CHAPTER - II

ACCEPTANCE SAMPLING PLANS BASED ON TRUNCATED GAMMA DISTRIBUTION
2.1 INTRODUCTION TO CASP - CUSUM

It is obvious that the prime concern of any industrialist, to survive in this competitive world, is the Quality of his manufactured product. We come across various difficulties while controlling the quality of the product. To overcome these difficulties and complexity involved, various researchers to attain the same in different situations introduce different methods. The word 'quality' has come to have a set of different meanings in addition to "fitness for use". If the quality of the items in a particular sample
A number of papers has been appeared in many statistical journals with the application of Cumulative Sum Charts to the control of continuously operating manufacturing plant with respect to time. In designing a CUSUM chart, it is important to know the ARL's corresponding to various possible choices of the mask parameters. In the sense, the applicability of CASP – CUSUM schemes for the finished products here forms the main theme for discussion. In the last three decades, several control chart procedures were developed.

Here, we concentrate on CASP – CUSUM schemes, which are optimized on the basis of the ARL in the acceptance zone, which in turn maximize the value of Type – C OC curve. The principal feature of CUSUM techniques is that successive values of a variable are compared with a pre-determined target or reference value and the CUSUM of deviations from this value is plotted on a chart or recorded in tabulation.

In the present chapter, we are proposing CASP – CUSUM charts when the variable under consideration follow a Truncated Two-Parametric Gamma Distribution. Hence, it is more appropriate to discuss some interesting properties of this distribution in the following section.
2.2 GAMMA DISTRIBUTION

2.2.1 DEFINITION:

A random variable ‘X’ is said to have a Two-Parametric Gamma Distribution with parameters ‘β’ and ‘η’, if the probability density function is given by:

\[ P(X = x) = f(x) = \frac{1}{(\beta \eta)^{\beta} \Gamma(\beta)} x^{\beta-1} e^{-x/\eta} \quad 0 < x < \infty, \; \eta > 0, \; \beta > 0 \]

\[ = 0, \text{otherwise} \]

(2.2.1)

Where ‘β’ is the shape parameter which has only positive integer numbers and ‘η’ is the scale parameter of the distribution.

2.2.2 PROPERTIES OF THE DISTRIBUTION:

Some of the important properties of this two – parametric gamma distribution are as follows:

1. The Mean of a two – parametric gamma distribution is given by “βη”.

2. The Variance is given by “βη^2”.

3. The Moment Generating Function of the random variable ‘X’ is given by:
\[ M_X(t) = \frac{1}{\Gamma(\beta)\eta^\beta} \int_0^\infty x^{(\beta-1)} e^{-\frac{x}{\eta}} dx = (1 - \eta t)^{-\beta} \]

4. The special case in which \( \beta = 1 \) leads to the exponential distribution with parameter \( \eta \). The probability density function of an exponentially distributed random variable is therefore given as:

\[ f(x) = \frac{1}{\eta} e^{-\frac{x}{\eta}}, \quad 0 < x < \infty, \quad \eta > 0 \]

\[ = 0, \text{ Otherwise.} \]

5. If \( X \) follows Gamma Distribution with \( \beta = 1 \) and \( \eta \) parameters, then, we have the moment of order \( r \) given by:

\[ r = r! \eta^r. \]

For this, Mean = \( \eta \)

Variance = \( \eta^2 \)

and \( M_X(t) = (1 - \eta t)^{-1} \)

6. Fig 2.1 gives the density function of Gamma Distributions for some values of \( \beta \) and \( \eta \).

2.2.3 APPLICATIONS OF THE DISTRIBUTION:

The Two - Parametric Gamma Distribution plays a very important role in Statistical Quality Control, particularly in reliability theory, in dealing with failure distributions. The nature of failures can be broadly classified into three categories.
Fig. 2.1. The Gamma Density $f_{\eta, \beta}(x)$ for some values of $\eta, \beta$. 

Graph showing three different curves for $f_{\eta, \beta}(x)$ with parameters:

- I: $\eta = 1$, $\beta = 1$
- II: $\eta = 2$, $\beta = 4$
- III: $\eta = 1$, $\beta = 4$
belonging to three phases of failure rates. A typical realization of the life history of populations of units of a complex product is given in Fig. 2.2, in which the three phases mentioned below are clearly explained, i.e., [1] De-bugging Phase

[2] Chance-Failure Phase and


The two important distributions that are applicable in the above three phases of failures are the Weibull Distribution and the Gamma Distribution.

Applicability of various distributions in different phases of lifetime distributions is given in Fig. 2.3, which reveals that the Weibull and Gamma Distributions are the only two distributions applicable in all the three phases by suitably imposing the conditions on scaling and shaping parameters ‘\( \eta \)’ and ‘\( \beta \)’ respectively.

