FUZZIFIED PREDICTOR - CORRECTOR ALGORITHM FOR APPROXIMATE SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

10.1 Introduction

An equation in which the unknown function is under the sign of the derivative or the differential is a differential equation. If the unknown function entering into a differential equation depends only on one independent variable, the differential equation is ordinary. The order of a differential equation is the highest order of the derivative (or of the differential) entering into the equation. In this chapter we study only ordinary differential equations.

In the most general case an ordinary differential equation of order n contains an independent variable, an unknown function and its derivatives or differentials upto order n inclusive and has the form

\[ F(x, y, y', y'', \ldots, y^{(n)}) = 0 \]  

In this equation x is an independent variable, y is an unknown function, y', y'', \ldots are derivatives of this function. If the left hand side of the differential equation (10.1.1) is a polynomial with respect to the derivative of the unknown function, then the degree of this polynomial is the degree of the differential equation. (See Danilina et al. p.423-424.)

The general solution of this equation contains n arbitrary constants or parameters. Thus the general solution is of the form

\[ F(x, y, c_1, c_2, \ldots, c_n) = 0 \]

If particular values are given to the constants c_1, c_2, \ldots, c_n, then the resulting solution is called a particular solution. To get a particular solution we must be given n conditions (e.g. the values of y or its derivatives for some specific values of x) to determine the n constants. If all the n conditions are specified in the same initial point x_0 then the problem is called an initial value problem. When the conditions are specified at two or more values of x, then the problem is called a boundary value problem.

Many ordinary differential equations can be solved by analytical methods discussed earlier giving closed form solutions i.e. expressing y in terms of a finite number of elementary functions of x. However, a majority of differential equations appearing in physical problems cannot be
solved analytically. Thus it becomes imperative to discuss their solution by numerical methods. (see Bali et al, 2004, P. 1292)

Ordinary differential equations (ODE) are the principal form of mathematical models encountered in Sciences and Engineering, and consequently their numerical solution is a very large area of study.

A general equation in fuzzified form of first order and first degree is \( \frac{dY}{dX} = f(X,Y) \). Here \( X = [x^{(1)}, x^{(2)}, x^{(3)}] \) and \( Y = [y^{(1)}, y^{(2)}, y^{(3)}] \) are triangular fuzzy variables \( X \) and \( Y \). In numerical methods, we do not proceed in the hope of finding a relation between \( X \) and \( Y \), but we find the numerical values of the dependent variable for certain values of the independent variable. There are methods for finding, to any desired degree of accuracy, the numerical solution of any ordinary differential equation having numerical coefficients with given initial conditions. Starting with the initial values, the solutions are then constructed for equal intervals \( \Delta X = H \) of \( X \), each step usually being checked by some method before proceeding to the next step, where \( H = [h^{(1)}, h^{(2)}, h^{(3)}] \) in triangular fuzzy form.

### 10.2 Fuzzified Form of Milne’s Algorithm of Solving Ordinary Differential Equations

A simple and reasonably accurate method of solving differential equations numerically has been devised by Milne (1941). It uses two Quadrature formulas - one for integrating ahead by extrapolation and the other by checking the extrapolation value. Predictor- corrector method is the method in which we first predict a value of \( Y_{n+1} \) by using a certain formula and then correcting this value by using a more accurate formula. Thus an initially crude estimate is refined.

Consider the differential equation \( \frac{dY}{dX} = f(X,Y) \), \( Y(X_0) = Y_0 \) (10.2.1)

We first find the fuzzy numerical values of \( Y_0 = Y(X_0), Y_1 = Y(X_0(+)H), Y_2 = Y(X_0(+) 2H) \), and then from (10.2.1), we compute \( Y'_0, Y'_1, Y'_2, \)------.

For equi-spaced arguments \( X_0, X_0(+)H, X_0(+)2H \), by Newton’s forward interpolation formula we know that

\[
U = U_0 (+) R(.)A U_0 (+) [R(.)R(-1)]/2!(.)A^2 U_0 (+)[R(.)R(-1)(R(-2)]/3!(.)A^3 U_0 (+) ------
\]

where \( R = (X(-)X_0)/H \)
which gives, \( X = X_0(+H(\cdot)R) \).

Replacing \( U \) by \( Y' \) we get

\[
Y' = Y_0' (+ R(\cdot)\Delta Y_0'/ (+ [R(\cdot)(R(-1))]/2(\cdot) \Delta^2 Y_0'/ (+ [R(R(-1))(R(-2))]/6(\cdot) \Delta^3 Y_0'/ (+ [R(R(-1))(R(-2))]/24(\cdot) \Delta^4 Y_0'/ (+ \ldots \right)
\]

(10.2.2)

In triangular fuzzy form \( Y_i = [y_i(1), y_i(2), y_i(3)] \) \((i = 0, 1, 2, \ldots)\), \( R = [r^{(1)}, r^{(2)}, r^{(3)}] \), \( X_0 = [x_0^{(1)}, x_0^{(2)}, x_0^{(3)}] \) etc.

Integrating (10.2.2) over the interval \( X_0 \) to \( X_0(+4H) \), we have

\[
\int_{X_0}^{X_0(+4H)} Y' dX = \int_{X_0}^{X_0(+4H)} [Y_0' (+R(\cdot)\Delta Y_0'/ (+ [R(\cdot)(R(-1))]/2(\cdot) \Delta^2 Y_0'/ (+ [R(R(-1))(R(-2))]/6(\cdot) \Delta^3 Y_0'/ (+ [R(R(-1))(R(-2))]/24(\cdot) \Delta^4 Y_0'/ (+ \ldots dX \tag{10.2.3}
\]

Left hand side = \( Y_4 (-) Y_0 \) as \( f(X_0(+4H)) = Y_4, f(X_0) = Y_0 \).

