^{(2)} - x_{ik}^{(1)}) + x_{ik}^{(3)}(x_{ij}^{(3)} - x_{ij}^{(2)}) + x_{ij}^{(3)}(x_{ik}^{(3)} - x_{ik}^{(2)}) - x_{i}(x_{ik}^{(2)} - x_{ik}^{(1)})a + (x_{ij}^{(1)}x_{ik}^{(1)} - x_{ij}^{(3)}x_{ik}^{(3)}) - x_{1}x_{ik}^{(1)} = 0
\end{align*}

(ix)
\[
\{ (x_j^3 - x_i^2)(x_{ik} - x_{ik}^2) - (x_j^2 - x_i^1)(x_{ik}^2 - x_{ik}^1) \} \alpha^2 - \{ x_{ik}^3 (x_j^3 - x_i^2) + x_j^3 (x_{ik}^3 - x_k^2) + x_{ik}^1 (x_j^1 - x_i^1) \} \alpha + \{ x_{ik}^3 (x_j^3 - x_{ik}^1) \} x_{ik} = 0
\]

We are to retain only two roots in \([0,1]\).

From (ix) we write,
\[
\alpha = \{ x_{ik}^1 (x_j^2 - x_i^1) + x_j^1 (x_{ik}^2 - x_{ik}^1) + x_{ik}^3 (x_j^3 - x_i^2) + x_j^3 (x_{ik}^3 - x_{ik}^2) \}
\]
and from (x) we write,
\[
\alpha = \{ x_{ik}^3 (x_j^3 - x_{ik}^2) + x_j^3 (x_{ik}^3 - x_{ik}^2) + x_{ik}^1 (x_j^2 - x_i^1) + x_j^1 (x_{ik}^2 - x_{ik}^1) - x_2 (x_{ik}^3 - x_{ik}^2) \}
\]

Thus we would write the f.m.f. of \(X_{ij}\) as
\[
\mu_{X_{ij}}(x) = \begin{cases} 
\{ x_{ik}^3 (x_j^3 - x_{ik}^2) + x_j^3 (x_{ik}^3 - x_{ik}^2) + x_{ik}^1 (x_j^2 - x_i^1) + x_j^1 (x_{ik}^2 - x_{ik}^1) - x_2 (x_{ik}^3 - x_{ik}^2) \} & x_{ik}^3 \leq x < x_{ik}^2 \\
0, & \text{otherwise}
\end{cases}
\]
3.7 A Numerical Example

Let us solve the following linear programming problem by means of the simplex procedure:

Minimize
\[ \mathbf{b}^T \mathbf{x} \]
subject to
\[ \mathbf{Ax} = \mathbf{b} \]
\[ x \geq 0. \]

Using the constraints we can write the problem as:

Minimize
\[ Z^1, Z^2, Z^3 \]
subject to
\[ \mathbf{Ax} = \mathbf{b} \]
\[ x \geq 0. \]

Let us see Table-a.
Our initial fuzzy basis consists of \( P_1, P_4 \) and \( P_6 \) and the corresponding solution is
\[ \delta = .5 \]

Let us see Table-a.
\[
X_0 = [X_0^{(1)} : X_0^{(2)} : X_0^{(3)}] = ([x_{10}^{(1)} : x_{10}^{(2)} : x_{10}^{(3)}], [x_{40}^{(1)} : x_{40}^{(2)} : x_{40}^{(3)}], [x_{60}^{(1)} : x_{60}^{(2)} : x_{60}^{(3)}])
= ([6.5, 7, 7.5], [11.5, 12, 12.5], [9.5, 10, 10.5]).
\]

Since \( C_1 = [c_1^{(1)} : c_1^{(2)} : c_1^{(3)}], C_4 = [c_4^{(1)} : c_4^{(2)} : c_4^{(3)}] \) and \( C_6 = [c_6^{(1)} : c_6^{(2)} : c_6^{(3)}] \) are all equal to \([- .5, 0, .5]\) the corresponding value of the objective function is
\[
Z_0 = \sum_{i=1}^{n} C_i X_{i0} = [-15.25, 0, 15.25] \text{ since }
\]

Max \( Z_j (-) C_j = Z_3 (-) C_3 = [-1.25, 3, 7.25] > [- .5, 0, .5], P_3 \) is selected to go into the basis.

