CHAPTER VI

POLYSEMY OF PERFECT VERTEX PATH GRAPHS
AND COMPACT VERTEX PATH GRAPHS.

6.1. Introduction

In this chapter, polysemic intersection representation is extended to
Perfect Vertex path (PV) and Compact Vertex path (CV) graphs. Two sub
classes of CV graphs are introduced and a recognition algorithm of a polysemic
intersection pair of PV graphs is constructed. Also we prove that PV graphs are
intersection polysemic with strongly chordal graphs.

Antonysamy [An 94] has identified two classes of intersection graphs
depending upon the nature of the vertex paths. They are the subclasses of vertex
path graphs, the CV graphs and the PV graphs. A family of paths in a tree T is
said to be Compact, if no edge of T is in more than one path of the family. The
intersection graph of a family of Compact vertex paths in a tree is called a
Compact Vertex path graph or CV graph. Also, a family of paths in a tree T is
said to be perfect, if no vertex of T is an internal vertex in more than one path of
the family. The intersection graph of a family of Perfect vertex paths in a tree is
called a Perfect Vertex Path graph of PV Graph. Various properties of PV and
CV graphs have been discussed in [An 94], and recognition algorithms for both
these intersection graphs have been introduced. The vertex disjoint paths in a
tree is further studied by Pancia and Mohanty [PM 95] and a polynomial time
recognition algorithm was also presented.
The following results characterize the CV graphs.

6.1.1. Atom Theorem for CV graphs [An 94]

(a) A complete graph is a CV graph.

(b) A chordal graph $G_i$ with exactly two cliques is a CV graph if and only if the cliques have exactly one common vertex.

**Proposition 6.1.2.** [An 94]

If $G_i$ is a CV graph; then

(a) $W(G_i)$ contains only one element where $W(G_i)$ is the set of $v \in C$, the separating clique, such that there exists vertex $w \in G_i$ for which the edge $(v, w) \in E(G)$.

(b) $G_i$ has exactly one relevant clique.

(c) No two separated subgraphs in CV graph are antipodal.

**Theorem 6.1.3.** [An 94]

CV graphs are precisely Block graphs.

**Theorem 6.1.4. Separator Theorem for CV graph**

$G$ is a CV graph if and only if each separated subgraph $G_i$ is a CV graph.

6.2. Characteristics of Polysemic Pair of CV Graphs

6.2.1. Definition (Polysemy of CV Graphs)

Consider $T$ an undirected (directed) tree as the underlying system. Let $P$ be the family of Compact vertex paths in $T$, directed (undirected). Let $G_1 (V, E_1)$ and $G_2 (V, E_2)$ be two simple finite graphs on a common vertex set $V$. Then the mapping $f : V \rightarrow P$ defined by $f(v) = P_v$ where $v \in V$ and $P_v \in P$ is the
compact path corresponding to \( v \) and if \((v, w) \in E_1\), for \( v, w \in V \) if and only if \( P_v \cap P_w \neq \emptyset \) and if \((v, w) \in E_2\), if and only if \( P_v \cup P_w \neq T \); then \( f \) is called the polysemic intersection representation with respect to \( T \) and \( G_1 (V, E_1), G_2 (V, E_2) \) are called the polysemic intersection pairs of Compact vertex path graphs.

**Example 6.2.2.**

CV graph \( G_1 \) and its polysemic image \( G_2 \).

6.2.3. A subclass of CV graphs

Let \( T \) be the underlying system. The intersection graph of a family of paths of length 2 with no common edges will be a subclass of CV graph.

Its polysemic image is a chordal atom.
6.2.4. Another subclass of CV graph

The intersection graphs of a family of paths of length two on a Star graph \( S \), will be a subclass of CV graphs and its polysemic image is also a chordal atom.

Now we propose certain characteristics of the polysemic intersection representation of CV graphs.

**Theorem 6.2.5.**

The color encoding of the polysemic pair of CV Graphs, over a Compact Vertex path representation \( \varphi \) on a tree \( T \), \( \gamma : V \rightarrow \varphi \) can be defined by,

\[
\gamma(u, v) = \begin{cases} 
\text{yellow}, & u \sim_1 v \text{ and } u \sim_2 v \\
\text{green}, & u \sim_1 v \text{ and } u \sim_2 v \\
\text{blue}, & u \sim_1 v \text{ and } u \sim_2 v.
\end{cases}
\]
where \( u \sim_1 v \) if and only if, \( P_u \cap P_v \neq \emptyset \) and \( u \sim_2 v \) if and only if \( P_u \cup P_v \neq T \), where \( \sim_1 \) and \( \sim_2 \) are adjacency in \( G_1 \) and \( G_2 \) respectively and \( P_u \) and \( P_v \) are paths in \( \mathcal{G} \).