In the right hand side, since \( X = X_0(+R(\cdot)H) \),

therefore \( dX = H(\cdot) dR \) (as \( H \) is width of interval it is fuzzy constant and \( R \) is fuzzy variable.)

\( X = X_0 \) gives, \( R = (X_0(-)X_0)/H = [(x_0^{(1)}-x_0^{(3)})/h^{(3)}, 0, (x_0^{(3)}-x_0^{(1)})/h^{(1)}] = [\delta', 0, \mu'] \) (say) \((a)\)

and \( X = X_0(+4H) \) gives, \( R = ((X_0(+4H)-X_0)/H = [(X_0(-)X_0 (+) 4H)/H

\( = (X_0(-)X_0)/H (+) 4H/H

\( = [\delta', 0, \mu'][(+4[h^{(1)}/h^{(3)}), 1, h^{(3)}/h^{(1)}] = [\delta, 4, \mu] \) (say) \((b)\)

So we have right hand side of (10.2.3),

\[
= H(\cdot) \int_{[\delta, \mu]/H}^{[\delta', \mu]}/H [Y_0' (+R(\cdot)\Delta Y_0'/ (+ [R(\cdot)(R(-1))]/2(\cdot) \Delta^2 Y_0'/ (+ [R(R(-1))(R(-2))]/6(\cdot) \Delta^3 Y_0'/ (+ [R(R(-1))(R(-2))]/24(\cdot) \Delta^4 Y_0'/ (+ \ldots dR \tag{10.2.4}
\]

Now,

\[
\Delta Y_0' = (Y_1(-)Y_0' \ldots \Delta^2 Y_0' = (Y_2(-)2Y_1(+Y_0') \ldots \Delta^3 Y_0' = (Y_3(-)3Y_2(+3Y_1(-)Y_0') \ldots
\]

Therefore

\[
Y_4 (-) Y_0 = H(\cdot)[Y_0' [\delta- \mu', 4, \mu-2]^2 (+) \{(Y_1(-)Y_0')/2 [\delta- \mu', 4, \mu-2]^3 \} (+) \{(Y_2(-)2Y_1(+Y_0') \ldots /12 [[2[\delta- \mu', 4, \mu-2]^3 (-3[\delta- \mu', 4, \mu-2]^2 \} (+) \{( Y_3(-)3Y_2(+3Y_1(-)Y_0')/24 \left[\right.\right.\right. \right.\]
\[
\mu', 4, \mu - \delta' [(-) 4[\delta - \mu', 4, \mu - \delta']^3 (+) 4[\delta - \mu', 4, \mu - \delta']^2]
\]

\[
= H/24(.) \{ [24[\delta - \mu', 4, \mu - \delta'] (-) 22[\delta - \mu', 4, \mu - \delta']^2 (+) 8[\delta - \mu', 4, \mu - \delta']^3 (-) [\delta - \mu', 4, \mu - \delta']^4 \} Y_0' (+) [36[\delta - \mu', 4, \mu - \delta']^2 (-) 20[\delta - \mu', 4, \mu - \delta']^3 (+) 3[\delta - \mu', 4, \mu - \delta']^4 \} Y_1' (+) [-18[\delta - \mu', 4, \mu - \delta']^2 (+) 16[\delta - \mu', 4, \mu - \delta']^3 (-) 3[\delta - \mu', 4, \mu - \delta']^4 \} Y_2' (+) [4[\delta - \mu', 4, \mu - \delta']^2 (-) 4[\delta - \mu', 4, \mu - \delta']^4 \} Y_3'
\]

(neglecting 4th and higher order differences.)

Therefore

\[
Y_4 = Y_0 (+) H/24(.) \{ [24[\delta - \mu', 4, \mu - \delta'] (-) 22[\delta - \mu', 4, \mu - \delta']^2 (+) 8[\delta - \mu', 4, \mu - \delta']^3 (-) [\delta - \mu', 4, \mu - \delta']^4 \} Y_0' (+) [36[\delta - \mu', 4, \mu - \delta']^2 (-) 20[\delta - \mu', 4, \mu - \delta']^3 (+) 3[\delta - \mu', 4, \mu - \delta']^4 \} Y_1' (+) [-18[\delta - \mu', 4, \mu - \delta']^2 (+) 16[\delta - \mu', 4, \mu - \delta']^3 (-) 3[\delta - \mu', 4, \mu - \delta']^4 \} Y_2' (+) [4[\delta - \mu', 4, \mu - \delta']^2 (-) 4[\delta - \mu', 4, \mu - \delta']^4 \} Y_3'
\]

(10.2.5)

This is Milne's Extrapolation algorithm or predictor algorithm.

**Note:** In Milne's method to find \(Y_4\) we require four prior values \(Y_0, Y_1, Y_2, Y_3\) of \(Y\). If these values are not given we determine them by using Runge-Kutta method of fuzzified form.

Let us write now fuzzy membership function of predictor algorithm.

Assuming \([\delta - \mu', 4, \mu - \delta']\) as \([v', 4, v]\) we can express (10.2.5) as

\[
\]

\[
= Y_0 (+) H/24(.) \{ [24v'^2-22v'^2-v^2, 0, 24v +8v^3-22v^2-v^4] Y_0' (+) [36v'^2+3v^4-20v^3, 64, 36v'^2+3v^4-20v^3] Y_1' (+) [16v'^2-18v^2-3v^4, -32, 16v'^2-18v^2-3v^4] Y_2' (+) [4v'^2+4v^4-4v^3, 64, 4v'^2+4v^4-4v^3] Y_3'
\]

(10.2.6)

We try to write fuzzy membership function(f.m.f) of coefficients of \(Y_0', Y_1', Y_2', Y_3'\), successively.

