CHAPTER 3

COMPLETE SET OF GENERA
OF COMPACT RIEMANN SURFACES
ADMITTING \( C_p \times D_m \),
AS AN AUTOMORPHISM GROUP

3.1 The conditions for the existence of a smooth epimorphism

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COMPLETE SET OF GENERA OF COMPACT RIEMANN SURFACES ADMITTING $C_p \times D_m$, AS AN AUTOMORPHISM GROUP *

It has already mentioned in Chapter 1 that a finite group can be represented as a group of automorphisms of a compact Riemann surface of genus $g \geq 2$ and the same group may act as a group of automorphisms of compact Riemann surfaces of different genera. The determination of the sets of genera of surfaces on which a given finite group acts as an automorphism group, is a very interesting problem.

The minimum genus problem, which we have discussed in Chapter 2, is a special case of this genera problem. The initiative in solving the genera problem was taken by Kulkarni [104], Kulkarni and Maclaclan [106] and McCullough and Miller[127], considering some elementary groups, namely, some special types of cyclic and non-cyclic abelian groups only. Further the genera problem was partially solved respectively by Chetiya and Patra; Chetiya, Dutta and Patra; Chetiya and Dutta; Das G for $K$-metacyclic groups [41], a special class of dihedral groups of order $2p$ [33], a triangular extension of triangular groups [36], a quadruple extension of quadruple groups [58].

In our thesis we consider the group $G = C_p \times D_m$, which we have shown to be a class of $z$-metacyclic groups. We determine the values of $g$, the genus of the compact Riemann surfaces on which the group $G$ acts as a group of automorphisms.

Before going to solve the genera problem, we recall in brief, the necessary and sufficient conditions for the existence of a smooth epimorphism $\phi$ from a Fuchsian group $\Gamma$ to the above mentioned group $G$. Then we determine the genera of the compact Riemann surfaces on which $G$ acts as a group of automorphisms.

3.1 The conditions for the existence of a smooth epimorphism

Suppose $\Gamma$ be a Fuchsian group with signature $\Delta(y; m_1, m_2, \ldots, m_r)$ and

It is mentioned in Chapter 2, that the group $G = C_p \times D_m$, under consideration, is generated by $s, t$ satisfying the relations

\[ s^m = t^{2p} = (st)^{2p} = 1, \quad t^{-1}st = s^{-1}. \]  

where $p$ is an odd prime and $m$ is a positive integer greater than one.

Now we recall the conditions for the existence of a smooth epimorphism, from a Fuchsian group $\Gamma$ onto the group $G$ for odd and even values of $m$, that we have already proved in Chapter 2 and rename as

**Theorem 3.1**

Let $\Gamma$ be a Fuchsian group with signature $\Delta(y; m_1, m_2, \ldots, m_r)$ and let $G = C_p \times D_m$, where $p$ is an odd prime and $m$ is an integer greater than one with $\text{hcf}(p, m) = 1$.

Let $\bar{m}_j = \frac{m}{d_j}$, where $d_j = \text{hcf}(m, j), 1 \leq j \leq m - 1$ and $p_j = \text{lcm}[p, \bar{m}_j]$.

Then there exists a smooth epimorphism $\phi: \Gamma \to G$ if and only if

1. The periods of $\Gamma$ (if any) take the values from the set $\{2, 2p, p, \bar{m}_j, p_j\}, 1 \leq j \leq m - 1,$

2. If $t_2, t_{2p}, t_p, t_{\bar{m}_j}, t_{p_j}$ denote the number of occurrences of periods $2, 2p, p, \bar{m}_j, p_j$ respectively in $\Gamma$, then

   (a) $t_2 + t_{2p}$ is even when $m$ is odd.

   (b) If one of $t_2$ and $t_{2p}$ is zero, then other must be even.

   (c) $\gamma = 0$ is possible in the following cases only

      (i) when $t_{2p} \geq 2$,

      or (ii) when $t_2 \geq 2$. In case $m$ is odd, at least one of $t_p$ and $t_{\bar{m}_j}$ should be even.
nonzero and \( t_x \geq 4 \) if \( t_r = 0 \).

(d) (i) If any two of \( t_{2p}, t_p \) and \( t_p \) are zero, then the third must be greater than or equal to 2.

(ii) \( t_x + t_{2p} \geq 4 \), if \( t_p = t_{\bar{m}} = 0 \).

(e) When \( \Gamma \) is a surface group and also when \( \Gamma \) contains \( p \) as its only period, we have \( \gamma \geq 2 \).