Proof

Let \( G_1 (V, E_1) \) be a CV - graph over the collection \( \mathcal{G} \) of undirected (directed) compact vertex paths of the underlying system \( T \) and let \( G_2 (V, E_2) \) be its polysemic intersection image, with respect to a polysemic intersection representation \( \gamma : V \rightarrow \mathcal{G} \). Let the vertices \( a \) and \( b \) in \( V \) have the respective images \( P_u \) and \( P_v \) in \( \mathcal{G} \). Let \( |\mathcal{G}| > 2 \). By the definition of compact vertex paths, if \( P_u \cap P_v = \emptyset \), definitely \( P_u \cup P_v \neq T \) and if, \( P_u \cup P_v = T \), since no edge is common to both the paths the representation can have only two paths in it. Hence \( P_u \cup P_v \neq T \). Therefore no pair of vertices is disjoint in both \( G_1 \) and \( G_2 \). Hence \( \gamma(u, v) = \text{yellow} / \text{green} / \text{blue only} \).

**Theorem 6.2.6**

The polysemic intersection image of a CV graph is always a complete graph in the same vertex set.

Proof

Let \( P \) be the family of Compact vertex paths in \( T \), directed / undirected. Let \( G_1 (V, E_1) \) be a CV graph on \( P \) and let \( f : V \rightarrow P \) be the polysemic intersection representation and let \( G_2 (V, E_2) \) be the polysemic image. By definition, a family of paths in a tree \( T \) is said to be Compact, if no edge of \( T \) is in more than one path of the family. Let the family of path contains more than two paths. Let \( P_i \) and \( P_j \) be two arbitrary paths. Then \( P_i \cup P_j \neq T \) for all \( i \) and \( j \). Hence by the
definition of graph polysemy, \( G_2 \) contains no disjoint pairs of vertices. Hence the polysemic image of a CV graph will always be a complete graph.

**Theorem 6.2.7**

The polysemic intersect on image of a CV graph is always a chordal graph in the same vertex set.

**Proof**

Let \( G_1 (V, E_1) \) be a CV graph on a CV path representation \( P \) and let \( f : V \rightarrow P \) be the polysemic intersection representation and let \( G_2 (V, E_2) \) be the polysemic image. To show that \( G_2 \) is a chordal graph. Let \( |V| = n > 2 \).

Suppose \( G_2 \) is chordal and has \( n \) vertices. Then, the case \( n = 1 \) is trivial. Suppose \( n > 1 \) and every chordal graph with fewer vertices has a Perfect Elimination Order. By Dirac's Lemma, every chordal graph has a simplicial vertex. Let \( G_2 \) has a simplicial vertex, say \( v \). Now \( G_2 \setminus \{v\} \) is a chordal graph with fewer vertices than \( G_2 \). Hence by induction it has a PEO say, \( \beta \). If \( \alpha \) orders the vertex \( v \) first, followed by the remaining vertices of \( G_2 \) in the order determined by \( \beta \), then \( \alpha \) is a PEO of \( G_2 \).

Conversely, suppose \( G \) has a PEO, say \( \alpha \), given by \( v_1, v_2, \ldots, v_n \). We seek a chord of an arbitrary cycle \( \mu \) in \( G_2 \) of length greater than three. Let \( v_i \) be the vertex on \( \mu \) whose label \( i \) is smaller than that of any other vertex on \( \mu \). Since \( \alpha \) is a PEO, monotone adjacency set of \( v_i \), \( (\text{madj}(v_i)) \) is complete; where \( (\text{madj}(v_i)) = \{v_j \in \text{adj}(v_i) | j > i\} \) from where we get, \( \mu \) has at least one chord namely, the edge joining the two neighboring vertices of \( v_i \) in \( \mu \). Hence \( G_2 \) is chordal.
Also, if \( u, v \in G_2 \) then \( P_u \cup P_v \neq T \). By the compactness of \( P_u \) and \( P_v \), no edge of \( T \) is in more than one of \( P_u \) or \( P_v \).

Hence it is not possible to get \( P_u \cup P_v = T \) when number of paths is \( > 2 \).

Hence \( G_2 \) has only a single clique. Hence it is a chordal atom.

**Theorem 6.2.8**

Every vertex of the polysemic image of a CV graph is simplicial.

Proof

Let \( G_1 (V, E_1) \) be a CV graph on a CV path representation \( P \) and let \( f : V \rightarrow P \) be the polysemic intersection representation and let \( G_2 (V, E_2) \) be the polysemic image. Let \( |V| = n > 2 \).

To show that every vertex in \( G_2 \) is simplicial. We have proved that \( G_2 \) is a chordal atom. Hence it has only a single clique which is a complete graph. Therefore \( \text{adj}(v_i) \forall i \) induces a complete subgraph of \( G \). Hence \( v_i \forall i \) is a simplicial vertex in \( G_2 \).

**Theorem 6.2.9**

The polysemic intersection image of a CV graph is a CV graph.