\[
\mu_{(\text{coefficient of } Y_0')}(x) = \begin{cases} 
\{x - (24v^2+8v^3-22v^2-v^4)/(-24v^2+8v^3-22v^2-v^4)\}, & 0 \leq x \leq 24v^2+8v^3-22v^2-v^4 \\
-x+(24v^2+8v^3-22v^2-v^4)/(24v^2+8v^3-22v^2-v^4), & 0 \leq x \leq 24v^2+8v^3-22v^2-v^4 \\
0, & \text{otherwise}
\end{cases}
\]

(c)
To compute the intervals of confidence for each level $a$, the triangular shapes will be described
by functions of $a$ in the following manner:

From (c),

$$a = \{x_1^{(a)} - (24\sqrt{v^3} - 22v^2 - 4^v) \}/ \{-(24\sqrt{v^3} - 22v^2 - 4^v)\}$$

and

$$a = \{-x_2^{(a)} + (24\sqrt{v^3} - 22v^2 - 4^v)\}/ \{(24\sqrt{v^3} - 22v^2 - 4^v)\}$$

So, (coefficient of $Y_0$) $^{(a)} = [x_1^{(a)}, x_2^{(a)}]$

$$= [\{-(24\sqrt{v^3} - 22v^2 - 4^v)\}a + (24\sqrt{v^3} - 22v^2 - 4^v), -(24\sqrt{v^3} - 22v^2 - 4^v)\}a + (24\sqrt{v^3} - 22v^2 - 4^v)]$$

Similarly f.m.f.s of coefficients of $Y_1$, would be

$$\mu(\text{coefficient of } Y_1^{(a)}(x)) = \begin{cases} \{x - (36v^2 + 3v^4 - 20v^3)\}/ \{64 - (36v^2 + 3v^4 - 20v^3)\}, & 36v^2 + 3v^4 - 20v^3 \leq x \leq 64 \\ \{x + (36v^2 + 3v^4 - 20v^3)\}/ \{(36v^2 + 3v^4 - 20v^3)\}, & 64 \leq x \leq 36v^2 + 3v^4 - 20v^3 \\ 0, & \text{otherwise} \end{cases}$$

(Coefficient of $Y_1^{(a)} = [(64 - (36v^2 + 3v^4 - 20v^3))a + (36v^2 + 3v^4 - 20v^3), -(36v^2 + 3v^4 - 20v^3) - 64]a + (36v^2 + 3v^4 - 20v^3]$

$$\mu(\text{coefficient of } Y_2^{(a)}(x)) = \begin{cases} \{x - (16v^3 - 18v^2 - 3v^4)\}/ \{-32 - (16v^3 - 18v^2 - 3v^4)\}, & 16v^2 - 18v^2 - 3v^4 \leq x \leq -32 \\ \{x + (16v^3 - 18v^2 - 3v^4)\}/ \{(16v^3 - 18v^2 - 3v^4) - 32\}, & -32 \leq x \leq 16v^2 - 18v^2 - 3v^4 \\ 0, & \text{otherwise} \end{cases}$$

(Coefficient of $Y_2^{(a)} = [(-32 - (16v^3 - 18v^2 - 3v^4))a + (16v^3 - 18v^2 - 3v^4), -(16v^3 - 18v^2 - 3v^4) + 32]a + (16v^3 - 18v^2 - 3v^4]$]

$$\mu(\text{coefficient of } Y_3^{(a)}(x)) = \begin{cases} \{x - (4v^2 + v^4 - 4v^3)\}/ \{64 - (4v^2 + v^4 - 4v^3)\}, & 4v^2 + v^4 - 4v^3 \leq x \leq 64 \\ \{x + (4v^2 + v^4 - 4v^3)\}/ \{(4v^2 + v^4 - 4v^3) - 64\}, & 64 \leq x \leq 4v^2 + v^4 - 4v^3 \\ 0, & \text{otherwise} \end{cases}$$

(Coefficient of $Y_3^{(a)} = [(64 - (4v^2 + v^4 - 4v^3))a + (4v^2 + v^4 - 4v^3), -(4v^2 + v^4 - 4v^3) - 64]a + (4v^2 + v^4 - 4v^3)]$

Now, $Y_4^{(a)} = Y_0^{(a)} + (\text{coefficient of } Y_0^{(a)}) (.) Y_0^{(a)} (+) (\text{coefficient of } Y_1^{(a)}) (.) Y_1^{(a)} (+) (\text{coefficient of } Y_2^{(a)}) (.) Y_2^{(a)} (+) (\text{coefficient of } Y_3^{(a)}) (.) Y_3^{(a)}$

(10.2.7)
As here $Y' = f(X,Y)$, $Y_0' = f(X_0,Y_0)$ etc., which are to be computed from given function with
given conditions we could find f.m.f. of $Y_4$ only with the help of a numerical example.

To get the checking formula we integrate (10.2.1) from $X_0$ to $X_0(+2H)$,
or, from $R = [\delta', 0, \mu']$ to $R = [\delta', 0, \mu'] (+) 2[h^{(1)}/h^{(3)}, 1, h^{(3)}/h^{(1)}] = [\delta, 2, \mu]$.
Then

$$Y_2 (-) Y_0 = H(.) \int_{[\delta, 0, \mu]} \{ Y_0' (+) R \Delta Y_0' (+) [R(R-1)]/2 \Delta^2 Y_0' (+) [R(R-1)(R-2)]/6 \Delta^3 Y_0' \} dR$$

$$= H(.) \int_{[\delta, 0, \mu]} \{ R^2/2 \Delta Y_0' (+) [R^3/6(-)R^2/4] \Delta^2 Y_0' (+) [R^4/4 (-) R^3(+R^2)]/6 \Delta^3 Y_0' \}$$

$$= H(.) \int_{[\delta, 0, \mu]} \{ [\delta - \mu', 2, \mu-\delta'] Y_0' (+)/12[\delta - \mu', 2, \mu-\delta']^2 \Delta Y_0' (+) 1/12[2[\delta - \mu', 2, \mu-\delta']^3 (-) 3$$

