CHAPTER 2

DETERMINATION OF MINIMUM GENERA OF
COMPACT RIEMANN SURFACES ADMITTING
$C_p \times D_m$,

$p$ IS AN ODD PRIME AND

$m$ IS A POSITIVE INTEGER ($\geq 2$),
AS IT'S GROUP OF AUTOMORPHISMS

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DETERMINATION OF MINIMUM GENERA OF COMPACT RIEMANN SURFACES ADMITTING \( C_p \times D_m \), \( p \) IS AN ODD PRIME AND \( m \) IS A POSITIVE INTEGER (\( \geq 2 \)), AS IT'S GROUP OF AUTOMORPHISMS

The minimum genus problem, already introduced in chapter 1, is to determine the minimum value of the genus \( g \) of a compact Riemann surface on which a given finite group acts as its group of automorphisms. Harvey [82] obtained the minimum genus for a finite cyclic group and Wiman's result [160] follows immediately from the result of Harvey. Moreover the minimum genus problem for non-cyclic abelian groups, was solved by Macclachlan [119,124]. Glover and Sjerve [69] solved it for PSL \((2, p)\). Recently some researchers solved the minimum genus problem for a class of k-metacyclic groups [41]; a subfamily of quaternion groups [06]; a sub-class of non-abelian metacyclic groups of order \( pq \), where \( p, q \) are primes and \( q \) divides \((p-1) \) [35]; a dihedral group of order \( 2n \) [42]; two special classes of metacyclic groups \( G_{n}^{0}(n) = \langle a,b : a^n = 1 = b^2, b^{-1}ab = a^{-1} \rangle \), \( n \) is odd and \( G_{n}^{1}(n) = \langle a,b : a^n = 1, a^2 = b^2, b^{-1}ab = a^{-1} \rangle \), \( n \) is even [132]; and a class of zs-metacyclic groups of order \( 4p^2 \), \( p \) is an odd prime [58].

In our study, we investigate the minimum genera of compact Riemann surfaces admitting the group \( G = C_p \times D_m \) as an automorphism group. Here \( C_p \) is a cyclic group of order \( p \), where \( p \) is an odd prime and \( D_m \) is a dihedral group of order \( 2m \), where \( m \) is a positive integer (\( \geq 2 \)). In the first section we establish a definite presentation for \( G = C_p \times D_m \) and determine orders and the forms of the elements of \( G \). We also verify that \( G \) is a class of metacyclic group. In section 2, we determine a set of necessary and sufficient conditions on the genus and the periods of a Fuchsian

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group $\Gamma$ such that the group $G$ is isomorphic to $\Gamma_K$ where $K$ is a Fuchsian surface group. In section 3, we determine the minimum genera of compact Riemann surfaces admitting $G$ as an automorphism group.

2.1 Preliminaries

Let $G = C_p \times D_m$, where $C_p$ is a cyclic group of order an odd prime $p$ and $D_m$ is a dihedral group of order $2m$, where $m$ is any positive integer ($\geq 2$). We first obtain the generating elements and the possible presentation for $G$. For this we can take $C_p$ as a group, generated by two elements $u$ and $v$ satisfying the relation

$$u^p = v^p = uv = 1.$$ 

Similarly if $x$ and $y$ are the generators of $D_m$, then

$$x^m = y^2 = (xy)^2 = 1.$$ 

Now let $G = C_p \times D_m$ be generated by $(u,1)$, $(v,1)$, $(1,x)$ and $(1,y)$.

We claim that $(1,x)$ and $(v,y)$ will generate $G$, i.e. the above four generators of $G$ can be obtained with the help of these two generators. We have

$$(v,y)^{p-1} = (v^{p-1}, y^{p-1}) = (v^p v^{-1}, 1) = (v^{-1}, 1) = (u,1),$$

$$(v,y)^{p+1} = (v^{p+1}, y^{p+1}) = (v,1),$$

$$(v,y)^p = (v^p, y^p) = (1,1),$$

$$(1,x)(v,y)^{p-1} = (1,x)(u,1) = (u,x).$$

The order of $(1,x) = m$ and the order of $(v,y) = 2p$.

Taking $s = (1,x)$, $t = (v,y)$ we can say that $G = C_p \times D_m$ is generated by $s$, $t$ satisfying the relations

$$s^m = t^{2p} = (st)^{2p} = 1, \quad t^{-1}st = s^{-1}. \quad (2.1.1)$$

By using relation $t^{-1}st = s^{-1}$, it can be shown that any element of $G$ is of the form $t^i s^j$: $1 \leq i \leq 2p$, $1 \leq j \leq m$. Next we show that $G$ is a metacyclic group. Let $t^a s^b$, $t^c s^d \in G$; $1 \leq a, c \leq 2p$, $1 \leq b, d \leq m$. 

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Let us consider any commutator $[t^a s^b, t^c s^d]$ of $G$.