Proof

By Clique Tree Theorem for CV Graphs [An 94], a Graph \( G(V, E) \) is a CV graph, if and only if there exists a tree \( T \) with vertex set \( C \), such that for every \( v \in V, T(C_v) \) is a path in \( T \) and every edge in \( T \) contained in exactly one path of \( T(C_v) \).

Let \( G_1 (V, E_1) \) be a CV graph on a CV path representation \( P \) and let
f : V \rightarrow P be the polysemic intersection representation and let \( G_2 (V, E_2) \) be the polysemic image. To show that \( G_2 \) is a CV graph. Let \( |V| = n > 2 \).

Let \( T \) be a tree satisfying the conditions given in the theorem. Clearly the family of paths \( P = T(C_v) \) for all \( v \in V \), satisfies the definition of CV graph. Hence \( G_2 \) is a CV graph.

Conversely, suppose \( G_2 \) is a CV graph. Let \( T \) be the tree with minimum number of vertices such that \( G_2 \) is the intersection graph of compact paths of \( T \). We claim that there is a one to one correspondence between the cliques of \( G_2 \) and the vertices of \( T \). Since a family of paths in a tree satisfies Helly's property, for every clique of \( G_2 \), there is a node in \( T \). Let \( C_1 \) and \( C_2 \) be two complete subgraphs of \( G_2 \) such that \( C_1 \subseteq C_2 \). Let \( c_1 \) and \( c_2 \) be the corresponding nodes in \( T \). Let \( c_3 \) be the node next to \( c_1 \) in the path from \( c_1 \) to \( c_2 \) in \( T \). Let \( C_3 \) be the corresponding complete sub graph of \( G_2 \). Since \( C_1 \subseteq C_2 \), every path in \( C_1 \) also passes through \( C_2 \) and hence through \( C_3 \). Obtain a new tree \( T' \) from \( T \) by coalescing \( c_1 \) and \( c_3 \) and eliminating the edge between them. Any path that contains \( c_1 \) also contains \( c_3 \). Therefore \( T' \) is also a CV graph tree representation for \( G_2 \). But \( T' \) contains one node less than \( T \). This contradicts the fact that \( T \) is a tree representation for \( G_2 \) with minimum number of nodes. Hence, the nodes of \( T \) correspond to distinct cliques and \( T \) is the clique tree representation for \( G_2 \).

**Theorem 6.2.10**

The polysemic image of a CV graph is a block.

Proof

Let \( G_1 (V, E_1) \) be a CV graph on a CV path representation \( P \) and let
f : V → P be the polysemic intersection representation and let \( G_2 (V, E_2) \) be the polysemic image. To show that \( G_2 \) is a block.

A cut point of a graph is one whose removal increases the number of components of the graph. A non separable graph is connected, non trivial and has no cut vertex. A block of a graph is a maximal non separable subgraph. \( G_2 \) is a complete graph. A complete graph contains no cut vertices. Therefore it is the maximal non separable subgraph of \( G_2 \), and hence \( G_2 \) is a block.

6.3. Characteristics PV Graphs

**Proposition 6.3.1** [An 94]

If \( G \) is a PV graph,

(a) Intersection of three cliques in \( G \) is utmost a singleton

(b) There do not exist more than one pair of antipodal subgraphs with respect to a separating clique \( C \).

**Lemma 6.3.2** [An 94]

Let \( G \) be a PV graph and let \( G_i \) and \( G_j \) be two separated subgraphs of \( G \). If \( G_i \) and \( G_j \) are opposite to each other then they must appear in two different branches of \( T \).

**Theorem 6.3.3** [An 94]

If \( G \) is a PV graph and \( G_i \) dominates \( G_j \), where \( G_i \) and \( G_j \) are two separated subgraphs of \( G \), then

(a) \( W(G_j) \) is a singleton set.

(b) For every \( v \in W(G_i) \setminus W(G_j) \), \( v \) is in exactly one relevant clique of \( G \).

(c) \( G \) has exactly one relevant clique intersecting \( W(G_i) \setminus W(G_j) \).
(d) There don't exist two pairs of cliques antipodal with respect to any relevant clique of \( G_i \).

**Theorem 6.3.4. Separator Theorem** [An 94]

\( G \) is a PV graph if and only if the following are true;

(a) Each \( G_i \) is a PV graph.

(b) There do not exist more than one pair of antipodal sub graphs.

(c) Whenever \( G_i \) dominates \( G_j \), \( W(G_j) \) is a singleton and there is exactly one relevant clique in \( G_i \) intersecting antipodal cliques with respect to any relevant cliques of \( G_i \).

(d) There do not exist more than two pairs of opposite sub graphs.

Now we define the polysemic intersection representation of PV graphs.