$$[\delta - \mu', 2, \mu-\delta']^2 \Delta Y_0' (+) 1/24 \{[\delta - \mu', 2, \mu-\delta']^4 (-) 4[\delta - \mu', 2, \mu-\delta']^3 (+) 4[\delta - \mu', 2, \mu-\delta']^2 \Delta^2 Y_0' \} \}$$

$$= H/12(.) \{ 12[\delta - \mu', 2, \mu-\delta'] (-) 9[\delta - \mu', 2, \mu-\delta']^2 (-) 2[\delta - \mu', 2, \mu-\delta']^3 Y_0' (+) 12[\delta - \mu', 2, \mu-\delta']^2 (+) 4[\delta - \mu', 2, \mu-\delta']^3 Y_1' (+) -3[\delta - \mu', 2, \mu-\delta']^2 (+) 2[\delta - \mu', 2, \mu-\delta']^3 Y_2' \}$$

Therefore neglecting $3^{rd}$ and higher order differences we can write,

$$Y_2 = Y_0 (+) H/12(.) \{ 12[\delta - \mu', 2, \mu-\delta'] (-) 9[\delta - \mu', 2, \mu-\delta']^2 (+) 2[\delta - \mu', 2, \mu-\delta']^3 \} Y_0' (+) \{ 12[\delta - \mu', 2, \mu-\delta']^2 (-) 4[\delta - \mu', 2, \mu-\delta']^3 \} Y_1' (+) \{-3[\delta - \mu', 2, \mu-\delta']^2 (+) 2[\delta - \mu', 2, \mu-\delta']^3 \} Y_2' \}$$

(10.2.9)

Formula (10.2.9) is called the Corrector Formula. The value of $Y_4$ obtained from (10.2.5) can
be checked using (10.2.9).

Exactly in the same way as in (10.2.7) we can find

$$Y_2^{(a)} = Y_0^{(a)} (+) (\text{coefficient of } Y_0'/)^{(a)}(.) Y_0'/^{(a)} (+) (\text{coefficient of } Y_1'/)^{(a)}(.) Y_1'/^{(a)}$$

$$(+) (\text{coefficient of } Y_2'/)^{(a)}(.) Y_2'/^{(a)}.$$
five consecutive values of \(X\), formulas (10.2.5) and (10.2.9) may be written in the more general forms:

\[
Y_{n+1} = Y_{n-3} + \frac{h}{24} \left\{ 24(\delta-\mu', 4, \mu-\delta')^2 + 8(\delta-\mu', 4, \mu-\delta')^3 \right\} + \frac{h}{12} \left\{ 12(\delta-\mu', 2, \mu-\delta')^2 + 3(\delta-\mu', 2, \mu-\delta')^3 \right\} + \frac{h}{24} \left\{ 4(\delta-\mu', 4, \mu-\delta')^3 \right\} Y_{n+1}' + \frac{h}{24} \left\{ 3(\delta-\mu', 4, \mu-\delta')^3 \right\} Y_{n+1}'' + \frac{h}{24} \left\{ 18(\delta-\mu', 4, \mu-\delta')^3 \right\} Y_{n+1}''' + \frac{h}{24} \left\{ 20(\delta-\mu', 4, \mu-\delta')^3 \right\} Y_{n+1}'''
\]

and

\[
Y_{n+1} = Y_{n-3} + \frac{h}{12} \left\{ 12(\delta-\mu', 2, \mu-\delta')^2 + 3(\delta-\mu', 2, \mu-\delta')^3 \right\} Y_{n+1}' + \frac{h}{12} \left\{ 4(\delta-\mu', 2, \mu-\delta')^3 \right\} Y_{n+1}'' + \frac{h}{12} \left\{ 3(\delta-\mu', 2, \mu-\delta')^3 \right\} Y_{n+1}''' + \frac{h}{12} \left\{ 18(\delta-\mu', 2, \mu-\delta')^3 \right\} Y_{n+1}'''
\]

which are the final forms of the Milne's formulas.

For error, let \(Y_{n+1}^{(1)}\) and \(Y_{n+1}^{(2)}\) denote the values of \(Y\) given by (10.3.1) and (10.3.2). Then if the value of \(H\) is such that the inherent error in each formula is given by its remainder term involving \(\Delta^4 Y\), the true value of \(Y\) at \(X = X_{n+1}\) is either

\[
Y = Y_{n+1}^{(1)}(+)\{[\delta-\mu', 4, \mu-\delta']^3/5(-)3/2[\delta-\mu', 4, \mu-\delta']^4 (+)11/3[\delta-\mu', 4, \mu-\delta']^3 (-)3[\delta-\mu', 4, \mu-\delta']^2}A4Y/24\quad\text{(using(10.2.4))}
\]

or

\[
Y = Y_{n+1}^{(2)}(+)\{[\delta-\mu', 2, \mu-\delta']^3/5(-)3/2[\delta-\mu', 2, \mu-\delta']^4 (+)11/3[\delta-\mu', 2, \mu-\delta']^3 (-)3[\delta-\mu', 2, \mu-\delta']^2}A4Y/24\quad\text{(using(10.2.8))}
\]

which gives

\[
Y_{n+1}^{(1)}(-)Y_{n+1}^{(2)} = ([\delta-\mu', 2, \mu-\delta']^3/5(-)3/2[\delta-\mu', 2, \mu-\delta']^4 (+)11/3[\delta-\mu', 2, \mu-\delta']^3 (-)3[\delta-\mu', 2, \mu-\delta']^2}\cdot\Delta^4Y/24
\]

or

\[
\Delta^4Y/24 = (Y_{n+1}^{(1)}(-)Y_{n+1}^{(2)}/(([\delta-\mu', 2, \mu-\delta']^3/5(-)3/2[\delta-\mu', 2, \mu-\delta']^4 (+)11/3[\delta-\mu', 2, \mu-\delta']^3 (-)3[\delta-\mu', 2, \mu-\delta']^2}(-)\{[\delta-\mu', 4, \mu-\delta']^3/5(-)3/2[\delta-\mu', 4, \mu-\delta']^4 (+)11/3[\delta-\mu', 4, \mu-\delta']^3 (-)3[\delta-\mu', 4, \mu-\delta']^2}24))
\]

where \(E = \Delta^4Y/24\) denotes the principal part of the error.

