CHAPTER - 1

Introduction...
In the theory of rings, the study of both associative and nonassociative rings has evoked great interest and assumed importance. In particular, the study of derivations in prime or semiprime rings has yielded many interesting results. These results have applications in other branches.

Many mathematicians of recent years studied derivations on rings with keen interest and their investigations throw light on the study of derivations on ideals. Among these mathematicians Posner, Herstein, Bell, Daif, Martindale, Lee, Yen, Chung and Luh are the ones whose contributions to this field are outstanding.

In this work, we prove that if $R$ admits a nonzero derivation $d$ such that $d([x,y]) \in Z$ or $d([d(x),y]) \in Z$ or $d([x^2,y]) \in Z$, then $R$ is commutative. Also we prove that if $d$ is a nonzero $U^{-1}$ or $U^{**}$ derivation, then either $R$ is commutative or $d^2(U) = \{0\} = d(U)d(U)$. Further we prove the nilpotency of derivations contained in the nucleus.

The first chapter is devoted to the necessary algebraic concepts. We give a brief survey of the work done by Posner, Bell, Herstein, Daif, Lee and Yen on derivations in prime rings.

Chapter 2 consists of some results on derivations and commutativity in prime rings. In section 2.1 we present some elementary properties of derivations in prime rings. We show that if $d$ is a nonzero derivation of a prime ring $R$ such that $ad(a) - d(a)a$ is equal to zero for all $a \in R$, then $R$ is commutative. In section 2.2, we prove that if $R$ admits a nonzero derivation $d$ such that $d([x,y]) \in Z$ or
d([d(x),y]) \in Z or d([x^2,y]) \in Z, then R is commutative. In section 2.3. We present the proof of the theorem which states that if d is a nonzero U-* or U-** derivation, then either R is commutative or d^2(U)={0} =d(U)d(U).

In chapter 3 we discuss nilpotent derivations and commutativity with some conditions. Section 3.1 consists of some results on nilpotency of derivations on an ideal due to Chung and Luh [4]. In section 3.2, we discuss the nilpotency of derivations on subrings and we prove that if K is a subring of R generated by d(R) such that d^n(K) \subseteq Z or if H is a commutative subring of R such that d^n(H) \subseteq Z, then R is commutative or d^n = 0. In section 3.3, We show that if d is a derivation of a nonassociative prime ring R such that d^n(R) is contained in the nucleus, then either R is associative or d^{3n-1} = 0.

Chapter 4 contains some problems for possible further work. We want to study some more properties of prime or semiprime rings with reverse derivation using the work of Samman and Alyamani [17].

Now we give certain basic definitions of some algebraic concepts which are indispensable for our further work.

**Associative ring**: An associative ring R, sometimes called a ring in short, is an algebraic system with two binary operations addition ' + ' and multiplication '.' such that

1. the elements of R form an abelian group under ' + ' and a semigroup under ' . ';
2. Multiplication '.' is distributive on the right as well as on the left over addition ' + ' , that is (x + y)z = xz + yz, z(x + y) = zx + zy; for all x, y, z in R.
**Nonassociative ring**: A nonassociative ring $R$ is an additive abelian group in which multiplication is defined, which is distributive over addition, on the left as well as on the right, that is $(x + y)z = xz + yz$, $z(x + y) = zx + zy$, for all $x, y, z$ in $R$.

A nonassociative ring differs from an associative ring in that the full associative law of multiplication is no longer assumed to be associative. That is, it is not necessarily associative. Strictly speaking, the associative law of multiplication has not been done away with, it has merely weakened.

The well-known examples of nonassociative rings are alternative rings, Lie rings and Jordan rings.

**Alternative ring**: An alternative ring $R$ is a ring in which $(xx)y = x(xy)$, $y(xx) = (yx)x$, for all $x, y$ in $R$. These equations are known as the left and right alternative laws respectively.

**Lie ring**: A Lie ring $R$ is a ring in which the multiplication is anticommutative, that is, $x^2 = 0$ (implying $xy = -yx$) and the Jacobi identity

$$(xy)z + (yz)x + (zx)y = 0,$$

for all $x, y, z$ in $R$ is satisfied.