Next,
\[
[\theta_0^{(1)}, \theta_0^{(2)}, \theta_0^{(3)}] = \text{min} \{ ([x_{10}^{(1)} : x_{10}^{(2)} : x_{10}^{(3)}] (+) [x_{13}^{(1)} : x_{13}^{(2)} : x_{13}^{(3)}])
= \text{min} \{ ([11.5, 12, 12.5] (+) [3.5, 4, 4.5], [9.5, 10, 10.5] (+) [2.5, 3, 3.5])
= \text{min} \{ [2.56, 3, 3.57], [2.71, 3.33, 4.2] \} = [2.56, 3, 3.57]
\]

And hence \( P_4 \) is removed.

We transform the Table-a to Table-b (see Table-b) using the formulae (3.6.4) – (3.6.6) and obtain
\[
X_{10} = X_{10} (-) (X_{40} (+) X_{43})(-) X_{13} = [7.78, 10, 12.86], \text{ } X_{30} (-) X_{40} (+) X_{43} = [2.56, 3, 3.57] \text{ and }
X_{60} = X_{60} (-) (X_{40} (+) X_{43})(-) X_{63} = [-3, 1, 4]
\]

Hence the new solution is
\[
X_0' = [X_0^{(1)} : X_0^{(2)} : X_0^{(3)}] = ([x_{10}^{(1)} : x_{10}^{(2)} : x_{10}^{(3)}], [x_{30}^{(1)} : x_{30}^{(2)} : x_{30}^{(3)}], [x_{60}^{(1)} : x_{60}^{(2)} : x_{60}^{(3)}])
= ([7.78, 10, 12.86], [2.56, 3, 3.57], [-3, 1, 4])
\]

and the value of the objective function is
\[
Z_0' = \sum_{i=1}^{n} X_{i0} (-) C_i = [-20.36, -9, 1.46].
\]

In the second step, since
\[
\text{max} Z_j (-) C_j = Z_2' (-) C_2 = Z_2 (-) C_2 (-) (X_{42} (+) X_{43} ) (-) (Z_3 (-) C_3)
= \sum_{i=1}^{n} X_{12} (-) C_i (-) C_2 = [-1.57, -5, 2.61]
\]
and \( [\theta_0^{(1)}, \theta_0^{(2)}, \theta_0^{(3)}] = X_{10}' (+) X_{12}' = [7.78, 10, 12.86](+) [1.67, 2.5, 3.29] = [2.36, 4, 7.7]
\( P_2 \) is introduced into the basis and \( P_1 \) is removed.

We transform the second step values of Table-b and obtain the third solutions as
\[
X_{20}' = X_{10}' (+) X_{12}' = [2.53, 4, 6.63],
X_{30}' = X_{30}' (-) (X_{10}' (+) X_{12}') (-) X_{32}' = [3.57, 5, 7.86] 
\]
\[ X_{60}'' = X_{60}' \times X_{10}' \times X_{12}' \times X_{62}' = [-.47, 11, 28.37] \]
So \[ X_0'' = ([2.53, 4, 6.63], [3.57, 5, 7.86], [-.47, 11, 28.37]) \] with a value of the objective function
\[ Z_0'' = \sum_{i=1}^{m} X_i'(\cdot)C_i = [-39.49,-11,15.61]. \]
Since \( \max([z_i''(1), z_i''(2), z_i''(3)](-)[c_j^{(1)}, c_j^{(2)}, c_j^{(3)}]) \) = fuzzy zero, this solution is a minimum feasible solution in fuzzified form.