This simple formula enables the computer to test the accuracy of each computed result.
10.4 A Numerical Example

To tabulate using Milne’s algorithm the numerical solution of
\[ \frac{dY}{dX} = X(Y) \] with \( X_0 = [-.01, 0, 01], Y_0 = [.99, 1, 1.01] \) for
\[ [.35, .4, .45] \leq X \leq [.44, .5, .56] \] with \( H = [.09, .1, .11] \).

Here \( f(X,Y) = X(Y) \).

From Runge-Kutta fourth order formula, we get the following steps:

**Step-1**

\[ K_1 = H(.)f(X_0,Y_0) = [.09, .1, .11](.)([-.01, 0, 01] + [.99, 1, 1.01]) = [.0882, .1, .1122] \]
\[ K_2 = H(.)f(X_0(+)H/2, Y_0(+)K_1/2) = [.09, .1, .11](.)([.035, .05, .065] + [.10341, 1.05, 1.0661]) \]
\[ = [.0962, .110, .1244] \]
\[ K_3 = H(.)f(X_0(+)H/2, Y_0(+)K_2/2) = [.09, .1, .11](.)([.08, .1, .12] + [.108658, 1.1105, 1.13509]) \]
\[ = [.09658, .1105, .12509] \]
\[ K_4 = H(.)f(X_0(+)H, Y_0(+)K_3) = [.09, .1, .11](.)([1.0731, 1.105, 1.1372]) \]
\[ = [.0962, .1105, .12509] \]

From which we get \( K = (K_1 + 2K_2 + 2K_3 + K_4)/6 = [.096458, .110342, .12487] \).

Therefore \( Y_1 = Y_0(+)K = [1.086458, 1.110342, 1.13487] \) where \( X_1 = X_0(+)H = [.08, .1, .12] \).

**Step-2**

\[ K_1 = H(.)f(X_1,Y_1) = [.09, .1, .11](.)([1.166458, 1.210342, 1.25487]) \]
\[ = [.094981, .121034, .1380357] \]
\[ K_2 = H(.)f(X_1(+)H/2, Y_1(+)K_1/2) = [.09, .1, .11](.)([1.125, .15, .175] + [.1138948, 1.170859, 1.203888]) \]
\[ = [.113755, .132086, .151678] \]
\[ K_3 = H(.)f(X_1(+)H/2, Y_1(+)K_2/2) = [.09, .1, .11](.)([1.268336, 1.326385, 1.385709]) \]
\[ = [.114150, .132638, .152428] \]
\[ K_4 = H(.)f(X_1(+)H, Y_1(+)K_3) = [.09, .1, .11](.)([1.17, .2, .23] + [.1200608, 1.24298, 1.287298]) \]
\[ = [.123355, .144298, .166903] \]

From which we get \( K = (K_1 + 2K_2 + 2K_3 + K_4)/6 = [.096458, .110342, .12487] \).

Therefore \( Y_2 = Y_1(+)K = [1.200482, 1.242805, 1.287062] \)

and \( X_2 = X_1(+)H = [.17, .2, .23] \).

**Step-3**

\[ K_1 = H(.)f(X_2,Y_2) = [.09, .1, .11](.)([1.370482, 1.442805, 1.517062]) \]
\[ = [.123343, .144280, .1668768] \]

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\[ K_2 = H(.)f(X_2(+)H/2, Y_2(+)K_1/2) = [.09, .1, .11](.)([.215, .25, .285](+)1.262154, 1.314945, 1.370500)) = [.132944, .156494, .182105] \]
\[ K_3 = H(.)f(X_2(+)H/2, Y_2(+)K_2/2) = [.09, .1, .11](.)([1.481954, 1.571052, 1.663114] = [.133376, .157105, .182942] \]
\[ K_4 = H(.)f(X_2(+)H, Y_2(+)K_3/2) = [.09, .1, .11](.)([1.333858, 1.39991, 1.470004]) = [.143447, .169999, .1991] \]

From which we get \[ K = (K_1(+)2K_2(+)2K_3(+)K_4)/6 = [.133238, .156912, .182678] \]
Therefore \[ Y_3 = Y_2(+)K = [1.33372, 1.399716, 1.469740] \]
and \[ X_3 = X_2(+)H = [.26, .3, .34]. \]
These are given in the table below:

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>f(X,Y) (-Y')</th>
</tr>
</thead>
<tbody>
<tr>
<td>[.01, 0, .01]</td>
<td>[.99, 1, 1.01]</td>
<td>[.98, 1, 1.02]</td>
</tr>
<tr>
<td>[.08, .1, .11]</td>
<td>[.1086458, 1.110342, 1.13487]</td>
<td>[1.166458, 1.210342, 1.24487]</td>
</tr>
<tr>
<td>[.17, .2, .23]</td>
<td>[.1200482, 1.242805, 1.287062]</td>
<td>[1.370482, 1.442805, 1.517062]</td>
</tr>
<tr>
<td>[.26, .3, .34]</td>
<td>[.133372, 1.399716, 1.46974]</td>
<td>[1.59372, 1.699716, 1.80974]</td>
</tr>
<tr>
<td>[.35, .4, .45]</td>
<td>[-39.784297, 1.583643, 43.081829]</td>
<td>[-39.49297, 1.983643, 43.531829]</td>
</tr>
</tbody>
</table>

To obtain \( Y([.35, .4, .45]) \) we are to use the predictor formula (10.2.5).