In the following sections, we are explaining the operation of the CASP – CUSUM Chart along with the method of solution of ARL and Type – C OC-Curves.

2.3 DESCRIPTION OF THE PLAN, ARL AND TYPE – C OC CURVES

Beattie [2] has suggested the method for constructing the continuous acceptance sampling plan. The procedure, suggested by him consists of a chosen decision
Fig. 2.2: Typical life history of a population of units of a complex product (not to scale regarding phase lengths).
Fig. 2.3: Possible life distributions by phase
interval namely, 'return interval' with the length h', above the decision line is taken as shown in Fig 1.7.

We plot on the chart the sum $S_m = \sum (X_i = k)$, $X_i$'s ($i = 1, 2 \ldots$) are distributed independently and 'k' is the reference value. The product is accepted if the sum lies in the area of normal chart, and the product is rejected if it lies in the area of return chart, subject to the following assumptions.

1. When the recently plotted point on the chart touches the decision line, then the next point to be plotted at the maximum, i.e. $h + h'$.

2. When the decision line is reached or crossed from above the next point on the chart is to be plotted from the base line.

When the CUSUM falls in the return chart, rework or a change of a specification may be employed rather than outright rejection. The procedure in brief is given below:

1. Start plotting the CUSUM at 0.

2. The product is accepted when $S_m = \sum (X_i = k) < h$; when $S_m < 0$, return cumulation to 0.
3. When \( h < S_m < h + h' \) the product is rejected; when \( S_m \) crosses \( h \), i.e. when \( S_m > h \) restart cumulation at \( h + h' \) and continue rejecting product until \( S_m > h + h' \) return cumulation to \( h + h' \).

The Type-C OC function, which is defined as the probability of acceptance of an item as a function of incoming quality, when sampling rate is same in acceptance \( P(A) \) is given by

\[
P(A) = \frac{L(0)}{L(0) + L'(0)}
\]  

(2.3.1)

Where, \( L(0) = \) Average Run Length in acceptance zone and \( L'(0) = \) Average Run Length in rejection zone.

Page [28] has introduced the formulae for \( L(0) \) and \( L'(0) \) as

\[
L(0) = \frac{N(0)}{1 - P(0)} \quad \ldots \quad (2.3.2)
\]

\[
L'(0) = \frac{N'(0)}{1 - P'(0)} \quad \ldots \quad (2.3.3)
\]

Where, \( P(0) = \) Probability for the test starting from zero on the normal chart,

\( N(0) = \) ASN for the test starting from zero on the normal chart,

\( P'(0) = \) Probability for the test on the return chart,
and \( N'(0) = ASN \) for the test on the return chart.

Page [28], further obtained integral equations for the quantities \( P(0), N(0), P'(0), N'(0) \) as follows:

\[
P(z) = F(k - z) + \int_0^h P(y) f(y + k - z) \, dy, \quad \ldots \quad (2.3.4)
\]

\[
N(z) = 1 + \int_0^h N(y) f(y + k - z) \, dy, \quad \ldots \quad (2.3.5)
\]

\[
P'(z) = \int_0^{k+z} f(y) \, dy + \int_0^h P'(y) f(-y + k + z) \, dy, \quad \ldots \quad (2.3.6)
\]

\[
N'(z) = 1 + \int_0^h N'(y) f(-y + k + z) \, dy, \quad \ldots \quad (2.3.7)
\]

Where \( F(X) = \int_A^h f(x) \, dx \); \( F(k - z) = \int_A^{k - z} f(y) \, dy \), and \( z \) is the distance of the starting of the test in the normal chart from zero.

In the above equation \( f(.) \) represents the density function of the distribution of random variable ‘\( X_i \)’ where \( X_i \) is the response measured from the \( i \)th unit submitted for inspection. Further, throughout the dissertation it is assumed that \( X_i \) ‘s are continuous random variables. It is already mentioned that the chief objective of this dissertation is to obtain solutions of ARL and Type-C OC curves for various CASP-CUSUM charts when \( X_i \) ‘s follow different truncated distributions. In this dissertation we have assumed that the random variable \( X_i \) ‘s follows the Truncated Two – Parametric Gamma Distribution.
It is a well-known fact that the Gamma Distribution plays a vital role in Statistical Quality Control, in particular when we are dealing with the failure rate of products having a guarantee period. Since the life of the product under operation is assured up to the guarantee period, which means that if a failure occurs before this period replacement or repairing the unit free of cost is guaranteed by the manufacturer, it is reasonable to assume that the life of the unit is greater than the guarantee period.