**Table-a Initial step**

<table>
<thead>
<tr>
<th>i</th>
<th>Basis</th>
<th>C</th>
<th>( P_0 )</th>
<th>( C_1 )</th>
<th>( C_2 )</th>
<th>( C_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>( [-.5, 0, .5] )</td>
<td>( [.5, 1, 1.5] )</td>
<td>( [-2.5, 3, 3.5] )</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>( P_1 )</td>
<td>( C_1 )</td>
<td>( X_{10} )</td>
<td>( X_{11} )</td>
<td>( X_{12} )</td>
<td>( X_{13} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( [6.5,7,7.5] )</td>
<td>( [.5, 1, 1.5] )</td>
<td>( [2.5,3,3.5] )</td>
<td>( [-1.5,-1,-5] )</td>
</tr>
<tr>
<td>2</td>
<td>( P_4 )</td>
<td>( C_4 )</td>
<td>( X_{40} )</td>
<td>( X_{41} )</td>
<td>( X_{42} )</td>
<td>( X_{43} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( [11.5,12,12.5] )</td>
<td>( [-5, 0, .5] )</td>
<td>( [-2.5,-2,-1.5] )</td>
<td>( [3.5,4,4.5] )</td>
</tr>
<tr>
<td>3</td>
<td>( P_6 )</td>
<td>( C_6 )</td>
<td>( X_{60} )</td>
<td>( X_{61} )</td>
<td>( X_{62} )</td>
<td>( X_{63} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( [9.5,10,10.5] )</td>
<td>( [-5, 0, .5] )</td>
<td>( [-4.5,-4,-5] )</td>
<td>( [2.5,3,3.5] )</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>( Z_0 )</td>
<td></td>
<td>( Z_1(-)C_1 )</td>
<td>( Z_2(-)C_2 )</td>
<td>( Z_3(-)C_3 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( [-15.25,0,15.25] )</td>
<td>( [-.75,0,1.75] )</td>
<td>( [-6.75,-1,4.75] )</td>
<td>( [-.25,3,7.25] )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( C_4 )</th>
<th>( C_5 )</th>
<th>( C_6 )</th>
<th>( \Theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( [-.5, 0, .5] )</td>
<td>( [1.5,2,2.5] )</td>
<td>( [-.5, 0, .5] )</td>
<td>---</td>
</tr>
<tr>
<td>( P_4 )</td>
<td>( P_5 )</td>
<td>( P_6 )</td>
<td>---</td>
</tr>
<tr>
<td>( X_{14} )</td>
<td>( X_{15} )</td>
<td>( X_{16} )</td>
<td>( X_{16} )</td>
</tr>
<tr>
<td>( [-.5, 0, .5] )</td>
<td>( [1.5, 2, 2.5] )</td>
<td>( [-.5, 0, .5] )</td>
<td>( -- )</td>
</tr>
<tr>
<td>( X_{44} )</td>
<td>( X_{45} )</td>
<td>( X_{46} )</td>
<td>( X_{46} )</td>
</tr>
<tr>
<td>( [.5, 1, 1.5] )</td>
<td>( [-.5, 0, .5] )</td>
<td>( [-.5, 0, .5] )</td>
<td>( [2.56, 3, 3.57] )</td>
</tr>
<tr>
<td>( X_{64} )</td>
<td>( X_{65} )</td>
<td>( X_{66} )</td>
<td>( X_{66} )</td>
</tr>
<tr>
<td>( [-.5, 0, .5] )</td>
<td>( [7.5, 8, 8.5] )</td>
<td>( [.5, 1, 1.5] )</td>
<td>( [2.71, 3.33, 4.2] )</td>
</tr>
<tr>
<td>( Z_4(-)C_4 )</td>
<td>( Z_5(-)C_5 )</td>
<td>( Z_6(-)C_6 )</td>
<td>( Z_6(-)C_6 )</td>
</tr>
<tr>
<td>( [-1.75,0,1.75] )</td>
<td>( [-8.25,-2,4.25] )</td>
<td>( [-1.75,0,1.75] )</td>
<td>( [2.56, 3, 3.57] )</td>
</tr>
</tbody>
</table>
### Table-b  Second step

<table>
<thead>
<tr>
<th>i</th>
<th>Basis</th>
<th>C</th>
<th>P₀</th>
<th>P₁</th>
<th>P₂</th>
<th>P₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>P₁</td>
<td>C₁</td>
<td>X₁₀</td>
<td>X₁₁</td>
<td>X₁₂</td>
<td>X₁₃</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>{.5,0,.5}</td>
<td>[.34, 1, 1.67]</td>
<td>[1.68, 2.5, 3.29]</td>
<td>[-1.01, 0, 0.98]</td>
</tr>
<tr>
<td>2</td>
<td>P₃</td>
<td>C₃</td>
<td>X₃₀</td>
<td>X₃₁</td>
<td>X₃₂</td>
<td>X₃₃</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>[-3.5, -3, -5]</td>
<td>[-11, 0, .14]</td>
<td>[-56, -5, -43]</td>
<td>[.78, 1, 1.28]</td>
</tr>
<tr>
<td>3</td>
<td>P₆</td>
<td>C₆</td>
<td>X₆₀</td>
<td>X₆₁</td>
<td>X₆₂</td>
<td>X₆₃</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>[.5,1,1.5]</td>
<td>[11, 0, .14]</td>
<td>[-11, 0, 1]</td>
<td>[-.66, -2.5, -1]</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td>[-20.36, -9, 1.46]</td>
<td>[-1.2, 0, 3.22]</td>
<td>[-.56, 1, 1.61]</td>
<td>[-3.24, 0, 2.81]</td>
</tr>
</tbody>
</table>

