CHAPTER III

POLYSEMY OF SUBTREE GRAPHS

3.1. Introduction

For any topologic graph, we can characterize the intersection graphs of finite families of arcs in trees. Consider an undirected tree as a topological pattern. A connected portion of a tree is called a subtree. The intersection graph of a family of subtrees in an undirected tree is called a Subtree graph. The Interval Graphs and the Circular -Arc graphs are the important types of subtree graphs.

Gavril has proved that, a graph $G$ is a subtree graph if and only if it is a chordal graph [Ga 74]. In chapter II, we proved that the Interval Graphs and the Circular Arc graphs are intersection polysemic with chordal graphs on the same vertex set [2. 12.5., 2. 12.6.].

Chordal graphs can be recognized efficiently by polynomial time algorithms. For Chordal graphs, the minimum coloring, maximum clique, minimum clique cover, and maximum independent set problems are solved successfully by Gavril [Ga 72]. Complete characterizations of interval graphs were given by Gilmore and Hoffman [GH 64], Lekkerkerker and Boland [LB 62], and Fulkerson and Gross [FG 65]. Fulkerson and Gross [FG 65] also proposed the most efficient algorithm for constructing a representation by intervals of a given graph, if one exists. The interval graphs have applications in genetics, psychophysics [Ro 65], archeology [LB 62], and in ecology. The intersection
graph of a family of arcs on a circularly ordered set is called a circular – arc graph. The problem of characterizing these graphs appears in [KL 69]. Tucker [Tu 71] had found out a characterization of the circular – arc graphs by means of adjacency matrix. He also gave an efficient algorithm for identifying the proper circular arc graphs.

In this chapter, we extend the notion of graph Polysemy to Subtree graphs and prove that, the polysemic intersection representation of a subtree graph also is a subtree graph and they have the same clique tree representation, if one of them is not a single clique. Also graph polysemy is introduced to bi-subtree graphs and proper subtree graphs. Among the subtree graphs, we are interested in Interval Graphs and Circular – arc graphs and we have defined graph polysemy for these two types of graphs and some characterizing results are also proved.

3.2. Characteristics of Subtree graphs

Gavril has given an efficient algorithm for constructing a representation by a family of subtrees, to a given chordal graph [Ga 74]. It is well known, in an undirected tree, that for every two points there exists a unique simple path connecting them. If the family F is a representation of a graph G, and $T_v$ is the subtree corresponding to a vertex $v$, then, we have the following lemma.

**Lemma 3.2.1 [Ga 74]**

For every completely connected set $A$ of $G$, $\bigcap_v T_v \neq \emptyset$ for all $v \in A$. 
It has been proved in [Ga 74] that, every chordal graph $G(V)$ has a vertex $v$, so that $\Gamma v$ (the set of vertices connected to $v$) is completely connected; $v$ is called a simplicial vertex. The following theorem has been proved in [Ga 74] and [MW 86].

**Theorem 3.2.2** [Ga 74, MW 86]

A graph $G(V)$ is a subtree graph if and only if there exists a tree $T$ whose vertex set is $\mu(G)$, the set of cliques in $G$, so that for every $v \in V$, $T(\mu_v(G))$, where $\mu_v(G)$ is the set of cliques containing the vertex $v$, is connected.

**3.2.3. Maximum Cardinality Search** [BP 93]

The maximum cardinality search algorithm for chordal graphs orders the vertices in reverse order beginning with an arbitrary vertex $v \in V$ for which it sets $\alpha(v) = n$. At each step the algorithm selects as the next vertex to label an unlabeled vertex adjacent to the largest number of labeled vertices, with ties broken arbitrarily. In [TY 84] Tarjan et al have given an effective algorithm for the recognition of chordal graphs in $O(n + e)$ time.

We have the following theorem due to [TY 84]

**Theorem 3.2.4.**

Every maximum cardinality search ordering of a chordal graph $G$ is a perfect elimination ordering.
Theorem 3.2.5 [Go 80]

Let G be an undirected graph. Then the following statements are equivalent.