**6.4. Polysemny of PV Graphs**

**6.4.1. Definition**

Consider \( T \) an undirected (directed) tree as the underlying system. Let \( P \) be the family of Perfect vertex paths in \( T \), directed (undirected). Let \( G_1 (V, E_1) \) and \( G_2 (V, E_2) \) be two simple finite graphs on a common vertex set \( V \). Let the mapping \( f : V \rightarrow P \) be defined by \( f(v) = P_v \) where \( v \in V \) and \( P_v \in P \) is the perfect vertex path corresponding to \( v \). Now if for \( v, w \in V \), \( (v, w) \in E_1 \), if and only if \( P_v \cap P_w \neq \emptyset \) and \( (v, w) \in E_2 \), if and only if \( P_v \cup P_w \neq T \). Then \( f \) is called the polysemic intersection representation with respect to \( T \) and \( G_1 (V, E_1) \), \( G_2 (V, E_2) \) are called the polysemic intersection pairs of Perfect vertex path graphs.
Example 6.4.2

PV graph $G_1$ and its polysemic pair $G_2$.

Clique tree representations of $G_1$ and $G_1$.

The following results characterize the polysemic intersection image of PV graphs.

Theorem 6.4.3

The polysemic intersection pair of a PV graph is triangulated.

Proof

Let $G_1 (V, E_1)$ be a PV graph on a PV path representation $P$ and let $f : V \rightarrow P$ be the polysemic intersection representation and let $G_2$ be the polysemic image of a PV graph, on vertices $v_0, v_1, \ldots, v_n$, and let $P_1, P_2, P_3, \ldots, P_n$ be the corresponding perfect vertex paths in $P$ of the tree $T$. Suppose $G_2$ contains a chordless cycle $C = \{v_0, v_1, \ldots, v_{i-1}, v_0\}$. Let $P_k$ denotes the perfect
vertex path corresponding to the vertex $v_k$. Choose the vertices $v_{i-1} & v_0 \in P_{i-1} \cap P_0$. Since $P_{i-1}$ and $P_0$ are perfect vertex paths $v_{i-1} & v_0$ cannot be internal vertices in both the paths. They can be end vertices in only one of the paths, at a time, for if there are two paths with same end vertices that will form a cycle in $T$ contradicting the fact that $T$ is a tree. Therefore there is no possibility of having an edge between $v_{i-1} & v_0$, contradicting the fact that, $C$ is a cycle. Hence the theorem.

**Theorem 6.4.4**

The polysemic intersection graph of a PV graph is also a PV graph.

Proof

Let $G_1 (V, E_1)$ be a PV graph on a FV path representation $P$ and let $f : V \rightarrow P$ be the polysemic intersection representation and let $G_2 (V, E_2)$ be the polysemic image.

By the Clique Tree Theorem of PV Graph [An 94], a graph $G (V, E)$ is a PV graph, if and only if there exists a tree $T$ with the vertex set $C$, such that for every $v \in V$, $T(C_v)$ is a path in $T$ and no vertex of $T$ is an internal vertex of more than one path of $T(C_v)$. To show that $G_2$ is a PV graph.

Let $T$ be a tree satisfying all the conditions of the given theorem. We can construct the family of paths $P = T(C_v)$ for all $v \in V$ to obtain a PV representation. Clearly, this family of paths satisfy all the conditions of the PV graph, hence the given graph $G_2$ is a PV graph.

Conversely, let $G$ be a PV graph. Let $T$ be the tree with minimum number of nodes such that $G_2$ is the intersection graph of a family of perfect paths of $T$. We
claim that there is a one to one correspondence between the cliques of $G_2$ and the nodes of $T$. Since a family of vertex paths in a tree satisfy Helly's property for every clique of $G_2$, there is a node in $T$ for every clique of $G_2$.

Let $C_1$ and $C_2$ be two completely connected subgraphs of $G_2$ such that $C_1 \subseteq C_2$. Let $c_1$ and $c_2$ be the corresponding nodes in $T$. Let $c_3$ be the node next to $c_1$ in the path from $c_1$ to $c_2$ in $T$. Let $C_3$ be the corresponding complete subgraph of $G_2$. Since $C_1 \subseteq C_2$, every path in $C_1$ also passes through $C_2$ and hence through $C_3$, $C_1 \subseteq C_3$.

Obtain a new tree $T'$ from $T$ by coalescing $c_1$ and $c_3$ and eliminating the edge between them. Any path that contains $c_1$ also contains $c_3$. Therefore, $T'$ is also a PV tree representation for $G_2$. But $T'$ contains one node less than $T$. This contradicts the fact that $T$ is a tree representation for $G_2$ with minimum number of nodes. Hence, the nodes of $T$ correspond to distinct cliques and $T$ is the clique tree representation for $G_2$.

**Theorem 6.4.5**

If the polysemic image $G_2$ of a PV graph $G_1$ has exactly two cliques, then it is also a PV graph.

Proof

By the atom theorem for PV graphs [An 94] it is clear that, a graph $G$, with exactly two cliques is a PV graph.