In this chapter, it is assumed that the random variable ‘X,’ follows truncated two parametric gamma distribution, which is defined as follows:

A random variable ‘X’ is said to follow a truncated two parametric gamma distribution, if the probability density function is:

\[ f_\beta(x) = \frac{1}{\eta - e^{-\beta/\eta}} \sum_{j=0}^{\beta} \frac{(-1)^j (B/\eta)^{\beta-j}}{(j!)^{\beta-j}} x^{\beta-j} e^{-x/\eta}, \quad 0 < x < B; \beta > 0; \eta > 0 \]  

Where ‘\beta’ is the shape parameter which has only positive integer numbers and ‘\eta’ is the scale parameter of the distribution.

Now we proceed to obtain solutions for the equations from (2.3.4) to (2.3.7) when \( f(.) \) is the density function given in the equation (2.3.8).
2.4 METHOD OF SOLUTION

Several methods are available to find the derivative of a function \( f(x) \) or to evaluate the definite integral \( \int_{a}^{b} f(x) \, dx \) where \( a, b \) are real finite constants, in the closed form. When \( f(x) \) is a complicated function or when it is given in tabular form, we use numerical methods. We use numerical methods for approximating the derivative \( f(x) \), \( r \geq 1 \) of a given function \( f(x) \) and for evaluation of the integrals \( \int_{a}^{b} f(x) \, dx \) where \( a, b \) may be finite or infinite. Numerical differentiation methods are obtained using one of the following three techniques: 1. Methods based on interpolation. 2. Methods based on finite difference operators. 3. Methods based on undetermined coefficients. There are several methods to solve the integral equations given from (2.3.4) to (2.3.7). Here we have considered a method suggested by Krishna Murthy, and Sen [21] to obtain the numerical solution for \( P(A) \).

In this dissertation two methods are considered for obtaining the values of \( P(A) \). They are as follows: 1. Gauss-Chebyshev integration method. 2. Lobatto integration method. These methods, which are used in this dissertation, come under the methods based on undetermined coefficients. In Gauss-Chebyshev integration method when \( W(x) = \frac{1}{\sqrt{(1-x^2)}} \), the methods of the form
\[
\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x) \, dx = \sum_{k=0}^{n} \lambda_k f_k
\]

are called Gauss-chebyshev integration methods. These methods are exact for polynomial of degree up to 2n+1. The nodes \( x_k \)'s are found to be the roots of the chebyshev polynomials.

In Lobatto integration method when \( W(x) = 1 \) and the two end points \(-1\) and \(+1\) are always taken as nodes. The remaining \( n-1 \) nodes are to be determined. The integration method of the form is

\[
\int_{-1}^{1} f(x) \, dx = \lambda_0 f(-1) + \lambda_n f(1) + \sum_{k=1}^{n-1} \lambda_k f_k
\]

called the Lobatto integration method. Since there are \( 2n \) unknowns (\( n-1 \) and \( n+1 \) weights) this method can be made exact for polynomials of degree up to \( 2n-1 \).

Gauss formula does not contain the values of the function at the end points of the interval of integration. In certain type of problems it is highly advantageous to utilize the end values of the function. Lobatto therefore modified Gauss’s formula so as to include the end values and also the value of the function at the mid point of the interval. The inherent errors in the formulas of Gauss’s, Lobatto, are usually given in terms of the coefficients in a power series. The methods are briefly explained and they are as follows:
METHOD-I

We first express the integral equation (2.3.4) in the form

\[ F(x) = Q(x) + \int_C^d R(x, t) F(t) \, dt \]  \quad \text{(2.4.1)}

Where \( F(x) = P(z) \), \( Q(x) = F(k-z) \), \( R(x, t) = f(y + k \cdot z) \).

Let the integral \( I = \int_C^d f(x) \, dx \) be transformed to

\[ I = \frac{d-c}{2} \int_C^d f(y) \, dy = \frac{d-c}{2} \sum a_i f(t_i) \quad \text{(2.4.2)} \]

Where \( y = \frac{2x - (c-d)}{d-c} \) and \( a_i \)'s and \( t_i \)'s are respectively the weight factors and abscissa for the Gauss-Chebyshev polynomial given in Jain et al [17]. Using (2.4.1) and (2.4.2), (2.3.4) can be written as

\[ F(x) = Q(x) + \frac{d-c}{2} \sum a_i R(x, t_i) F(t_i) \]  \quad \text{(2.4.3)}

Since equation (2.4.3) should be valid for all values of \( x \) in the interval \((c, d)\), it must be true for \( x = t_i, i = 0 (1) n \). Then we obtain

\[ F(t_i) = Q(t_i) + \frac{d-c}{2} \sum a_i R(t_j, t_i) F(t_j); \quad j = 0 (1) n \]  \quad \text{(2.4.4)}