(i) G is triangulated

(ii) G has a perfect elimination scheme. Moreover, any simplicial vertex can start a perfect scheme.

(iii) Every minimal vertex separator induces a complete sub graph of G.

Lemma 3.2.6 [BP 93]

Every triangulated graph G(V, E) has a simplicial vertex. If G is not complete, then it has two nonadjacent simplicial vertices.

Theorem 3.2.7. [BP 93]

A graph G is chordal if and only if, every minimal vertex separator of G is complete in G.

Theorem 3.2.8. [BP 93]

A graph G is chordal if and only if G has a perfect elimination ordering.

Lemma 3.2.9 [BP 93]

A vertex is simplicial if and only if it belongs to precisely one clique.

Walter characterized chordal graphs as:

Theorem 3.2.10 [Wa 78]

Every chordal graph has a simplicial point.
Lemma 3.2.11 [Wa 78]

If G is a cycle of length at least four, then G cannot be represented on a tree.

Lemma 3.2.12 [Wa 78]

If the connected graph G is represented on the tree T and H is a complete subgraph of G, then the intersection of the representatives of the points of H is not empty.

Theorem 3.2.13 [Wa 78]

Let G be a connected graph, then there exists a tree on which G can be represented if and only if G is chordal.

Dale J. Skrien gives a comparative study of Triangulated graphs, Comparability Graphs, Proper Interval graphs, Proper Circular – arc Graphs, and Nested Interval Graphs.[Sk 82]. A comparative study of indifference graphs and interval graphs is seen in the paper, “Indifference graphs” of Fred S Roberts.[Ro 71]. The theory of chordless cycles are dealt by Irena Rasu [Ra 99]

3.3. Characterization of Clique Trees [BP 93]

Let G (V, E) be any graph. A clique of G is any maximal set of vertices that is complete in G, and thus a clique is properly contained in no other clique. For any chordal graph G, there exists a subset of the set of trees on KG, known as clique trees; where KG = {K₁, K₂, ..., Kₘ} denotes the set containing the cliques of G.
3.3.1. Clique Tree Representation [Ga 72, 74]

The tree constructed such that there is a bijection between the vertices of the tree $T$ and the collection of cliques of an undirected graph $G$, is called a clique tree of $G$. Let $G$ be the given graph and $T$ be its clique tree representation. If $G$ is a chordal graph then each vertex of $G$ will correspond to a subtree of $T$. A clique tree representation theorem for chordal graph was proved by Gavril as follows.

**Theorem 3.3.2** [Ga 74]

A graph $G$ is chordal if and only if there is an unrooted and undirected tree $T$ and a family of subtrees of $T$ indexed by the vertices of $G$, such that subtrees $T_v$ and $T_w$ share a node of $T$ if and only if the vertices $v$ and $w$ are adjacent. Furthermore, it is possible to construct $T$ in such a way that there is a bijection between the nodes of $T$ and the maximal cliques of $G$ where the subtree $T_v$ consists of all nodes that correspond to maximal cliques containing the vertex $v$.

3.3.3. Clique intersection property. [BP 93]

Assume that $G$ is a connected graph and consider its set of maximal cliques $K_G$. Then the clique intersection property states that, for any pair of distinct cliques $K, K' \in K_G$, the set $K \cap K'$ is contained in every clique on the path connecting $K$ and $K'$ in the tree.
Theorem 3.3.4 [BP 93]

A connected graph \( G \) is chordal, if and only if, there exists, a tree \( T = (K_G, E_T) \) for which the clique intersection property holds.

Monma and Wei have given a unique clique tree theorem.

**Theorem 3.3.5. (Clique Tree Theorem) [MW 86]**

A graph \( G (V, E) \) is chordal if and only if, there exists a tree \( T \) with vertex set \( C \) (set of cliques) such that for every \( v \in V \), \( T(C_v) \), the subgraph of a tree \( T \) with vertices or edges corresponding to \( C_v \), the set of cliques containing \( v \), is a subtree in \( T \).

Now, after these known results and theorems, let us investigate the notion of polysemy.