Let $G_2$ be a graph with exactly two cliques $C_1$ and $C_2$. Then the clique tree representation of $G_2$ is $K_2$, which is clearly a PV clique representation of $G_2$. Hence the graph $G_2$ is a PV graph.
**Theorem 6.4.6**

In the case of Perfect Vertex Path Graphs (PV - graphs), the Polysemic intersection representation does not satisfy red - adjacency.

**Proof**

Let $G_1$ be a PV - graph over the collection $\emptyset$ of undirected (directed) perfect vertex paths of the underlying system $T$ and let $G_2$ be its polysemic intersection image, with respect to a polysemic intersection representation $f : V \to \emptyset$, where $V$ is the vertex set of the intersection graph over $\emptyset$. Let the vertices $a$ and $b$ in $V$ have the respective images $P_a$ and $P_b$ in $\emptyset$. By the definition of Perfect vertex paths, if $P_a \cap P_b = \emptyset$, definitely $P_a \cup P_b \neq T$ and if, $P_a \cup P_b = T$, since no vertices are the internal vertices of more than one path, all other vertices will be adjacent in $G_2$, and $P_a \cap P_b \neq \emptyset$. Therefore $f$ is not conformed to red-adjacency.

**Theorem 6.4.7**

The clique tree representations of PV graphs and their polysemic intersection images are the same when the image is not a complete graph.

**Proof**

By Clique Tree Theorem of PV Graph. [An 94], a graph $G (V, E)$ is a PV graph, if and only if there exists a tree $T$ with the vertex set $C$, such that for every $v \in V$, $T(C_v)$ is a path in $T$ and no vertex of $T$ is an internal vertex of more than one path of $T(C_v)$.

Let $G_1 (V, E_1)$ be a PV graph on a PV path representation $P$ and let
Let \( f: V \to P \) be the polysemic intersection representation and let \( G_2(V, E_2) \) be the polysemic image.

Let \( T \) be a tree satisfying all the conditions of the given theorem. We can construct the family of paths \( P = T(C_v) \) for all \( v \in V \) to obtain a PV representation. Clearly, this family of paths satisfy all the conditions of the PV graph hence the given graph \( G \) is a PV graph.

Conversely, let \( G_1 \) be a PV graph. Let \( T \) be the tree with minimum number of nodes such that \( G_1 \) is the intersection graph of a family of perfect paths of \( T \). We claim that there is a one to one correspondence between the cliques of \( G_1 \) and the nodes of \( T \). Since a family of vertex paths in a tree satisfy Helly's property for every clique of \( G_1 \), there is a node in \( T \) for every clique of \( G_1 \).

Let \( C_1 \) and \( C_2 \) be two completely connected subgraphs of \( G_1 \) such that \( C_1 \subseteq C_2 \). Let \( c_1 \) and \( c_2 \) be the corresponding nodes in \( T \). Let \( c_3 \) be the node next to \( c_1 \) in the path from \( c_1 \) to \( c_2 \) in \( T \). Let \( C_3 \) be the corresponding complete subgraph of \( G_1 \). Since \( C_1 \subseteq C_2 \), every path in \( C_1 \) also passes through \( C_2 \) and hence through \( C_3 \), \( C_1 \subseteq C_3 \).

Obtain a new tree \( T' \) from \( T \) by coalescing \( c_1 \) and \( c_3 \) and eliminating the edge between them. Any path that contains \( c_1 \) also contains \( c_3 \). Therefore, \( T' \) is also a PV tree representation for \( G_1 \). But \( T' \) contains one node less than \( T \). This contradicts the fact that \( T \) is a tree representation for \( G_1 \) with minimum number of nodes. Hence, the nodes of \( T \) correspond to distinct cliques and \( T \) is the clique tree representation for \( G_1 \).
By the symmetric property of the polysemic intersection representation, $G_2$ is also a PV graph in the same representation $P$ and $F^1$ is the intersection representation of $G_2$ and hence $G_2$ is also having the same clique tree representation $T$.

**Theorem 6.4.8**

A PV graph and its polysemic intersection image, have equal number of pairs of antipodal subgraphs, when its image is not complete.

**Proof**

By previous theorem it is clear that, the clique tree representation of a PV graph and its polysemic image are the same. By separator theorem for PV graphs it is true that, there do not exist more than one pair of antipodal subgraphs with respect to a separating clique $C$ in a PV graph. Also the polysemic image of a PV graph is a PV graph. Hence the result.

**Lemma 6.4.9**

The color encoding of the polysemic pair of PV Graphs, over a Perfect Vertex path representation $\phi$ on a tree $T$, $\gamma : V \to \phi$ can be defined by,

\[
\gamma(u, v) = \begin{cases} 
\text{yellow, } u \sim_1 v \text{ and } u \not\sim_2 v \\
\text{green, } u \sim_1 v \text{ and } u \sim_2 v \\
\text{blue, } u \sim_1 v \text{ and } u \sim_2 v.
\end{cases}
\]
where $u \sim_1 v$ if and only if, $P_u \cap P_v \neq \emptyset$ and $u \sim_2 v$ if and only if $P_u \cup P_v \neq T$, where $\sim_1$ and $\sim_2$ are adjacency in $G_1$ and $G_2$ respectively and $P_u$ and $P_v$ are paths in $\mathcal{P}$.