### Table-c  Third step

<table>
<thead>
<tr>
<th>i</th>
<th>Basis</th>
<th>C</th>
<th>P₀</th>
<th>P₁</th>
<th>P₂</th>
<th>P₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>P₂</td>
<td>C₂</td>
<td>X₂₀</td>
<td>X₂₁</td>
<td>X₂₂</td>
<td>X₂₃</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>[5,1,1.5]</td>
<td>[.51, 1, 1.97]</td>
<td>[44, 1, 1.61]</td>
<td>[-3.1, 0, 0.59]</td>
</tr>
<tr>
<td>2</td>
<td>P₃</td>
<td>C₃</td>
<td>X₃₀</td>
<td>X₃₁</td>
<td>X₃₂</td>
<td>X₃₃</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>[-2.5,3,3.5]</td>
<td>[-.06, .1, .7]</td>
<td>[-3.4, 0, .69]</td>
<td>[44, 1, 1.61]</td>
</tr>
<tr>
<td>3</td>
<td>P₆</td>
<td>C₆</td>
<td>X₆₀</td>
<td>X₆₁</td>
<td>X₆₂</td>
<td>X₆₃</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>[-5,0,.5]</td>
<td>[-.9, 1, 4.66]</td>
<td>[-3.16, 0, 6.21]</td>
<td>[-4.21, 0, 3.69]</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td>[-36.49, -1,15.61]</td>
<td>[-5.26, -2.4, 54]</td>
<td>[-6.7, 0, 6.79]</td>
<td>[-5.71, 0, 5.39]</td>
</tr>
<tr>
<td></td>
<td>( P_4 )</td>
<td>( P_5 )</td>
<td>( P_6 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-------</td>
<td>----------------------</td>
<td>----------------------</td>
<td>----------------------</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( X_{24}'' )</td>
<td>[(-.13), (.1), (.60)]</td>
<td>( X_{25}'' )</td>
<td>[(.41), (.8), (1.59)]</td>
<td>( X_{26}'' )</td>
<td>[(-.20), (0), (.40)]</td>
<td></td>
</tr>
<tr>
<td>( X_{34}'' )</td>
<td>[(-.04), (.3), (.77)]</td>
<td>( X_{35}'' )</td>
<td>[(.06), (.4), (1.04)]</td>
<td>( X_{36}'' )</td>
<td>[(-.33), (0), (.36)]</td>
<td></td>
</tr>
<tr>
<td>( X_{44}'' )</td>
<td>[(-2.94), (-.5), (.24)]</td>
<td>( X_{65}'' )</td>
<td>[(7.4), (10), (14.82)]</td>
<td>( X_{66}'' )</td>
<td>[(-1.45), (1), (3.45)]</td>
<td></td>
</tr>
<tr>
<td>( Z_4''(-)C_4 )</td>
<td>[(-4.86), (-.8), (3.01)]</td>
<td>( Z_5''(-)C_5 )</td>
<td>[(-13.34), (-2.4), (8.14)]</td>
<td>( Z_6''(-)C_6 )</td>
<td>[(-2.68), (0), (3.98)]</td>
<td></td>
</tr>
</tbody>
</table>

Let us now write fuzzy membership function \((\text{f.m.f.})\)s of \( X_{10}'' , \ X_{30}'' , \ X_{12}'' , \ X_{32}'' \) and then of \( X_{30}'' \) as:

\[
X_{30}'' = X_{30}'' (-) (X_{10}'' (+)X_{12}'' ) (.X_{32}'' ), \text{as follows:}
\]

\[
\mu_{X_{10}}''(x) = \begin{cases} 
(x - 7.78)/2.22, & 7.78 \leq x \leq 10 \\
(-x + 12.86)/2.86, & 10 \leq x \leq 12.86 \\
0, & \text{otherwise}
\end{cases}
\]  \hspace{1cm} (ix)