3.4. Polysemic Intersection Representation of Subtree graphs

3.4.1. Definition

Consider an undirected tree \( T \), as a topological pattern or as the underlying system and \( \{T_v\} \), a family of subtrees of \( T \) and let \( G_1 (V, E_1) \) and \( G_2 (V, E_2) \) be two simple finite graphs on a common vertex set \( V \). The intersection representation, \( f : V \rightarrow \{T_v\} \), is called a polysemic intersection representation if for \( v_i, v_j \in V \), \( (v_i, v_j) \in E_1 \) if \( T_v_i \cap T_v_j \neq \emptyset \) and \( (v_i, v_j) \in E_2 \) if \( T_v_i \cup T_v_j \neq T \). \( G_1 (V, E_1) \) and \( G_2 (V, E_2) \) are called polysemic intersection pairs.
Example 3.4.2.

A tree $T$ and the Subtree Representation $\{T_i\}$

The Subtree graph [Chordal graph] $G_1$ on $\{T_i\}$
Polysemic Intersection Pair $G_2$

Clique Tree representation of $G_1$

Clique tree representation of $G_2$ is a trivial graph.
We have proved the following theorems on the polysemey of subtree graphs.

**Theorem 3.4.3.**

**The polysemic intersection pair of subtree graph is also a subtree graph.**

**Proof**

Let $G_1 (V, E_1)$ be an intersection graph of a collection of subtrees $F$ of a tree $T_1$. Let $f : V \rightarrow F$ be a mapping, such that for $u, v \in V$, (i) $(u, v) \in E_1 \iff T_u \cap T_v \neq \emptyset$, and (ii) $(u, v) \in E_2 \iff T_u \cup T_v \neq T_1$, then $f$ is called a polysemic intersection representation and $G_1, G_2$ are called polysemic intersection pairs.

By construction, $G_1$ is a subtree graph. In order to show that $G_2$ is also a subtree graph; if $G_2$ is a complete graph, it is trivial, since it has a clique tree representation, a trivial graph. Assume that $G_2$ is not complete. Let the set of cliques in $G_2$ be, $\mu(G_2) = \{C_1, C_2, \ldots, C_k \}$. By lemma 3.2.1. for every clique $C_i$, we get, $S_i = \bigcap_{v} \mu(G_2) \neq \emptyset$, $v \in C_i$ and for $i \neq j$, $S_i \cap S_j = \emptyset$. For every $C_i$, take a point in $S_i$ and denote it as $C_i$. With the points $C_1, C_2, \ldots, C_k$ on $T_1$, we construct a graph $T$ in the following way: we connect in $T$ the points $C_i$ and $C_j$ by an edge if and only if the simple path connecting $C_i$ with $C_j$ on $T_1$ does not contain other points of $C_1, C_2, \ldots, C_k$. That is, we connect $C_i$ and $C_j$, if they are neighbors on $T_1$. Clearly $T$ is a tree, and for every $v \in V$, $T (\mu_v(G_2))$ is connected.
Conversely, let \( T \) be a tree whose set of vertices is \( \mu(G_2) \), so that for every \( v \in V, T (\mu_v (G_2)) \) is connected. Let \( F = \{ T (\mu_v (G_2)) \mid v \) is a vertex of \( G_2 \} \). If \( T(\mu_v(G_2)) \cap T(\mu_u(G_2)) \neq \emptyset \), then there exists a vertex \( C_i \) of \( T \) so that \( C_i \in T(\mu_v(G_2)) \cap T(\mu_u(G_2)) \). Hence \( C_i \in \mu_v(G_2), C_i \in \mu_u(G_2) \), thus \( u,v \in C_i \), and therefore \( v \) is connected to \( u \) in \( G_2 \). On the other side, if \( v \) is connected to \( u \) in \( G_2 \), then there exists a clique \( C_i \), so that \( u,v \in C_i \) and hence \( C_i \in T(\mu_v(G_2)) \cap T(\mu_u(G_2)) \). Thus \( G_2 \) is the intersection graph of \( F \), and therefore it is a subtree graph by theorem 1.13.1. "A graph \( G(V, E) \) is a subtree graph if and only if there exists a tree \( T \) whose set of vertices is \( \mu(G) \), the set of cliques, so that, for every \( v \in V, T(\mu_v(G)) \) where \( \mu_v(G) \) is the set of cliques containing \( v \), is connected".