Proof

Let $G_1 (V, E_1)$ be a PV-graph over the collection $\mathcal{P}$ of undirected (directed) perfect vertex paths of the underlying system $T$ and let $G_2 (V, E_2)$ be its polysemic intersection image, with respect to a polysemic intersection representation $\gamma : V \rightarrow \mathcal{P}$. Let the vertices $a$ and $b$ in $V$ have the respective images $P_a$ and $P_b$ in $\mathcal{P}$. By the definition of Perfect vertex paths, if $P_u \cap P_v = \emptyset$, definitely $P_u \cup P_v \neq T$ and if, $P_u \cup P_v = T$, since no vertices are the internal vertices of more than one path, all other vertices will be adjacent in $G_2$, and $P_u \cap P_v \neq \emptyset$. Therefore no pair of vertices are disjoint in both $G_1$ and $G_2$.

Hence $\gamma(u, v) = \text{yellow} / \text{green} / \text{blue}$ only.

6.5. Recognition of Polysemic Pair of PV Graphs

Theorem 6.5.1

Let $G_1 = (V, E_1)$, and $G_2 = (V, E_2)$ be PV graphs on the vertex set $V$, with intersection representations $f$ and $g$ respectively. Then $(G_1, G_2)$ is a polysemic intersection pair if and only if, their intersection representations are disjoint and the blue edges of their joint graph $J(G_1, G_2)$ can be directed to create a transitive tournament having no restrictions isomorphic to any of the forbidden graphs given in Lemma 4.2.2.

Proof
If \( f \) is a polysemic intersection representation of \((G_1, G_2)\), over Perfect Vertex path representation on a tree \( T \), then \( f \) conforms to some joint tournament \( \Pi \), provided that the perfect vertex paths \( P_x \) and \( P_y \) have the property, \( P_x \cap P_y \neq \emptyset \) and \( P_x \cup P_y \neq T \), \((x, y)\) is directed from \( x \) to \( y \); and this implies that if there exists a polysemic pair of path graphs \((G_1, G_2)\), their intersection representation cannot intersect, they are of the form \( f \) and \( f^c \), the complement of \( f \).

Assume without loss of generality, that the vertex paths in \( f(V) \) are labeled as, \( P_1, P_2, P_3, P_4, \ldots \) where, \( V = \{1, 2, 3, 4, \ldots\} \) is the vertex set of \( G_1 \) and \( G_2 \).

Now, direct each blue edge from the vertex corresponding to the path with a label of small integer to the vertex corresponding to the path having a large integer as label. The direction of every edge in the resulting tournament \( T \) conforms to the total order, so \( T \) is transitive.

Thus \( \Pi \) is a joint tournament of \((G_1, G_2)\), because, blue - edges of \( J(G_1, G_2) \) can be directed to create a joint tournament \( T \).

By induction on the size of \( V \), we show that \((G_1, G_2)\), is a polysemic pair of path graphs and has a representation conforming to \( T \). This claim is true for \( |V| < 3 \), so assume it holds whenever \( |V| < k \) and let \( |V| = k \).

Let \( z \) be the sink of \( \Pi \) - the unique vertex with out - degree equal to 0.
Since the necessary conditions are hereditary, we know by the induction hypothesis that \((G_1 - z, G_2 - z)\) has a polysemic vertex path representation \( f \) that maps every \( x \) of \( V' = V - \{z\} \) to a vertex path \( P_x \), and conforms to \( \Pi - z \).
Assume again without loss of generality, that the \((k - 1)\) paths of \(f(V')\) are distinct. Since \(z\) is maximal in \(G_1\) and minimal in \(G_2\), the three sets \(\sim_1(z)\), \(\sim_2(z)\), and \((\sim_1 \text{ and } \sim_2)(z)\), form a partition of \(V'\).

One can always augment \(f\) with the path \(P_z\) corresponding to \(z\), thereby creating a polysemic interval representation for \((G_1, G_2)\), that conforms to \(\Pi\).

6.5.2. Algorithm

6.5.2.1 Subroutine I

To check whether \(G_1\) and \(G_2\) are PV Graphs
- check whether they are triangulated,
- check whether every cycle of length greater than or equal to 3 has a chord,
- Construct a tree representation.
- If possible, accept.

6.5.2.2 Subroutine II

-To construct a joint graph \(J(G_1, G_2)\)
-Construct the complete graph on \(V\)
-direct the edge \((x, y)\) from \(x\) to \(y\), if \(x \sim_1 y\) and \(x \sim_2 y\).