If \((p, m) = p\), then \( p_j = \bar{m}_j \) and in this case, we have the following Theorem.

**Theorem 3.2**

Let \( \Gamma \) be a Fuchsian group with signature \( A(\gamma; m_1, m_2, \ldots, m_r) \) and \( G = C_p \times D_m \), where \( p \) is an odd prime and \( m \) is an integer greater than one with \((p, m) = p\). Then there exists a smooth epimorphism \( \phi: \Gamma \to G \) if and only if

1. The periods of \( \Gamma \) (if any) take the values from the set \( \{2, 2p, \bar{m}_j\} \).
2. If \( t_2, t_{2p}, t_{\bar{m}} \) denote the number of occurrences of periods \( 2, 2p, \bar{m}_j \) respectively in \( \Gamma \) then

(a) \( t_x + t_{2p} \) is even when \( m \) is odd.

(b) If one of \( t_x \) and \( t_{2p} \) is zero, other must be even.

(c) \( \gamma = 0 \) is possible in the following cases only

(i) when \( t_{2p} \neq 0 \). Here \( t_{2p} \geq 2 \) if \( t_{\bar{m}} = 0 \).

or (ii) when \( t_x \geq 4 \) and \( t_{\bar{m}} \neq 0 \).

After getting the conditions for the existence of a smooth epimorphism, for odd and even \( m \) separately, we now going to determine the genera of the compact Riemann surfaces on which \( G \) acts as a group of automorphisms, for positive integral values of \( m \), combining the two cases for odd and even \( m \).

**3.2 Determination of genera**

Let \( g \) be the genus of a compact Riemann surface on which \( G \) acts as a group of automorphisms. Then there exists a Fuchsian group \( \Gamma \) of genus \( \gamma \), with
presentation (3.1.1) and a smooth epimorphism $\phi : \Gamma \to G$ such that the kernel of $\phi$ is a surface group of genus $g$. As stated in Chapter 1, we get

$$2(g-1) = |G| \left\{ 2 \gamma - 2 + \sum \left( 1 - \frac{1}{m_i} \right) \right\}$$

or

$$g = 1 + pm \left\{ 2 \gamma - 2 + \sum \left( 1 - \frac{1}{m_i} \right) \right\}.$$  \hspace{1cm} (3.2.1)

where $p$ is any odd prime, $m$ is a positive integer greater than one and $m_i$ are the periods of $\Gamma$.

It is proved that for the existence of a smooth epimorphism $\phi : \Gamma \to G$, the periods and the genus of $\Gamma$ must satisfy the conditions of Theorem (3.1) and Theorem (3.2).

As defined in Chapter 1 the equation (3.2.1) gives

$$g = 1 + pm \left[ 2(\gamma - 1) + \left( \frac{1}{2} t_2 + \frac{1}{2p} t_{2p} + \frac{1}{p} t_p + \sum \frac{1}{m_j} \right) \right]$$

$$\text{or} \quad g = 1 + pm \left[ 2(\gamma - 1) + \sum \frac{t_{m_j} - \frac{t_{2p}}{2p}}{p - \frac{t_{2p}}{2p}} - \sum \frac{t_{p_j}}{p_j} \right].$$

Therefore we can write $g = \lambda + m \alpha + pm \beta$. \hspace{1cm} (3.2.2)
where
\[
\lambda = 1 - pm \left[ \frac{t_p}{p} + \sum \frac{t_{\bar{m}_j}}{m_j} + \sum \frac{t_{\bar{p}_j}}{p_j} \right]
\]
\[
\alpha = 2p\gamma - 2p + \frac{pk + (p - 1)t_{2p}}{2}
\]
\[
\beta = t_p + \sum t_{\bar{m}_j} + \sum t_{\bar{p}_j}
\]
Here \( \lambda, \alpha, \beta \) are integers, not simultaneously zero.

Clearly \( \lambda \leq 1 \)

\(\beta \geq 0\)

\(\alpha < 0 \) when \( \gamma = 0, \ t_{2p} = 0, \ t_2 = 2, \)

\(= 0 \) when \( \gamma = 1, \ t_{2p} = t_2 = 0, \)

\(> 0 \) otherwise.