Here \[ [\delta', 0, \mu'] = (X_0(-)X_0)/H = [.02/.11, 0/.1, .02/.09] = [.181818, 0, .222222] \] (using (a))
and \[ [\delta- \mu', 4, \mu-\delta'] = [\delta', 0, \mu'] (+) 4 [.09/.11, .1/.1, .11/.09] (=) [2.868684, 4, 5.29293] \] (using (b))

Coefficient of \( Y_0' \)
\[ = [.09, .1, .11]/24 (.) (24[2.868684, 4, 5.29293] (-) 22[2.868684, 4, 5.29293])^2 (+) \]
\[ 8[2.868684, 4, 5.29293]^3 (-)[2.868684, 4, 5.29293]^4 \] = [-5.234163, 0, 4.876624]

Coefficient of \( Y_1' \)
\[ = [.09, .1, .11]/24 (.) (36[2.868684, 4, 5.29293]^2 (-) 20[2.868684, 4, 5.29293]^3 (+) \]
\[ 3[2.868684, 4, 5.29293]^4 \] = [-11.34556, .266667, 13.233254]

Coefficient of \( Y_2' \)

Coefficient of \( Y_3' \)
\[ = [.09, .1, .11]/24 (.) 4[2.868684, 4, 5.29293]^2 (-) 4[2.868684, 4, 5.29293]^3 (+) [2.868684, 4, 5.29293]^4 \]
Now for the formula (10.2.4),
\[ Y_4 = Y_0 \quad (+) \quad (\text{coefficient of } Y_0) (\cdot) Y_0 \quad (+) \quad (\text{coefficient of } Y_1) (\cdot) Y_1 \quad (+) \quad (\text{coefficient of } Y_2) (\cdot) Y_2 \quad (+) \quad (\text{coefficient of } Y_3) (\cdot) Y_3 \]
\[ = [0.99, 1, 1.01] \quad (+) \quad [-5.234163, 0, 4.876624] \quad (\cdot) \quad [0.98, 1, 1.02] \quad (+) \quad [-11.34556, 0.266667, 13.233254] \quad (\cdot) \quad [1.166458, 1.210342, 1.24487] \quad (+) \quad [-11.358182, -0.133333, 9.256656] \quad (\cdot) \quad [1.370482, 1.442805, 1.517062] \quad (+) \quad [-2.25482, 0.266667, 3.636473] \quad (\cdot) \quad [1.59372, 1.699716, 1.80974] \]
\[ = [-39.784297, 1.58364, 43.081829] \quad (10.4.1) \]

Here \( Y_0, Y_1, Y_2, Y_3 \) are taken from the table above.

We now write the fuzzy membership functions (f.m.f) of each term one by one.

\[ \mu_{Y_0}(x) = \begin{cases} 
(x - 0.99)/0.01, & \text{if } 0.99 \leq x \leq 1 \\
(-x + 1.01)/0.01, & \text{if } 1 \leq x \leq 1.01 \\
0, & \text{otherwise}
\end{cases} \]

The intervals of confidence for each level \( \alpha \) is
\[ Y_0(\alpha) = [x_1(\alpha), x_2(\alpha)] = [0.01\alpha + 0.99, -0.01\alpha + 1.01] \]

Similarly f.m.f.s of \( Y_1^i, Y_2^i, Y_3^i \) (i = 0,1,2,3) would be as follows:

\[ \mu_{Y_1}(x) = \begin{cases} 
(x - 1.166458)/0.043884, & \text{if } 1.166458 \leq x \leq 1.210342 \\
(-x + 1.24487)/0.034528, & \text{if } 1.210342 \leq x \leq 1.24487 \\
0, & \text{otherwise}
\end{cases} \]

\[ Y_1(\alpha) = [0.043884\alpha + 1.166458, -0.034528\alpha + 1.24487] \]

\[ \mu_{Y_2}(x) = \begin{cases} 
(x - 1.370482)/0.072323, & \text{if } 1.370482 \leq x \leq 1.442805 \\
(-x + 1.517062)/0.074258, & \text{if } 1.442805 \leq x \leq 1.517062 \\
0, & \text{otherwise}
\end{cases} \]

\[ Y_2(\alpha) = [0.072323\alpha + 1.370482, -0.074258\alpha + 1.517062] \]
\( \mu_{Y_3'}(x) = \begin{cases} (x - 1.59372)/.105996, & 1.59372 \leq x \leq 1.699716 \\ (-x + 1.80974)/.110024, & 1.699716 \leq x \leq 1.80974 \\ 0, & \text{otherwise} \end{cases} \)

\( Y_3^{(a)} = [.105996a + 1.59372, -.110024a + 1.80974] \)

Next,

\[ \mu(\text{coefficient of } Y_0') (x) = \begin{cases} (x + 5.234163)/5.234163, & -5.234163 \leq x \leq 0 \\ (-x + 4.876624)/4.876624, & 0 \leq x \leq 4.876624 \\ 0, & \text{otherwise} \end{cases} \]

\[ (\text{coefficient of } Y_0')^{(a)} = [5.234163a - 5.234163, -4.876624a + 4.876624] \]

\[ \mu(\text{coefficient of } Y_1') (x) = \begin{cases} (x + 11.34556)/11.34556, & -11.34556 \leq x \leq 11.34556 \\ (-x + 13.233254)/12.966587, & 11.34556 \leq x \leq 13.233254 \\ 0, & \text{otherwise} \end{cases} \]

\[ (\text{coefficient of } Y_1')^{(a)} = [11.234163a - 11.34556, -12.966587a + 13.233254] \]

\[ \mu(\text{coefficient of } Y_2') (x) = \begin{cases} (x + 11.358182)/11.358182, & -11.358182 \leq x \leq 11.358182 \\ (-x + 9.256656)/9.389989, & 9.256656 \leq x \leq 11.358182 \\ 0, & \text{otherwise} \end{cases} \]

\[ (\text{coefficient of } Y_2')^{(a)} = [11.224849a - 11.358182, -9.389989a + 9.256656] \]

\[ \mu(\text{coefficient of } Y_3') (x) = \begin{cases} (x + 2.25482)/2.521487, & -2.25482 \leq x \leq 2.25482 \\ (-x + 3.636473)/3.369806, & 3.636473 \leq x \leq 2.25482 \\ 0, & \text{otherwise} \end{cases} \]

\[ (\text{coefficient of } Y_3')^{(a)} = [2.521487a - 2.25482, -3.369806a + 3.636473] \]