Substituting \( F(t_i) = F_{t_i}, Q(t_i) = Q_{t_i}, I = 0 (1) n \), in equation (2.4.4), we get
\[ F_0 = Q_0 + \frac{d-c}{2} \left( a_0 R(t_0, t_0) F_0 + a_1 R(t_0, t_1) F_1 + \ldots + a_n R(t_0, t_n) F_n \right) \]

\[ F_1 = Q_1 + \frac{d-c}{2} \left( a_0 R(t_1, t_0) F_0 + a_1 R(t_1, t_1) F_1 + \ldots + a_n R(t_1, t_n) F_n \right) \quad \ldots \quad (2.4.5) \]

\[ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \]

\[ F_n = Q_n + \frac{d-c}{2} \left( a_0 R(t_n, t_0) F_0 + a_1 R(t_n, t_1) F_1 + \ldots + a_n R(t_n, t_n) F_n \right) \]

In the system of equations except \( F_i \), \( i = 0(1)n \) are known and hence can be solved for \( F_i \). We solve the system of equations by the method of iteration. For this we write the system (2.4.5) as

\[ \left( 1 - \frac{d-c}{2} a_0 R(t_0, t_0) \right) F_0 = Q_0 + \frac{d-c}{2} \left( a_0 R(t_0, t_0) F_0 + a_1 R(t_0, t_1) F_1 + \ldots + a_n R(t_0, t_n) F_n \right) \]

\[ \left( 1 - \frac{d-c}{2} a_1 R(t_1, t_1) \right) F_1 = Q_1 + \frac{d-c}{2} \left( a_0 R(t_1, t_0) F_0 + a_1 R(t_1, t_1) F_1 + \ldots + a_n R(t_1, t_n) F_n \right) \quad \ldots \quad (2.4.6) \]

\[ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \]

\[ \left( 1 - \frac{d-c}{2} a_n R(t_n, t_n) \right) F_n = Q_n + \frac{d-c}{2} \left( a_0 R(t_n, t_0) F_0 + a_1 R(t_n, t_1) F_1 + \ldots + a_n R(t_n, t_n) F_n \right) \]

To start with the iteration process, let us put \( F_1 = F_2 = \ldots = F_n = 0 \) in the first equation of (2.4.6), we then obtain a rough value of \( F_0 \). Putting this value of \( F_0 \) and \( F_2 = F_3 = \ldots = F_n = 0 \) in the second equation, we get a rough value of \( F_1 \) and so on. This gives the first set of values \( F_i \), \( i = 0,1,2,\ldots,n \). Repeating the process as above we get a second set of values of \( F_i \), \( i = 0,1,2,\ldots,n \) which are just the
refined values of $F_i$, $i = 0, 1, 2, \ldots, n$. The process is continued until two consecutive sets of values are obtained up to a certain degree of accuracy. In the similar way solutions of $P'(0), N(0), N'(0)$ can be obtained.

METHOD-II

In this method of solution the procedure explained in the above Method-I is as usual except the difference that Lobatto Method of Integration is used to evaluate the integral in equation (2.4.4) in the place of Gauss-Chebyshev polynomial. In other words in this method the nodes and weight factors for various values of $n$ are given in the following table.

After evaluating the integrals with above weights and nodes the other iterative procedure to solve the system of equations given in (2.4.5) is as usual as explained in the method I. On similar lines integrals (2.3.5) through (2.3.7) are evaluated by Lobatto method of integration and the values of $N(0), P'(0), N'(0)$ are evaluated, which in turn are useful in evaluating the expression of $P(A)$ given in the equation (2.3.1).
Nodes and Weights for Lobatto Integration Method

<table>
<thead>
<tr>
<th>n</th>
<th>Nodes $x_k$</th>
<th>Weights $\lambda_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\pm 1.00000000$ 0.00000000</td>
<td>0.33333333</td>
</tr>
<tr>
<td></td>
<td>1.33333333</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$\pm 1.00000000$ 0.44721360</td>
<td>0.16666667</td>
</tr>
<tr>
<td></td>
<td>0.83333333</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$\pm 1.00000000$ 0.65465367 0.00000000</td>
<td>0.10000000</td>
</tr>
<tr>
<td></td>
<td>0.54444444 0.71111111</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$\pm 1.00000000$ 0.76505532 0.28523152</td>
<td>0.06666667</td>
</tr>
<tr>
<td></td>
<td>0.37847496 0.55485837</td>
<td></td>
</tr>
</tbody>
</table>