Let \( D = \alpha + pm\beta - |\lambda| \)

\[
= 2p\gamma - 2p + \frac{pk + (p - 1)t_{2p}}{2} + pm\left( t_p + \sum t_{\bar{m}_j} + \sum t_{\bar{p}_j} \right) - pm\left( \frac{t_p}{p} + \sum \frac{t_{\bar{m}_j}}{m_j} + \sum \frac{t_{\bar{p}_j}}{p_j} \right) + 1
\]

\[
= 1 + 2p(\gamma - 1) + \frac{p(t_2 + t_{2p})}{2} + \frac{p - 1}{2}t_2 + pm\left[ t_p \left( 1 - \frac{1}{p} \right) + \sum t_{\bar{m}_j} \left( 1 - \frac{1}{m_j} \right) + \sum t_{\bar{p}_j} \left( 1 - \frac{1}{p_j} \right) \right]
\]

If \( \gamma \geq 1 \), then clearly \( D > 0 \).

If \( \gamma = 0 \), then from Theorem (3.1) and Theorem (3.2), we get

(i) when \( t_{2p} = 0, \ t_2 \neq 0 \), then at least one of \( t_p \) and \( t_{\bar{p}} \) is non zero.

If \( t_2 \geq 4 \), then clearly \( D > 0 \). If \( t_2 \geq 2 \), then \( t_p + t_{\bar{p}} \geq 2 \) and so also \( D > 0 \).

(ii) when \( t_{2p} = 2 \), then either \( t_2 \geq 2 \) or \( t_p + t_{\bar{p}} \geq 1 \) and in both cases \( D > 0 \).

(iii) when \( t_{2p} \geq 4 \), then clearly \( D > 0 \).

Therefore in every possible cases we get \( D > 0 \)

or \( \alpha + pm\beta - |\lambda| > 0 \) or \( |\lambda| < \alpha + pm\beta \).

Now we see that \( g \geq 2 \) for every possible values of \( \lambda, \alpha \) and \( \beta \).

When \( \Gamma \) is a surface group, then \( \gamma \geq 2 \) and then \( \lambda = 1, \ \alpha \geq 4p - 2p = 2p, \ \beta = 0. \)
Here \( g = \lambda + m\alpha + pm\beta \geq 2pm > 2 \).

When \( \Gamma \) is not surface group, then the following two cases arise.

**Case I :** \( y = 0 \) and **Case II :** \( y \neq 0 \).

**Case I :** \( y = 0 \).

Recall that \( y = 0 \) is possible only when (i) \( t_{2p} \neq 0 \) or (ii) \( t_2 \neq 0 \) with \( t_p \) or \( t_{p_i} \) is non zero.

We first consider \( t_{2p} \neq 0 \). Here we have the following possibilities.

(a) \( t_p + \sum t_{m_i} + \sum t_{p_i} = 0 \). By Theorem (3.1) and Theorem (3.2) we get \( t_2 + t_{2p} \geq 4 \) and \( t_{2p} \geq 2 \). Here we have \( \lambda = 1, \ \alpha = -2p + \frac{4p + 2(p-1)}{2} = p - 1, \) \( \beta = 0 \) and \( g = \lambda + m\alpha + pm\beta \geq 1 + m(p-1) + pm > 2 \).

For example, for \( G = C_3 \times D_5 \), if \( t_2 + t_{2p} = 2k \), \( k = 2, 3, \ldots \) and \( t_{2p} = 2u \), \( u = 1, 2, 3, \ldots \); we get \( \lambda = 1, \ \alpha = -2.3 + \frac{3.2k + (3-1)2u}{2} = (3k - 6) + 2u, \) \( \beta = 0 \) and \( g = 1 + 5 [(3k - 6) + 2u] + 15.0 > 2 \).

(b) \( t_2 = 0 \) and \( t_p + \sum t_{m_i} + \sum t_{p_i} = 0 \). By Theorem 3.1 and 3.2, we get \( t_{2p} \geq 4 \). Let \( t_{2p} = 2u \), \( u = 2, 3, 4, \ldots \). Then

\[
\lambda = 1, \ \alpha = -2p + \frac{2p-1}{2}2u = (2p-1)u - 2p, \ \beta = 0 \text{ and hence } g = 1 + m [(2p - 1) u - 2p + pm \cdot 0 = 1 + m[2p(u - 1) - u] > 2, \text{ for all possible values of } p \text{ and } m. 
\]

For example, for \( G = C_3 \times D_5 \), we get

\[
g = 1 + 5 [2.3(u - 1) - u] = 1 + 5 [5u - 6] > 2, \text{ for } u = 2, 3, 4, \ldots .
\]