**Theorem 3.4.4**

Let \( G_1 \) and \( G_2 \) be subtree graphs and \( G_2 \) is the polysemic image of \( G_1 \). Then \( G_1 \) and \( G_2 \) have same clique tree representation if one of them is not a single clique.

**Proof**

Given \( G_1 \) and \( G_2 \) are subtree graphs and \( G_2 \) is the polysemic image of \( G_1 \). Assume that \( G_2 \) is not a single clique. Then, at least one pair of vertices in \( G_2 \) exists, which are not adjacent to each other and let those vertices be \( v_1 \) and \( v_2 \). Gavril has proved that a graph is a subtree graph if and only if it is a chordal graph [Ga 74]. Hence \( v_1 \) and \( v_2 \) are simplicial vertices. Therefore there exist cliques \( C_{v_1} \) and \( C_{v_2} \). Since \( G_2 \) is a connected graph there exists a path between \( v_1 \) and \( v_2 \) that is \( C_{v_1} \) and \( C_{v_2} \) have non-empty intersection. Hence we can draw a tree with
vertices $V_i$ which is connected by theorem 1.13.1 and is called the clique tree representation of $G_i$. Since $G_1$ is obtained by the dual operation used to form $G_2$, $G_1$ will also have same clique tree representation.

### 3.4.5 Polysemy of bi-subtree graphs

Let $S$ be a family of subtrees of a tree $T$, partitioned into subfamilies $S_1$ and $S_2$. Let the bi-subtree graphs on $(S_1, S_2)$ be $G_1(V, E_1)$ and $G_2(V, E_2)$. The mapping $f: V \to S$ defined as for $u, v \in V$, $(u, v) \in E_i \iff T_u \in S_1, T_v \in S_2$ and $T_u \cap T_v \neq \emptyset$ and $(u, v) \in E_2, \iff T_u \in S_1, T_v \in S_2$ and $T_u \cup T_v \neq T$ where $T_u$ and $T_v$ are subtrees corresponding to $u$ and $v$, is called a polysemic intersection representation and $G_1(V, E_1)$ and $G_2(V, E_2)$ are called bi-subtree polysemic intersection pairs.

### 3.4.6 Polysemy of proper subtree graphs

Let $F$ be a proper subtree representation of a tree $T$. Let $G_1(V, E_1)$ and $G_2(V, E_2)$ be the intersection graphs over $F$. Then, if a map $f: V \to F$ exists such that, $(u, v) \in E_1 \iff T_u \cap T_v \neq \emptyset$ and $(u, v) \in E_2 \iff T_u \cup T_v \neq T$ where $T_u$ and $T_v$ are proper subtrees corresponding to the vertices $u$ and $v$ in $V$, then $f$ is called a polysemic intersection representation with respect to $F$ and $G_1$ and $G_2$ are called the polysemic intersection pairs.

The most important types of Subtree (Chordal) graphs are The Interval Graphs and The Circular Arc Graphs. We are interested in these two types of intersection graphs.
3.5. Interval graphs

An undirected graph $G$ is called an *interval graph* if its vertices can be put into one to one correspondence with a set of intervals. We have the following theorems according to Golumbic.

**Theorem 3.5.1** [Go 80]

An undirected graph $G$ is an interval graph if and only if $G$ is triangulated and its complement $G^c$ is a comparability graph.

**Theorem 3.5.2** [Go 80]

Let $G$ be an undirected graph. The following statements are equivalent.

(i) $G$ is an interval graph.

(ii) $G$ contains no chordless 4-cycles and its complement $G^c$ is a comparability graph.

(iii) The maximal cliques of $G$ can be linearly ordered such that, for every vertex $x$ of $G$, the maximal cliques containing $x$ occur consecutively.

3.5.3. *Proper interval graphs* [Ga 74]

A graph is called a *proper interval graph*, if it is the intersection graph of a family of intervals on a line, so that no one of the intervals is contained in another.