So, \( Y_4^{(a)} = Y_0^{(a)} (+) (\text{coefficient of } Y_0')^{(a)} (+) (\text{coefficient of } Y_1')^{(a)} (+) (\text{coefficient of } Y_2')^{(a)} (+) (\text{coefficient of } Y_3')^{(a)} \)

\[ = [-1.60529a^2 + 42.549755a - 39.784297, 1.613984a^2 - 43.112185a + 43.081829] \]

The equations to be solved are

\[-1.60529a^2 + 42.549755a - 39.784297 - x_1 = 0 \quad \text{(i)}
\]
\[1.613984a^2 - 43.112185a + 43.081829 - x_2 = 0 \quad \text{(ii)}
\]

We are to retain only two roots in \([0,1] \). From (i), \( a = \{-42.549755 + \sqrt{(42.549755)^2 - 4\times 1.60529 (39.784297 + x_1)}\}/2 \times 1.60529 \)
From (ii), 
\[
a = \frac{43.112185 - \sqrt{((-43.112185)^2 - 4*1.613984(43.081829 - x^2))}}{2} * 1.613984
\]
So,
\[
\mu_y(x) = \begin{cases} 
\frac{-42.549755 + \sqrt{((42.549755)^2 - 4*1.60529(39.84297 + x))}}{-2} * 1.60529, & \text{for } -39.84297 \leq x \leq 1.58364 \\
43.112185 - \sqrt{((-43.112185)^2 - 4*1.613984(43.081829 - x))}/2 * 1.613984, & \text{for } 1.58364 \leq x \leq 43.081829 \\
0, & \text{otherwise}
\end{cases}
\]

We next discuss the **corrector formula** (10.2.9)

For the corrector formula, 
\[
[\delta - \mu , 2, \mu - \delta'] = [1.454544, 2, 2.666666]
\]

Coefficient of \(y'_0\)
\[
= \begin{bmatrix} .09, & .1, & .11 \end{bmatrix}/12 \begin{bmatrix} 12[1.454544, 2, 2.666666] - 9[1.454544, 2, 2.666666]^2 & + 2[1.454544, 2, 2.666666]^3 \end{bmatrix} = \begin{bmatrix} -.370248, & .033333, & .466442 \end{bmatrix}
\]

Coefficient of \(y'_1\)
\[
= \begin{bmatrix} .09, & .1, & .11 \end{bmatrix}/12 \begin{bmatrix} 12[1.454544, 2, 2.666666] - 4[1.454544, 2, 2.666666]^2 \end{bmatrix} = \begin{bmatrix} -.462581, & .133333, & .669385 \end{bmatrix}
\]

Coefficient of \(y'_2\)
\[
= \begin{bmatrix} .09, & .1, & .11 \end{bmatrix}/12 \begin{bmatrix} 12[1.454544, 2, 2.666666] - 2[1.454544, 2, 2.666666]^3 \end{bmatrix} = \begin{bmatrix} -.138918, & .033333, & .289472 \end{bmatrix}
\]

\(Y_2 = Y_0 + (\text{coefficient of } y'_0) \cdot Y'_0 + (\text{coefficient of } y'_1) \cdot Y'_1 + (\text{coefficient of } y'_2) \cdot Y'_2\)

\[
= \begin{bmatrix} 1.200482, & 1.242805, & 1.287062 \end{bmatrix} + \begin{bmatrix} -.370248, & .033333, & .466442 \end{bmatrix} \cdot 1.370482, 1.442805, 1.517062 + \begin{bmatrix} -.462581, & .133333, & .669385 \end{bmatrix} \cdot 1.59372, 1.699716, 1.80974 + \begin{bmatrix} -.138918, & .033333, & .289472 \end{bmatrix} \cdot 1.983642, 43.531829
\]

\[
= \begin{bmatrix} 1.200482, & 1.242805, & 1.287062 \end{bmatrix} + \begin{bmatrix} 1.370482, 1.442805, 1.517062 \end{bmatrix} + \begin{bmatrix} 1.59372, 1.699716, 1.80974 \end{bmatrix} + \begin{bmatrix} 1.983642, 43.531829 \end{bmatrix}
\]

\(Y_2 = Y_0 + (\text{coefficient of } y'_0) \cdot Y'_0 + (\text{coefficient of } y'_1) \cdot Y'_1 + (\text{coefficient of } y'_2) \cdot Y'_2\)

(Here \(Y'_2\) is \(Y'_4\), \(Y'_1\) is \(Y'_3\), \(Y'_0\) is \(Y'_2\) and \(Y_0\) is \(Y_2\) of the predictor formula (10.2.5). So we would consider \(Y'_0\) to be \(Y_2\) and \(Y'_1\) to be \(Y_3\) of predictor formula.)