(c) \( t_2 = 0 \) and \( t_p + \sum t_{m_i} + \sum t_{p_i} \neq 0 \). By Theorem 3.1 and Theorem 3.2 we get \( t_{2p} \geq 4 \). Let \( t_{2p} = 2u \), \( u = 2, 3, 4, \ldots \) and let \( t_p + \sum t_{m_i} + \sum t_{p_i} = v \). Then

\[
\lambda = 1 - pm \left[ \frac{t_p}{p} + \sum \frac{t_{m_i}}{m_i} + \sum \frac{t_{p_i}}{p_i} \right]
\]
\[
\alpha = -2p + \frac{p^2u + (p - 1)2u}{2} = -4p + pu + (p - 1)u = (2p - 1)u - 4p
\]

\[\beta = v\]

Obviously \(\lambda + pm\beta > 1\) and \(\alpha > 1\) for all possible values of \(p, u\) and \(v\). Hence \(g > 2\).

As above, for the group \(G = C_3 \times D_5\), we get

\[\overline{m}_j = 5, \ p_j = 15\] and hence \(\lambda = 1 - 15\left(\frac{t_3}{3} + \frac{t_5}{5} + \frac{t_{15}}{15}\right) = 1 - (5t_3 + 3t_5 + 14t_{15})\).

\[\alpha = 5u - 12\] and \(\beta = t_3 + t_5 + t_{15}\). Then

\[g = 1 - (5t_3 + 3t_5 + 14t_{15}) + 5(5u - 12) + 15( t_3 + t_5 + t_{15})\]

\[= 1 + 10t_3 + 12t_5 + 14t_{15} + 5(5u - 12) > 2,\]

for all possible values of \(u, t_3, t_5\) and \(t_{15}\).

(d) \(t_2 \neq 0, \ t_2 p \neq 0 \) and \(t_p + \sum t_{m_j} + \sum t_{p_j} \neq 0\). Here also we can show that \(g \geq 2\) for all possible values of \(p, m\) and \(t_i\)'s.

Now we consider \(t_2 p = 0, \ t_2 \neq 0\) and at least one of \(t_p\) and \(t_{p_j}\) is non-zero. The possible cases are

(a) \(t_2 \neq 0, \ t_p \neq 0, \ t_{p_j} = 0\) and \(t_{\overline{m}_j} = 0\). By Theorem 3.1 and Theorem 3.2, we get \(t_2 (\geq 4), \ t_p (\geq 2)\) are both even. Let \(t_2 = 2u, \ u = 2, 3, \ldots, t_p = 2v, \ v = 1, 2, \ldots\).

Here \(\lambda = 1 - 2uv, \ \alpha = -2p + pu, \ \beta = 2v\) and hence

\[g = (1 - 2uv) + m(-2p + pu) + pm2v = 1 + 2m(p - 1)v + pm(u - 2) > 2.\]

(b) \(t_2 \neq 0, \ t_p = 0, \ t_{p_j} \neq 0\) and \(t_{\overline{m}_j} = 0\). We have \(t_2 (\geq 2)\) and \(t_{p_j} (\geq 2)\) are both even. Let \(t_2 = 2u, \ u = 1, 2, \ldots, \sum t_{p_j} = 2v, \ v = 1, 2, \ldots\).

Hence \(\lambda = 1 - w, \ \alpha = -2p + pu, \ \beta = 2v\); where \(w = pm\sum t_{p_j} \) and hence

\[g = (1 - w) + m(-2p + pu) + 2pmv = 1 + 2pm(v - 1) + pm(u - \sum t_{p_j}) \geq 2\]

for all possible values of \(p, m, u\) and \(v\).

(c) \(t_2 \neq 0, \ t_p \neq 0, \ t_{p_j} \neq 0\) and \(t_{\overline{m}_j} = 0\). We have \(t_2 (\geq 2)\) is even.

Let \(t_2 = 2u, \ u = 1, 2, \ldots, t_p = x, \ \sum t_{p_j} = y\), where \(x, y\) are positive integers. Here
also \( g \geq 2 \) for all values of \( p, m, u, x, y \). Because we get, 
\[
\lambda \leq 1 - pm \left( \frac{1}{p} + \frac{1}{p_j} \right),
\]
\( \alpha \geq -2p + p = -p, \beta \geq 2, \) taking \( t_p \geq 1, \sum t_p \geq 1 \) and so \( g = 1 - pm \left( \frac{1}{p} + \frac{1}{p_j} \right) + m(-p) + 2pm \geq 2 \) as \( \frac{1}{p} + \frac{1}{p_j} < 1 \).