3.5.4. Polysemy of Interval graphs

Let $I$ be a family of intervals on a linearly ordered set $L$ and let $G_1 (V, E_1)$ and $G_2 (V, E_2)$ be simple finite graphs on a common vertex set $V$. Then the
mapping \( f: V \rightarrow I \) such that \( u, v \in V, (u, v) \in E_1 \Leftrightarrow I_u \cap I_v \neq \emptyset \) and \( (u, v) \in E_2, \Leftrightarrow I_u \cup I_v \neq L \), is called the polysemic intersection representation of interval graphs with respect to \( L \) and \( G_1 (V, E_1) \) and \( G_2 (V, E_2) \) are polysemic intersection pairs of interval graphs (Polysemic Interval Pairs).

Now we obtain the following results.

**Theorem 3.5.5.**

The polysemic intersection image of an interval graph is an interval graph.

**Proof**

Let \( (G_1, G_2) \) be polysemic intersection pairs of the polysemic intersection representation \( f \) over the interval representation \( I \) of intervals of the linearly ordered set \( L \). Let \( G_1 \) be an interval graph. To show that \( G_2 \) is an interval graph.

By definition, \( (v_i, v_j) \in E_2 \Leftrightarrow I_i \cup I_j \neq L \). There are two possibilities for \( G_2 \). (i) \( I_i \cup I_j \neq L \) for all \( i \) and \( j \); (ii) \( I_i \cup I_j \neq L \) for at least one pair of vertices \( i \) and \( j \).

In the first case, \( G_2 \) will be a complete graph, hence is chordal, and contains no asteroidal triple. Hence it is an interval graph.

In the second case, because of the conformity property of polysemic intersection representation, no vertex can be isolated, and the disjoint vertices will be simplicial and hence no cycle of length more than three in \( G_2 \). Hence it is an interval graph.

**Theorem 3.5.6**

The interval graph and its polysemic intersection pair have same clique tree representations.
Proof

Let $G_1(V, E_1)$ be an intersection graph on the set of intervals $I$ of a linearly ordered set $L$.

Let $f: V \rightarrow I$ be a polysemic intersection representation. Then for $u, v \in V$ there exists $I_u$ and $I_v \in I$ such that $(u, v) \in E_1 \iff I_u \cap I_v \neq \emptyset$ and $(u, v) \in E_2, \iff I_u \cup I_v \neq L$, and $G_2(V, E_2)$ is a intersection polysemic with $G_1(V, E_1)$.

Let $T$ be the clique tree of $G_1$, whose set of vertices is $\mu(G_1)$ so that $\forall v \in V, T(\mu_v(G_1))$ is connected. Let $F = \{ T(\mu_v(G_1)) | v \text{ is a vertex of } G_1 \}$. If $T(\mu_v(G_1)) \cap T(\mu_u(G_1)) \neq \emptyset$, then there exists a vertex $A_i$ of $T$ so that $A_i \in T(\mu_v(G_1)) \cap T(\mu_u(G_1))$. Hence $A_i \in \mu_v(G_1)$ and $A_i \in \mu_u(G_1)$, thus $u, v \in A_i$ and therefore $v$ is connected to $u$ in $G_1$. But the vertex set of $G_1$ and $G_2$ are the same.

Taking the dual of the statement, $T(\mu_v(G_1)) \cap T(\mu_u(G_1)) \neq \emptyset$, we get, $T(\mu_v(G_1)) \cap T(\mu_u(G_1)) \neq L$, which means that $u$ and $v$ are connected in $G_2$ and then, there exists a clique $A_i$ so that $u, v \in A_i$ and hence $A_i \in T(\mu_v(G_2)) \cup T(\mu_u(G_2))$.

Hence $G_2$ too has the clique tree representation $T$.

Corollary 3.5.7

The proper interval graph has chordal atom as its polysemic intersection image, but they have different clique tree representations.

Proof

Let $I$ be a proper interval representation of the graph $G_1(V, E)$, and let $G_2$ be its polysemic image under the polysemic intersection representation $f: V \rightarrow I$.