**Jordan ring**: A Jordan ring $R$ is a ring in which the products are commutative, that is, $xy = yx$ and satisfy the Jordan identity $(xy)x^2 = x(yx^2)$ for all $x, y$ in $R$.

**Associator**: The associator $(x,y,z)$ is defined by $(x,y,z) = (xy)z - x(yz)$, for all $x, y, z$ in a ring.
This plays a key role in the study of nonassociative rings. It can be viewed as a measure of the nonassociativity of a ring. This definition is due to Max Zorn [23] where he proved that a finite alternative division ring is associative.

In terms of associators, a ring is called left alternative if \((x,x,y) = 0\); right alternative if \((y,x,x) = 0\) for all \(x,y\) in \(R\) and alternative if both the conditions hold.

**Commutator**: The commutator \((x,y)\) is defined by \((x,y) = xy-yx\), for all \(x,y\) in a ring. This can be considered to be a measure of noncommutativity of a ring.

**Commutative ring**: If the multiplication in a ring \(R\) is such that \(x.y = y.x\), for all \(x,y\) in \(R\), then we call \(R\) as a commutative ring.

A noncommutative ring differs from a commutative ring in that the multiplication is not assumed to be commutative. That is, we do not assume \(x.y = y.x\), for all \(x,y\) in \(R\) as an axiom. However, it does not mean that there always exist elements \(x,y\) in \(R\) such that \(x.y \neq y.x\).

The ring of 2x2 matrices over rationals and ring of real quaternions due to Hamilton are the examples of noncommutative rings.

**Left nucleus**: The set of all elements \(n\) in \(R\) such that \((n,R,R) = 0\) is called a left nucleus of a ring \(R\).

**Right nucleus**: The set of all elements \(n\) in \(R\) such that \((R,R,n) = 0\) is called a right nucleus of a ring \(R\).
**Middle nucleus**: The set of all elements \( n \) in \( R \) such that \( (R,n,R) = 0 \) is called a middle nucleus of a ring \( R \).

**Nucleus**: By the nucleus \( N \) of a ring \( R \), we mean the set of all elements \( n \) in \( R \) such that \( (n,R,R) = (R,n,R) = (R,R,n) = 0 \).

**Center**: By the center \( Z \) of \( R \), we mean the set of all elements \( z \) in \( N \) such that \( (z,R) = 0 \).

It is easily verified that \( N \) is a subring of \( R \) and \( Z \) is a subring of \( N \). Obviously, we note that \( N = R \) if and only if \( R \) is an associative ring and \( Z = R \) if and only if \( R \) is associative and commutative.

**Characteristic of a ring**: If there exists a positive integer \( n \) such that \( nx = 0 \) for every element \( x \) of a ring \( R \), the smallest such positive integer is called the characteristic of \( R \). If \( R \) is of characteristic not equal to 2, then \( 1 + 1 \neq 0 \) and \( 2x = 0 \) implies \( x = 0 \).

**Associator ideal**: The associator ideal \( I \) of \( R \) is the smallest ideal which contains all associators in \( R \).

**Minimal ideal**: An ideal \( I \) in a ring \( R \) is called minimal if \( I \neq (0) \) and \( J \) is a nonzero ideal of \( R \) contained in \( I \), then \( J = I \).

**Nil ideal**: An ideal \( A \) in a ring \( R \) is called a nil ideal if each element of \( A \) is nilpotent.

**Nilpotent ideal**: An ideal \( A \) in a ring \( R \) is called nilpotent if \( A^n = (0) \) for some positive integer \( n \).
**Derivation**: An additive map \( d \) from a ring \( R \) to \( R \) is called a derivation on \( R \) if \( d(xy) = d(x)y + xd(y) \) for all \( x, y \in R \).

**Inner derivation**: An additive map \( d \) from a ring \( R \) to \( R \) is called an inner derivation on \( R \) if \( d(a)x = xa - ax \) for all \( a, x \in R \).

**Skew derivation or s-derivation**: An additive mapping \( a \) from \( R \) to \( R \) is called a skew derivation or a s-derivation if \( d(xy) = d(x)y + s(x) \, d(y) \) holds for all \( x, y \in R \), where \( s \) is an automorphism of \( R \).

