CHAPTER-1

GENERAL INTRODUCTION
A set with one binary operation satisfying the law of associativity. A semigroup is a generalization of the concept of groups: only one of the group axioms is retained—associativity; this is the explanation of the term "semigroup". Semigroups are called monoids if they have, in addition, an identity element. The theory of semigroups is one of the relatively young branches of algebra. The earliest studies of semigroups date back to the 1920's and are associated with the name of A.K. Sushkevich, who, in particular, determined the structure of the kernel (the minimal ideal) of a finite semigroup and thus, in particular, the structure of any finite semigroup without proper ideal. This result was later generalized by D.Rees to arbitrary completely simple semigroups and refined by the introduction of the concept of matrices over a group. Rees' theorem, which may be regarded as a sort of analogue of Wedderburn's theorem for simple algebras, is one of the fundamental propositions of the theory of the semigroups. Other early research on semigroups was done by A.Clifford; one of his first significant achievements was the introduction and investigation of semigroups which are unions of groups; these semi-groups are now known as completely regular or Clifford semigroups. By the end of the 1950's, the theory of semi-groups had become a self-contained branch of modern algebra with a rich store of problems, a broad range of methods and strong links with many fields of mathematics, both properly algebraic (primarily the theory of groups and rings) and others, such as functional analysis (semi-groups of operators on Banach spaces), differential geometry (semi-groups of partial transformations), and the algebraic theory of automata (semi-groups of automata).
The purpose of this dissertation is to present some structures theorems on archimedean semigroups, weakly commutative, congruence semigroups, \(\sigma\)-reflexive semigroups using the properties like medial, permutative, weakly separative, quasi separative and separative. The first step in the archimedean semigroups has been made by Tamura T [17]. Since then lot of papers has been presented on the theory of archimedean semigroups like Naoki Kimura [18], Miyuki Yamada [18] and many others.

The research of separative semigroups was being began form the famous paper of Howitt and Zulcevman [6]. Where in particular, they proved that any commutative separative semigroup is isomorphic to a semilattice of cancellative semigroups. A generalization for the non commutative case has been made by Burmistrorich [9], and independently by Petrich [11]. Drazin [17] introduced the term 'quasi-separativity' and studied connections between it and other semigroup properties. We shall follow the the terminology proposed by Drazin.

We also investigate the structures of a class of semigroups which are weakly commutative, weakly cancellative and weakly separative. The notion weakly separative is weaken version of a cancellative semigroup. N.P.Mukharjee [15] studied some properties of quasi commutative semigroups. We shall extend his results on weakly commutative, commutative semigroups to separative semigroups. Finally we consider the maximal separative homomorphic image of a quasi commutative semigroups.
Chapter 1 deals with the general introduction, preliminaries and summary of the results. Chapter 2 contains some results on archimedean semigroups and weakly commutative semigroups with some properties. Section 2.1 is mainly concerned with archimedean semigroups with two variables and prove some results using the properties like medial, permutable, weakly balanced and weakly separative. Section 2.2 deals with the archimedean semigroups defined in terms of three variables and prove that it is cancellative by assuming different conditions on semigroups. In section 2.3 it is prove that weakly commutative semigroup is weakly cancellative it is cancellative using the condition separative and without separative property.

In chapter 3 we discuss some congruence relations on semigroups. Congruence relation on semigroup is an equivalence relation with compatible condition. In section 3.1 we define a relation 'σ' on commutative semigroup and prove that 'σ' is weakly separative congruence on the semigroup and also prove that it is separative congruence on weakly commutative permutable semigroup. Section 3.2 concerned with the properties of σ-reflexive semigroups. The concept of σ-reflexive is due to Chacron and Thierren (1972) [9]. We prove that σ-reflexive semigroup is cancellative using the property permutable and separative.
Basic definitions and Examples

Definition: A semigroup is a non empty set 'S' together with an associative binary operation from $S \times S \rightarrow S$. The associative condition on $S$ states that $a(b \ c) = (a \ b) \ c$, for $a, b, c$ in $S$.

Examples: 1. The set of natural numbers with usual multiplication.
2. Any group.

Definition: A non-empty subset 'A' of a semigroup $(S, \cdot)$ is called a subsemigroup if $(A, \cdot)$ is a semigroup by itself.

Definition: A semigroup 'S' is called commutative if $a \cdot b = b \cdot a$ for all $a, b$ in $S$.

Definition: A system $(S, \leq)$, where the relation $\leq$ on $S$ satisfying the following axioms

1. Reflexivity: $a \leq a$;
2. Antisymmetry: $a \leq b, b \leq a$ imply $a = b$
3. Transitivity: $a \leq b, b \leq c$ imply $a \leq c$,
4. Linearity: $a \leq b$ or $b \leq a$
for all $a, b, c$ in $S$, is called a totally (linearly) ordered set.

Definition: An element 'a' in a semigroup $S$ is called an idempotent if $a^2 = a$.

If every element of $S$ is an idempotent then $S$ is called an idempotent semigroup.

Definition: An element $x$ in a semigroup $(S, \cdot)$ or (totally ordered set $(S, \cdot, \leq)$) is said to be

1. an identity if $ax = xa = a$.
2. a zero if $ax = xa = x$ for every $a$ in $S$. 
**Definition:** A semigroup \((S, .)\) with zero (usually we denote the zero element by the symbol "o".) is said to have no zero divisors if \(xy = o\) implies \(x = o\) or \(y = o\) for all \(x, y\) in \(S\).

**Definition:** An element \(x\) in a semigroup \((S, .)\) is

1. left cancellable; if \(xa = xb\) implies \(a = b\) for any \(a, b\) in \(S\).
2. right cancellable; if \(ax = bx\) implies \(a = b\) for any \(a, b\) in \(S\) and
3. cancellable if it is both left as well as right cancellable.

**Definition:** A semigroup \((S, .)\) all of whose elements are (left, right) cancellable is called a (left, right) cancellative semigroup.

**Definition:** A triple \((S, + , .)\) is a semiring if \(S\) is a non empty set, \(+ , .\) are binary operations on \(S\) satisfying that

i) \((S, +)\) is semigroup

ii) \((S, .)\) is semigroup

iii) \(a.(b+c) = a.b + a.c\) and \((b + c) .a = b. a + c . a\) for all \(a,b,c\) in \(S\)

* Through out this dissertation we consider permutable semigroup means semigroup satisfies the condition \(abc = bac=acb\) for every \(a, b, c\) in \(S\)