Let $V$ contains more than three vertices. (If $|V| \leq 3$ either one or both the
graphs will be complete). If \( f(u) = I_u \), then, \( I_u \cup I_v = L \) is not possible for any pair of vertices \( u \) and \( v \) in \( V \). Hence \( I_u \cup I_v \neq L \ \forall \ u \) and \( v \). Hence \( G_2 \) is a complete graph and its clique tree representation is a trivial graph.

But \( I_u \cap I_v \neq \emptyset \) exists for overlapping intervals.

Thus \( G_1 \) is not always complete.

### 3.5.8. Bi-interval graph polysemy

Let \( F \) be a family of intervals of \( L \), partitioned into subfamilies \( F_1 \) and \( F_2 \). Let \( G_1 (V, E_1) \) and \( G_2 (V, E_2) \) be simple finite bipartite graphs on same vertex set \( V \). The mapping \( f : V \to F \) defined as, \( u, v \in V, (u, v) \in E_1 \iff f(u) \in F_1 \ \& \ f(v) \in F_2 \) and \( f(u) \cap f(v) \neq \emptyset \) and \( (u, v) \in E_2 \iff f(u) \in F_1 \ \& \ f(v) \in F_2 \) and \( f(u) \cup f(v) \neq L \), where \( f(u) \) and \( f(v) \) are the intervals corresponding to \( u \) and \( v \) is called a polysemic intersection representation and \( G_1 (V, E_1) \) and \( G_2 (V, E_2) \) are called bi-interval polysemic intersection pairs.

**Example 3.5.9.**

An Interval representation \( \{I_i\} \) of a line \( L \):

```
I_1

I_2 ______ I_3 ______ I_4 ______

L

I_5

I_6
```


$G_1$. The Interval Graph on \{1, 2, ..., 7\}

$G_2$. The polysemic Image of $G_1$, $C_1$
3.6. The Circular – Arc Graphs

The intersection graphs obtained from a collection of arcs on a circle are called circular arc graphs.

3.6.1. Proper Circular-arc Graph [Go 80]

We call G, a proper circular-arc graph, if there exists a circular – arc representation for G in which no arc properly contains another.

The following theorems characterize the circular-arc graphs.

3.6.2. Theorem [Go 80]

An undirected graph G is an Helly circular arc graph if and only if its clique matrix has the circular 1’s property for columns.
3.6.3. Theorem [Go 80]

An undirected graph $G$ is a circular-arc graph, if its augmented adjacency matrix has the circular 1's property for columns. (The augmented adjacency matrix of a graph $G$ is obtained from the adjacency matrix by adding 1's along the main diagonal).

3.6.4. Theorem [Go 80]

If $G$ is a proper circular arc graph, then $G$ has a proper circular-arc representation in which no two arcs share a common endpoint and no two arcs together cover the entire circle, that is, they do not intersect at both ends.

3.6.5. Polysemy of Circular - Arc graphs

Let $A$ be a family of arcs of a circle $C$ and let $G_1 (V, E_1)$ and $G_2 (V, E_2)$ be simple finite graphs on a common vertex set $V$. Then, the intersection representation $f: V \rightarrow A$ is a polysemic intersection representation with respect to $C$, for $v_i, v_j \in V, (v_i, v_j) \in E_1 \iff A_i \cap A_j \neq \emptyset$, and $(v_i, v_j) \in E_2 \iff A_i \cup A_j \neq C$. Here $A = \{A_1, A_2, A_3, \ldots\}$ and $G_1 (V, E_1)$ and $G_2 (V, E_2)$ are polysemic intersection pairs.

Theorem 3.6.6

The polysemic intersection image of a circular arc graph is a circular arc graph.