6.5.2.3 Subroutine III

Check whether \(G_1\) and \(G_2\) satisfy the property of Compulsory containment.
- Call out the vertices 3 by 3 as \(\{u, v, x\}\) and check if \((u, x) \notin G_2\) then, \((v, x)\) is yellow and if \((v, x) \notin G_1\) then \((u, x)\) is green.

Algorithm

Input: Two intersection path graphs \(G_1\) and \(G_2\).
Output: Whether they are polysemic intersection pairs.

Step 1 Verify whether $G_1$ and $G_2$ have same number of vertices.

1.a. If no, abandon.

1.b. If yes, check whether both $G_1$ and $G_2$ are PV graphs.

Subroutine I

Step 2. Check, whether the corresponding intersection representation $f$ and $g$ of $G_1$ and $G_2$ respectively, are disjoint.

2.a. Represent $G_1$ and $G_2$ by their adjacency matrices and ensure that they are not equivalent. (No edge gets assigned both colors Yellow ($C_Y$) and Green ($C_C$).)

2.b. Also, for each $x \in V$ record $d^+_i(x)$ and $d^-_i(x)$, that is the number of $c_i$ edges in and out of the vertex $x$, respectively, $i = Y, G$.

Step 3. Construct the joint graph, $J(G_1, G_2)$.

Subroutine II

Step 4. Screens for the restrictions to check whether $J(G_1, G_2)$ is a transitive tournament. [Figure II in CH IV]

4.a. Call out the vertices in triples as $\{x, y, z\}$

4.b. If a triple matches $FR_{0a}$

4.b.1. Reject. $[(G_1, G_2) \text{ cannot be a polysemic intersection pair.}]$

4.c. If a triple matches $FR_{0b}$

4.c.1. Reject.

4.d. If a triple matches $FR_2$

4.d.1. Force
4.e. If a triple matches to the restriction acyclicity, 
    Force 
4.f. If a triple matches to the restrictions FR₁ and acyclicity, 
    Force. 
4.g. If any Cᵦ edge has already been directed contrary to this rule, 
    Reject. 
4.h. For any edge that has already been directed in the confirming 
    direction, 
    Do Nothing. 
4.i. Any other blue edges have not yet been directed, direct those in 
    the conforming direction, incrementing \( d⁻(\cdot) \) and \( d₊(\cdot) \) as 
    appropriate, and add them to the queue of newly forced edges. 
4.j. For triples with none of the pattern in Figure III in CH IV; forces 
    nothing. This is a default action. 

Step 5. Propagate any directions that were previously forced. 
5.a. While the queue of newly forced edges is non-empty, it de 
    queues an edge directed, as from \( u \) to \( v \) and checks every triple 
    \( \{u, v, w\} \) for \( w \in V - \{u, v\} \) 
5. b. If the queue is ever emptied, then propagation terminates. 

Step 6. If remain Cᵦ edges that are still undirected, 
6.a. Direct any such edge arbitrarily and enqueue it. 
6.b Repeat Step 4. 

Step 7. Check whether the properties are satisfied by \( (G₁, G₂) \). 
7.a. Compulsory Complementarity of vertices disjoint in both the
graphs.

No pair of vertices are disjoint in both the graphs.

7.b. Check whether \( \gamma(u, z) = \gamma(v, z) \) for all vertices \( u, v \) and \( z \) with \( u \neq v \).

No pair of vertices are red adjacent.

7.c. Check whether respecting compulsory containment.

Subroutine III

7.d. Check whether respecting extremality.

No vertex is isolated in \( G_1 \)

No vertex is isolated in \( G_2 \)

6.5.4. Correctness of the Recognition Algorithm

By rejecting a pair \((G_1, G_2)\), the algorithm essentially proves by contradiction that \((G_1, G_2)\), cannot be a polysemic pair of path graphs. The forcing rules states that if \( \Pi \) meets the conditions of theorem 6.5.1., then certainly blue edges must have specific directions. There are only two circumstances in which the algorithm rejects a pair, either \( J(G_1, G_2) \), contains \( FR_0 \) or two forcing rules conflict, which gives to a contradiction. Furthermore, if the algorithm rejects a pair after having made an arbitrary choice in step 4.b; then by theorem 6.5.1. every possible choice leads to rejection. On the other hand, the algorithm will accept \((G_1, G_2)\), only after it has built a joint tournament, in which case \((G_1, G_2)\) is a polysemic pair of path graphs.
6.5.5. Analysis of the recognition algorithm.

Theorem 6.5.5.1

The algorithm 6.5.2 can be implemented in polynomial time.

Proof

In order to check whether the graphs are PV graphs, the algorithm spends $O(1)$ time on each of the $nC_3$ triangles, where $n = |V|$. It spends $\Theta(n^2)$ times reading the input and building $J(G_1, G_2)$. To check the restrictions in Step.3 it takes $O(1)$ times for all $nC_3$ triangles. It spends $\Theta(n)$ times on each visit to a blue edge visiting none of the $O(n^2)$ edges more than once. Finally it takes $O(1)$ times each to check the two properties in groups of $nC_3$ triples, and the last $O(1)$ times for $2n$ vertices to check extremality. So the algorithm recognizes polysemic pair of path graphs in $O(n^3)$ times.