**Reverse derivation**: An additive mapping \( d \) from a ring \( R \) into itself satisfying \( d(xy) = d(y)x + yd(x) \), for all \( x, y \in R \), is called a reverse derivation on \( R \).

**U-* derivation**: If \( R \) is a ring having a nonzero left ideal \( U \) and \( d \) is a derivation on \( R \) such that \( d(x) \, d(y) + d(xy) = d(y) \, d(x) + d(yx) \) for all \( x, y \in U \) then we say that \( d \) is a U-* derivation.

**U-** derivation**: If \( R \) is a ring having a nonzero left ideal \( U \) and \( d \) is a derivation on \( R \) such that \( d(x) \, d(y) + d(yx) = d(y) \, d(x) + d(xy) \) for all \( x, y \in U \), then we say that \( d \) is a U-** derivation.

**Division ring**: A ring is said to be a division ring if its nonzero elements form a group with respect to multiplication.

**Prime ring**: A ring \( R \) is prime if whenever \( A \) and \( B \) are ideals of \( R \) such that \( AB = 0 \) then either \( A = 0 \) or \( B = 0 \). Also a ring \( R \) is called prime if and only if \( xa = 0 \) implies \( x = 0 \) or \( a = 0 \) for all \( x, a, y \in R \).
**Semiprime ring**: A ring $R$ is semiprime if for any ideal $A$ of $R$, $A^2 = 0$ implies $A = 0$. These rings are also referred as rings free from trivial ideals.

**Homomorphism**: A mapping $f$ from a ring $R$ into a ring $S$ such that

$$f(a + b) = f(a) + f(b)$$

and

$$f(ab) = f(a)f(b)$$

for all $a, b \in R$, is called a homomorphism of $R$ into $S$.

**Antihomomorphism**: Let $R$ and $S$ be rings. A mapping $f: R \rightarrow S$ is an antihomomorphism if for all $x, y \in R$, $f(x + y) = f(x) + f(y)$ and $f(xy) = f(y)f(x)$.

In 1957, Posner [16] studied derivations in prime rings and proved that if $d$ is a nonzero derivation of a prime ring $R$ such that $ad(a) - d(a)a$ is in the center of $R$, then $R$ is commutative.

Herstein [7] proved that if $n > 1$ and $x - x^n$ is in the center of a ring $R$ for all $x$ in $R$, then $R$ is commutative. Using this result, Bell and Daif [1] studied the derivations and commutativity in prime rings.

In 1978, Herstein [9] proved that if $R$ is a prime ring with char $R \neq 2$ and $R$ admits a nonzero derivation $d$ such that $[d(x), d(y)] = 0$ for all $x, y \in R$, then $R$ is commutative.

In 1982, Kelzan [10] studied the commutativity of semiprime PI - rings and proved that if $R$ is a prime ring satisfying an identity $q(x) = 0$, where $q(x)$ is a polynomial in a finite number of noncommuting indeterminates its coefficients being integers with highest common factor one and there exists no prime $p$ for
which the ring of $2 \times 2$ matrices over GF $(p)$ satisfies $q(x) = 0$, then $R$ is commutative.

Bell and Martindale in 1987 [3] studied centralizing mappings of semiprime rings and proved that if $d$ is a nonzero derivation of a prime ring $R$ such that $[d(x), x] = 0$ for all $x$ in a nonzero left ideal of $R$, then $R$ is commutative.

In 1995, Bell and Daif [1] proved that if $R$ is a prime ring admitting a nonzero derivation $d$ such that $d(xy - yx) = 0$ for all $x, y \in R$, then $R$ is commutative.

Lewin [14] and Trzepizur [20] studied nilpotency of derivations on rings. Lee and Lee [13] presented some properties of nilpotent derivations. It is proved that if $R$ is a prime ring with center $Z$, $I$ is a nonzero ideal of $R$, $n$ is a positive integer and $d$ is a derivation on $R$ such that $d^n(I) \subseteq Z$, then either $d^n = 0$ or $R$ is commutative.

The above studies of Posner, Herstein, Kelzan, Bell, Martindale, Daif, Lewin, Trzepizur and Lee have opened many avenues for further work. There have been several interesting studies of derivations in associative and non associative rings. In the next chapter we discuss some properties of derivations in prime rings.