To compute the interval of confidence for each level \(\alpha\), the triangular shapes will be described by functions of \(\alpha\) in the following manner:

From (i) \(\alpha = (x_1^{(a)} - 7.78)/2.22 \) and \(\alpha (-)(-x_2^{(a)} + 12.86)/2.86\)

Therefore \( x_{10}^{(a)} = [x_1^{(a)}, x_2^{(a)}] = [2.22a + 7.78, -2.86a + 12.86] \) \hspace{1cm} (x)

Exactly in the same way we write:

\[
\mu_{X_{30}}^{(a)}(x) = \begin{cases} 
(x - 2.56)/.44, & 2.56 \leq x \leq 3 \\
(-x + 3.57)/.57, & 3 \leq x \leq 3.57 \\
0, & \text{otherwise}
\end{cases}
\]  \hspace{1cm} (xi)

\( x_{30}^{(a)} = [.44a + 2.56, -.57a + 3.57] \)

\[
\mu_{X_{12}}^{(a)}(x) = \begin{cases} 
(x - 1.68)/.82, & 1.68 \leq x \leq 2.5 \\
(-x + 3.29)/.79, & 2.5 \leq x \leq 3.29 \\
0, & \text{otherwise}
\end{cases}
\]  \hspace{1cm} (xii)

\( x_{12}^{(a)} = [.82a + 1.68, -.79a + 3.29] \)

70
\[ \mu_{X_3}(x) = \begin{cases} \frac{x + .56}{.06}, & -.56 \leq x \leq -.5 \\ \frac{-x - .43}{.07}, & -.5 \leq x \leq -.43 \\ 0, & \text{otherwise} \end{cases} \]

\[ x_{03}^{(a)} = [-.07a + .43, -.06a + .56] \quad \text{(xiii)} \]

Therefore
\[ X_{30}^{(a)} = x_{03}^{(a)} (x_{10}^{(a)} + x_{12}^{(a)} X_{32}^{(a)}) \]
\[ = [.44a + 2.56, -.57a + 3.57] + [(2.22a + 7.78)(.07a + .43), (-2.86a + 12.86)(.82a + 1.68)] \]
\[ X_{30}^{(a)} = [11.7678 - 3.29x_1, 13.1992 - 1.68x_2] \quad \text{(xiv)} \]

The equations to be solved are:
\[ (.44a + 2.56)(-.79a + 3.29) + (2.22a + 7.78)(.07a + .43) - x_1(-.79a + 3.29) = 0 \quad \text{and} \]
\[ (-.57a + 3.57)(.82a + 1.68) + (-2.86a + 12.86)(.06a + .56) - x_2(.82a + 1.68) = 0 \]

or,
\[ -.1922a^2 + (.9244 + .79x_1)a + (11.7678 - 3.29x_1) = 0 \quad \text{(xv)} \]

and
\[ -.2958a^2 + (-.4034 - .82x_2)a + (13.1992 - 1.68x_2) = 0 \]

We are to retain only two roots in \([0,1]\).

From (xiv) we write
\[ a = (.9244 + .79x_1) + \sqrt{(\.9244 + .79x_1)^2 + 4*.1922 (11.7678 - 3.29x_1)}/2*(-.1922) \]

and from (xv) we write
\[ a = (-.4034 - .82x_2) - \sqrt{(-.4034 - .82x_2)^2 + 4*.2958 (13.1992 - 1.68x_2)}/2*(-.2958) \]

Therefore
\[ \mu_{X_{30}}(x) = \begin{cases} -.9244 + .79x + \sqrt{(.9244 + .79x)^2 + 4*.1922 (11.7678 - 3.29x)}/2*(-.1922), & 3.57 \leq x \leq 5 \\ -.4034 - .82x - \sqrt{(-.4034 - .82x)^2 + 4*.2958 (13.1992 - 1.68x)}/2*(-.2958), & 5 \leq x \leq 7.86 \\ 0, & \text{otherwise} \end{cases} \]

Exactly in the same way we could write the f.m.f. of \( X_{60}^{/}(x) \) and \( X_{20}^{/}(x) \).

### 3.8 Conclusion

We have thus seen that a fuzzy optimal solution of a fuzzy linear programming problem with a fuzzy variable vector can be found. We have however seen that fuzzification of the variable vector makes the problem computationally longer.

*****