Proof

Let $G_1 (V, E_1)$ and $G_2 (V, E_2)$ be polysemic intersection pair over the collection of arcs $A = \{A_i\}$ over a circle $C$ by the polysemic intersection
Let \( G_1 (V, E_1) \) be the intersection graph of the collection of arcs \( A = \{ A_i \} \) of the circle \( C \). Then, \( G_1 \) is a circular – arc graph and by theorem 3.6.3., its augmented adjacency matrix has the circular 1’s property for columns. Since \( f \) is an intersection representation, such as \( f(v_i) = A_i, f(v_j) = A_j; (v_i, v_j) \in E_1 \Leftrightarrow A_i \cap A_j \neq \emptyset \). In other words, \( A_i \cap A_j \neq \emptyset \Leftrightarrow (v_i, v_j) \in E_1 \Rightarrow a_{ij} = 1(i \neq j) \), in the augmented adjacency matrix of \( G_1 \). Also, \( A_i \cup A_j \neq C \Leftrightarrow (v_i, v_j) \in E_2 \Rightarrow a_{ij} = 1 (i \neq j) \), in the augmented adjacency matrix of \( G_2 \). Also \( A_i \cup A_j = C \Rightarrow a_{ij} = 0 \) and \( a_{ij} = 0 \) ( \( i \neq j \) ) in the augmented adjacency matrix of \( G_2 \).

By the permutation of rows, it can be easily shown that the augmented adjacency matrix of \( G_2 \) satisfies circular 1’s property (by replacing ith row by jth row). Hence \( G_2 \) is also a circular arc graph.

**Theorem 3.6.7**

The circular arc graphs and their polysemic intersection pairs are chordal graphs.

**Proof**

We have proved that (theorem 3.6.6) the polysemic pair of a circular arc graph is a circular arc graph. It is well known that a circular arc graph is a subtree graph, and Gavril [Ga 74] has proved that a subtree graph is a chordal graph.

Hence the theorem.

**3.6.8 Polysemy of Proper Circular arc graphs**

Let \( A \) be a family of proper circular arcs of a circle \( C \) and let \( G_1 (V, E_1) \) and \( G_2 (V, E_2) \) be simple finite intersection graphs over \( A \) with a common vertex
set $V$. Then, the intersection representation $f: V \rightarrow A$ is a polysemic intersection representation if and provided for $v_i, v_j \in V$, $(v_i, v_j) \in E_1 \iff A_i \cap A_j \neq \emptyset$, and $(v_i, v_j) \in E_2 \iff A_i \cup A_j \neq C$.

Here $A = \{A_1, A_2, A_3, \ldots\}$ and $G_1 (V, E_1)$ and $G_2 (V, E_2)$ are polysemic intersection pairs of proper circular arc graphs.

**Theorem 3.6.9**

Proper Circular arc graphs are intersection polysemic with the chordal atom induced by the same vertex set.

**Proof**

Let $G (V, E)$ be a proper Circular – Arc graph, over a family $A$ of Arcs of a circle $C$. By definition, no two arcs share a common end point, and no two arcs together cover the entire circle. Hence the adjacency in $G$ is defined as non-empty intersection, that is overlapping of intervals. In the polysemic intersection pair, the adjacency is defined as non-C union. Since no two arc together cover the entire circle, union of no two arcs will be $C$. Hence all vertices are incident with each other. That is, there is no pair of nonadjacent vertices. Hence the polysemic pair is a complete graph, which is having a single clique. A chordal atom too consists of a single clique. Hence the theorem.

**Theorem 3.6.10**

The proper Circular-arc graph is not chordal, but its polysemic intersection pair is a chordal graph.
Proof

In the proper Circular arc representation, we have $A_i \cap A_j \neq \emptyset$, only for consecutive values of $i$ and $j$. Hence $(v_i, v_j) \in E_1$, only when $A_i$ and $A_j$ are consecutive arcs. Hence $G_1$ will be of the form $C_n$, for $n \geq 4$ (where $C_n$ is a cycle of length $n$) and therefore $A_i \cap A_j = \emptyset$, for nonconsecutive values of $i$ and $j$. In $C_n$ it is not possible to have any chord. Hence $G_1$ is not chordal.

By the definition of proper circular arc representation, no two arcs fully cover the circle.

Hence $A_i \cup A_j \neq C$, or all $i$ and $j$. Hence $(v_i, v_j) \in E_2$ for all $i$ and $j$. That is $G_2$ is a complete graph, which is a chordal graph.

* * * * *