6.6. PV Graphs Are Intersection Polysemic with Strongly Chordal graphs.

Dirac and Rose [Di 61, Ro 70] have defined chordal graphs and strongly chordal graphs as follows - a graph $G$ is chordal if and only if every induced subgraph of $G$ has a simple vertex and a strongly chordal graph is a chordal graph in which every cycle on six or more vertices contains a chord joining two vertices with an odd distance in the cycle.

A similar statement like that of Dirac and Rose is proved by Farber [Fa 83] and we state the same without proof. A given graph is strongly chordal if and only if every induced subgraph of $G$ has a simple vertex.
Definition 6.6.1. [Fa 83]

An incompatible trampoline is a chordal graph $G$ on $2n$ vertices for some $n \geq 3$ whose vertex set can be partitioned into two sets $W = (w_1, w_2, \ldots, w_n)$ and $U = (u_1, u_2, \ldots, u_k)$ so that the following two conditions are satisfied.

1. $W$ is independent
2. For each $i$ and $j$, $w_i$ is adjacent to $u_j$ if and only if $i = j \pmod{n}$ or $i = j + 1 \pmod{n}$

Definition 6.6.2 [Fa 83]

An incompatible trampoline in which $G (u_1, u_2, \ldots, u_n)$ is a complete set is a trampoline.

Now, Farber [Fa 83] characterizes strongly chordal graphs as follows:

A graph $G$ is strongly chordal if and only if it contains no cycle of length greater than three and no trampoline as induced subgraph.

Kevin White et al [WFP 85] defined the strongly chordal graph as follows, “$G$ is strongly chordal if it is chordal and every even length cycle of length six or more has an odd chord”.

Further, Farber characterizes strongly chordal graphs thus:

1. Total graphs of trees are strongly chordal.
2. Interval graphs are strongly chordal graphs and
3. Block graphs are strongly chordal graphs.

After stating the above known results we prove the following theorem and the corollaries

Theorem 6.6.3
PV graphs are intersection polysemic with strongly chordal graphs.

Proof

Let $G_1 (V, E_1)$ be a PV graph on a PV path representation $P$ and let $f : V \rightarrow P$ be the polysemic intersection representation and let $G_2$ be the polysemic image.

First let us prove that PV graphs are strongly Chordal graphs.

Farber [Fa 83] has shown that a graph is strongly chordal if and only if it contains no cycle of length greater than three and no trampoline as induced subgraph. Hence to prove that PV graphs are strongly chordal, it is enough to prove that PV graphs have no trampoline. Suppose the PV graph $G_1$ has a trampoline $T$ on $2n$ vertices as an induced subgraph then by definition, the vertex set of $T$ is partitioned into two sets.

$U = \{u_1, u_2, \ldots, u_k\}$

$W = \{w_1, w_2, \ldots, w_n\}$ satisfying the following conditions.

(i) $W$ is independent

(ii) $G_1 (U)$ is a complete graph.

(iii) For each $i$ and $j$, $w_i$ is adjacent to $u_j$ if and only if $i = j \pmod{n}$ or $i = j + i\pmod{n}$.

Let $C$ be the maximal clique containing $G_1 (U)$. Without loss of generality we assume that $u_i$ and $u_{i+1}$ are adjacent to $w_i$. Let $C_i, 1 \leq i < n$ be the maximal clique containing the vertices $w_i, u_i, u_{i+1}$. In any clique tree representation of the PV graph $c$ is an internal node of the paths corresponding to the vertices $u_i, 1 \leq i \leq n$, which means that $c$ is an internal node in more than one path of the clique. Hence
the given PV graph does not have any trampoline as induced sub $\leq$graph which proves that PV graphs $G_1$ is a strongly chordal graph.

Since PV graphs are intersection polysemic with PV graphs it is trivial that $G_2$ is also strongly chordal. We can conclude that PV graphs are intersection polysemic with Strongly chordal graphs.

**Corollary 6.6.4.**

The Interval graphs are intersection polysemic with strongly chordal graphs.

**Proof**

By Farber [Fa 83], interval graphs are strongly chordal graphs. We have proved in CH III that the interval graphs are intersection polysemic with interval graphs in the same vertex set. Hence the interval graphs are intersection polysemic with strongly chordal graphs.

**Corollary 6.6.5.**

The Block graphs are intersection polysemic with strongly chordal graphs.

**Proof**

By Farber [Fa 83], block graphs are strongly chordal graphs. We have proved in CH II that the block graphs are intersection polysemic with single block in the same vertex set. Hence the block graphs are intersection polysemic with strongly chordal graphs.

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