Now,
\[
\mu_{y_0}(x) = \begin{cases} 
(x - 1.200482)/.042323, & 1.200482 \leq x \leq 1.242805 \\
-x + 1.287062)/.044257, & 1.242805 \leq x \leq 1.287062 \\
0, & \text{otherwise}
\end{cases}
\]
Therefore,

\[ Y_0^{(a)} = \begin{cases} 
0.042323a + 1.200482, & -39.49297 \leq x \leq 1.98364 \\
-x + 43.531828/141.548187, & 1.98364 \leq x \leq 45.531829 \\
0, & \text{otherwise}
\end{cases} \]

\[ \mu_{Y_2'}(x) = \begin{cases} 
(x + 39.49297)/41.476612, & -39.49297 \leq x \leq 1.98364 \\
-x + 43.531828/141.548187, & 1.98364 \leq x \leq 45.531829 \\
0, & \text{otherwise}
\end{cases} \]

\[ Y_2^{(a)} = \begin{cases} 
41.476612a - 39.49297, & -39.49297 \leq x \leq 0.33333 \\
-x + 43.531828/141.548187, & 0.33333 \leq x \leq 43.531829 \\
0, & \text{otherwise}
\end{cases} \]

\[ \mu_{\text{coefficient of } Y_0'}(x) = \begin{cases} 
(x + 0.370248)/0.403581, & -0.370248 \leq x \leq 0.03333 \\
-x + 0.466442/0.433109, & 0.03333 \leq x \leq 0.466442 \\
0, & \text{otherwise}
\end{cases} \]

\[ \mu_{\text{coefficient of } Y_1'}(x) = \begin{cases} 
(x + 0.462581)/0.595914, & -0.462581 \leq x \leq 0.13333 \\
-x + 0.669385/0.536052, & 0.13333 \leq x \leq 0.669385 \\
0, & \text{otherwise}
\end{cases} \]

\[ \mu_{\text{coefficient of } Y_2'}(x) = \begin{cases} 
(x + 0.138918)/0.172251, & -0.138918 \leq x \leq 0.03333 \\
-x + 0.289472/0.256139, & 0.03333 \leq x \leq 0.289472 \\
0, & \text{otherwise}
\end{cases} \]

\[ \mu_{\text{coefficient of } Y_2''}(a) = \begin{cases} 
0.403581a - 0.370248, & -0.370248 \leq x \leq 0.03333 \\
-x + 0.466442, & 0.03333 \leq x \leq 0.466442 \\
0, & \text{otherwise}
\end{cases} \]

So,

\[ Y_2^{(a)} = Y_0^{(a)} + (\text{coefficient of } Y_0')^{(a)} Y_0'^{(a)} + (\text{coefficient of } Y_1')^{(a)} Y_1'^{(a)} + (\text{coefficient of } Y_2')^{(a)} Y_2'^{(a)} \]

\[ = [-10.7919311a^2 + 23.933426\alpha - 11.630467, 10.73325a^2 - 24.956857a + 15.807342] \]

The equations to be solved are

-10.7919311a^2 + 23.933426\alpha - 11.630467 - x_1 = 0 \quad (iii) \\
10.73325a^2 - 24.956857a + 15.807342 - x_2 = 0 \quad (iv) 

We are to retain only two roots in [0,1].

From (iii),

\[ a = [-23.933426 + \sqrt{(23.933426)^2 - 4*10.7919311 (11.630467 + x_1)}/>2*10.697293 \]

From (iv),

\[ a = (24.956857 - \sqrt{(-24.956857)^2 - 4*10.73325 (15.807342 - x_2)})/2* 10.73325 \]
So,
\[
\mu_Y(x) = \begin{cases} 
-23.933426 + \sqrt{(23.933426)^2 - 4\times10.719311(11.630467 + x)} / -2\times10.697293, \\
(24.956857 - \sqrt{(-24.956857)^2 - 4\times10.73325(15.807342 - x)}) / 2\times10.73325, \\
0, \quad \text{otherwise}
\end{cases}
\]

For error, let \( Y_{nt+1}^{(1)} \) and \( Y_{nt+1}^{(2)} \) denote the values of \( Y \) given by (10.4.1) and (10.4.2).

Then if the value of H is such that the inherent error in each formula is given by its remainder term involving \( \Delta^4Y' \), the true value of \( Y \) at \( x = X_{nt+1} \) is either

\[
Y = Y_{nt+1}^{(1)} + \left[\frac{2.868684, 4, 5.29293}{5} - \frac{3.2[2.868684, 4, 5.29293]^4}{11/3}[2.868684, 4, 5.29293](-) 3[2.868684, 4, 5.29293]^2) \Delta^4Y_o/24 \right. \\
\left. \text{using (10.3.1),} \right)
\]

and

\[
Y = Y_{nt+1}^{(2)} + \left[\frac{1.454544, 2, 2.66666}{5} - \frac{3[1.454544, 2, 2.66666]^3}{11/3}[1.454544, 2, 2.66666](-) 3[1.454544, 2, 2.66666]^3) \Delta^4Y'/24 \right. \\
\left. \text{using (10.3.2),} \right)
\]

So, \( Y_{nt+1}^{(1)} - Y_{nt+1}^{(2)} = [-1332.85652, -7.733544, 1219.338594] \Delta^4Y_0(-1/24) \]

which gives,

\[
E = \frac{\Delta^4Y'/24 = \{-39.784297, 1.583643, 43.081829\}(-)\{-11.630467, 1.583647, 15.807342\}}{\{-1219.338594, 7.733544, 1332.85652\}} \\
\text{using (10.3.3)}
\]

= \{-55.591639, -.000004, 54.712296\}/\{-1219.338594, 7.733544, 1332.85652\}

= \{-55.591639/-1219.338594, -.000004/7.733544, 54.712296/1332.85652\}

= [.04559, 000000, .0410489]

10.5 Conclusion

Though it needs a lot of numerical computations, fuzzification of Milne's predictor-corrector algorithm is worth studying. We have studied the numerical values correct up to sixth
decimal places. We observe here that the variation in triangular fuzzy forms in the final results are larger than our initial assumptions.