Similarly in all of the following cases

(d) \( t_\alpha \neq 0, t_p \neq 0, t_p \neq 0 \) and \( t_{p_j} \neq 0 \),

(e) \( t_2 \neq 0, t_p = 0, t_{p_j} \neq 0 \) and \( t_{p_j} \neq 0 \),

(f) \( t_2 \neq 0, t_p = 0, t_{p_j} \neq 0 \) and \( t_{p_j} \neq 0 \),

we can show that \( g \geq 2 \) for all values of \( p, m, \lambda, \alpha \) and \( \beta \).

In this way it can be proved that \( g \geq 2 \) for \( \gamma = 1 \) and finally can be concluded that \( g \geq 2 \) for all possible values of \( \lambda, \alpha, \beta \) with possible \( \gamma \) and \( t, s \) satisfying the conditions of Theorem (3.1) and Theorem (3.2). Therefore the condition (3.2.2) is necessary.

We now show that this condition is also sufficient.

Let \( g \) be an integer \( \geq 2 \) such that \( g = \lambda + m\alpha + pm\beta \), \hspace{1cm} (3.2.3)

where \( \lambda, \alpha, \beta \) are integers not simultaneously zero and

\( \lambda \leq 1, \beta \geq 0, |\lambda| < \alpha + pm\beta, p \) is any odd prime and \( m \) is an integer \( \geq 2 \).

Since g.c.d. \((1, m, pm) = 1\), the equation (3.2.3) has solution. To solve this, let \( \alpha = Au + Bv, \beta = Cu + Dv \).

Taking \( B = \frac{pm}{(m, pm)} = \frac{pm}{m} = p \) and \( D = -\frac{m}{(m, pm)} = -\frac{m}{m} = -1 \).

We have \( AD - BC = 1 \) or \( -A - pC = 1 \) or \( A + pC = -1 \).

Now (3.2.2) gives,

\( g = \lambda + m(Au + Bv) + pm(Cu + Dv) = \lambda + m(u + pC) = \lambda + mu \)
or \( \lambda = g + mu \).

As \( g \geq 2, m \geq 3 \) and \( \lambda \leq 1 \), so \( u < 0 \) and (3.2.3) gives

\( g = g + mu + m\alpha + pm\beta \) or \( \alpha + p\beta = -u = t \) (say). \hspace{1cm} (3.2.4)
Therefore \( \alpha = t \mod p \) or \( \alpha = px + t \) and \( \beta = y - t \).

Where \( x \) and \( y \) are two particular solution of (3.2.4). [29]

Putting these values of \( \alpha \) and \( \beta \) in (3.2.4), we get

\[
px + t + py - pt = t \quad \text{or} \quad px + py = pt \quad \text{or} \quad x + y = t.
\]

We have \( \beta = y - t = -x = w \) (say) and \( \alpha = px + t = t - pw \)

We assume

\[
w = t_p + \sum \frac{t_{\pi_j}}{m_j} + \sum \frac{t_{p_j}}{p_j}, \quad t = 2p\gamma - 2p + pT - \frac{K}{2} \frac{p-1}{2} t_2
\]

where \( K = t_2 + t_{2p} \) and \( T = t_2 + t_{2p} + t_p + \sum t_{\pi_j} + \sum t_{p_j} \).

Then clearly \( \beta = w \geq 0 \),

\[
\alpha = t - pw = 2p\gamma - 2p + pT - \frac{K}{2} \frac{p-1}{2} t_2 - p \left( t_p + \sum t_{\pi_j} + \sum t_{p_j} \right)
\]

We take \( \lambda = 1 - pm \left( \frac{t_p}{p} + \sum \frac{t_{\pi_j}}{m_j} + \sum \frac{t_{p_j}}{p_j} \right) \leq 1 \).

Let \( D' = \alpha + pm\beta - |\lambda| \)

\[
= 2p\gamma - 2p + pT - \frac{K}{2} \frac{p-1}{2} t_2 - p \left( t_p + \sum t_{\pi_j} + \sum t_{p_j} \right) + pm \left( t_p + \sum t_{\pi_j} + \sum t_{p_j} \right)
\]

\[
- pm \left( \frac{t_p}{p} + \sum \frac{t_{\pi_j}}{m_j} + \sum \frac{t_{p_j}}{p_j} \right) + 1
\]

\[
= 2p(\gamma - 1) + pk - \frac{K}{2} \frac{p-1}{2} t_2 + pm \left[ t_p \left( \frac{1}{p} - \frac{1}{m_j} \right) + \sum t_{\pi_j} \left( \frac{1}{m_j} - \frac{1}{p_j} \right) \right] + 1
\]

\[
= 2p(\gamma - 1) + \frac{2p-1}{2} k - \frac{p-1}{2} t_2 + pm \left[ t_p \left( \frac{1}{p} - \frac{1}{m_j} \right) + \sum t_{\pi_j} \left( \frac{1}{m_j} - \frac{1}{p_j} \right) \right] + 1
\]

It can be shown that \( D' > 0 \) for \( \gamma \geq 0 \) in all possible cases by Theorem (3.1) and Theorem (3.2) and therefore \( \alpha + pm\beta - |\lambda| > 0 \) or \( |\lambda| < \alpha + pm\beta \).

For integral values of genus and periods of Fuchsian group \( \Gamma \), not simultaneously zero and satisfying the conditions of Theorem (3.1) and Theorem (3.2), we must have solutions for \( \lambda \), \( \alpha \), \( \beta \) not simultaneously zero satisfying the conditions \( \lambda \leq 1 \), \( \beta \geq 0 \), \( |\lambda| < \alpha + pm\beta \). This gives solution for \( g \geq 2 \).
Now
\[ \lambda = g - mt \]
\[ = g - m (2p \gamma - 2p + pT - \frac{K}{2} - \frac{p-1}{2} t_2 ) \]
\[ = g - (2pm \gamma - 2pm + pmT - \frac{mK}{2} - \frac{m(p-1)}{2} t_2 ) \]
\[ = g - (2pm \gamma - 2pm + pmT - \frac{pm}{2} t_2 - \frac{m}{2} t_{2p} ) \]
\[ = g - (2pm \gamma - 2pm + pmT - \frac{pm}{2} t_2 + \frac{t_{2p}}{2p} ) \]
\[ = g - [2pm \gamma - 2pm + pmT - pm(\frac{t_2}{2} + \frac{t_{2p}}{2p} + \frac{t_p}{p} + \sum \frac{t_{\bar{m}_i}}{m_j} + \sum \frac{t_{\bar{p}_j}}{p_j})] + pm(\frac{t_p}{p} + \sum \frac{t_{\bar{m}_i}}{m_j} + \sum \frac{t_{\bar{p}_j}}{p_j}) \]
\[ = g - [g' - 1] + pm(\frac{t_p}{p} + \sum \frac{t_{\bar{m}_i}}{m_j} + \sum \frac{t_{\bar{p}_j}}{p_j}) \]

where \( g' = 1 + 2pm \gamma - 2pm + pmT - pm(\frac{t_2}{2} + \frac{t_{2p}}{2p} + \frac{t_p}{p} + \sum \frac{t_{\bar{m}_i}}{m_j} + \sum \frac{t_{\bar{p}_j}}{p_j}) \)

Therefore \( g' \) gives the expression of the genus \( g \) of the compact Riemann surface on which our group \( G \) acts as a group of automorphisms as there exists a Fuchsian group of genus \( g' \) with signature

\[ \Delta (\gamma, 2, 2, 2, 2, 2, p, 2p, 2p, 2p, \ldots, p, p, \ldots, p, m, m_j, \ldots, m_j, p, p_j, p_j, \ldots, p_j) \]

satisfying all the conditions of Theorem (3.1) and Theorem (3.2) and there exists a smooth epimorphism \( \phi : \Gamma \rightarrow G \).

This completes the proof that the relation (3.2.2) is sufficient to exist a smooth epimorphism \( \phi : \Gamma \rightarrow G \).

We now concise the above discussion as

**Theorem 3.3**

Let \( G = C_p \times D_m \) be a group where \( C_p \) is a cyclic group with odd prime \( p \) and \( D_m \) is a dihedral group with positive integral values of \( m \geq 2 \). Then there exists a compact Riemann surface of genus \( g \geq 2 \) on which \( G \) acts as a group of automorphisms if and only if the integer \( g \) can be expressed as \( g = \lambda + m \alpha + pm \beta \),
where $\lambda, \alpha, \beta$ are integers, not simultaneously zero satisfying the conditions $\lambda \leq 1$, $\beta \geq 0$, $|\lambda| < \alpha + pm \beta$. 

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