We have 
\[ [t^a s^b, t^c s^d] = (t^a s^b)^{-1} (t^c s^d)^{-1} (t^a s^b) (t^c s^d) \]

\[ = s^{-b} t^{-a} s^{-d} t^{-c} t^o s^b t^c s^d \]

\[ = s^{-b} (t^{-a} s^{-d} t^{-c}) (t^{-c} s^b t^c) s^d \]

\[ = s^{-b} s^{-d} s^{-c} s^b s^d \]

Hence 
\[ [t^a s^b, t^c s^d] = s^{-b} s^{-d} s^b s^d, \text{ when } a, c \text{ both are even,} \]

\[ = s^{-b} s^{-d} s^{-d} s^b s^d = s^{-2b}, \text{ when } a \text{ is even, } c \text{ is odd,} \]

\[ = s^{-b} s^d s^b s^d = s^{2d}, \text{ when } a \text{ is odd, } c \text{ is even,} \]

\[ = s^{-b} s^d s^{-b} s^d = s^{2(b-d)}, \text{ when } a, c \text{ both are odd.} \]

*i.e.* the commutator subgroup of $G$ is a cyclic group generated by an even power of $s$.

The quotient group $G'/G \cong Z_{2p}$ and therefore it is a cyclic group of order $2p$ [55]. So $G$ will be a $z$-metacyclic group.

We now determine the orders of the elements of $G$. We have for any positive integer $k$,

\[ (t^i s^j)^k = t^{ki} s^{kj} \]

\[ = t^{ki} s^j, \text{ if } i \text{ is even} \]

\[ = t^k, \quad \text{if } i \text{ is odd, } k \text{ is even} \]

\[ = t^{ki} s^j, \quad \text{if both } i, k \text{ are odd.} \]

When $i$ is even, suppose order of $t^i s^j$ be $n$. This gives 
\[ (t^i s^j)^n = t^m s^m = 1. \]

Which implies $ni \equiv 0 \pmod{2p}$; $i$ is even, $p \geq 3$ is prime, $2 \leq i \leq 2p-2$

and $nj \equiv 0 \pmod{m}$; $1 \leq j \leq m-1$.

The first one gives, 
\[ 2nk \equiv 0 \pmod{2p}; \quad 1 \leq k \leq p-1 \]

or $nk \equiv 0 \pmod{p}$ or $p \mid n$ (p does not divide $k$)

or $n = px$, $x$ is any positive integer.

The second congruence gives, $m \mid nj$

*i.e.* $m \mid n$, $m \mid 2n$, $m \mid 3n$, $\ldots$, $m \mid (m-1)n$.

*i.e.* $m \mid n$, if $\text{hcf}(m, j) = 1$ and $\frac{m}{d_j} \mid n$, where $d_j = \text{hcf}(m, j)$.
Therefore $n = 0 \pmod{\text{lcm}[p,m]}$ if $\text{hcf}(m,j) = 1,$

$= 0 \pmod{\text{lcm}[p, \frac{m}{d_j}]}$ if $\text{hcf}(m,j) = d_j.$

It is clear that if $i$ is odd, the order of $t's'$ must be even i.e. $(t's')^k = 1,$ only if $k$ is even.

We have $(t's')^k = 1 \Rightarrow t^k = 1$ (k is even)

or $ki = 0 \pmod{2p}$

or $2hi = 0 \pmod{2p}$, taking $k = 2h$

or $hi = 0 \pmod{p}$, which gives $h = 1$ when $i = p$ and $h = p$ when $i \neq p$.

That is $k = 2$ for $i = p$ and $k = 2p$ for $i \neq p$.

Therefore for odd values of $i$, order of $t's'$ = 2, if $i = p,$

= $2p$, if $i \neq p$.

We can conclude that the order of any element $t's'$ of $G$ is

= 2, for $i = p$, $1 \leq j \leq m$.

= $2p$, for any odd value of $i (\neq p)$, $1 \leq j \leq m$.

= $p$, for $j = m$ and any even value of $i$.

= $\frac{m}{d_j}$, $d_j = \text{hcf}(m,j)$, $1 \leq j \leq m - 1$, when $i = 2p$

= $p_j$, where $p_j = \text{lcm}[p, \frac{m}{d_j}]$, when $i$ is any even integer $< 2p$.

Note that, when $\text{hcf}(p,m) = 1$, then $p_j = pm_j$.

We now determine the conditions for the existence of a smooth epimorphism $\phi$, from a Fuchsian group $\Gamma$ onto $G$.

### 2.2 Existence of a smooth epimorphism

Let $\Gamma$ be a Fuchsian group with presentation (1.3.5) and $G = C_p \times D_m$ be the group with presentation (2.1.1). Let $\phi : \Gamma \rightarrow G$ be a smooth epimorphism. We note that an epimorphism $\phi : \Gamma \rightarrow G$ will be smooth if and only if $\phi$ preserves the
periods of $\Gamma$ \[112].

The following two cases for odd and even integral values of $m$ are considered first.

Case A: $m$ is an odd positive integer ($\geq 3$) and $(p,m) = 1$.

Case B: $m$ is an even positive integer ($\geq 2$) and $(p,m) = 1$.

First we consider Case A: $m$ is an odd positive integer ($\geq 3$) with $(p,m) = 1$.

When $\Gamma$ is a surface group, then $\gamma \geq 2$ in view of (1.3.6). Since $\phi$ preserves the periods of $\Gamma$, and $G$ contains the elements of orders $2, 2p, p, \overline{m}_j, p_j$; so when $\Gamma$ is not a surface group, the possible periods of $\Gamma$ will take the values from \{2, 2p, p, \overline{m}_j, p_j\}, $1 \leq j \leq m - 1$ where $\overline{m}_j$ and $p_j$ are defined as in Section 2.1.

Suppose $\Gamma$ has the signature

$$\Gamma = \langle \gamma, 2, 2, 2, \ldots, 2p, 2, p, 2p, \ldots, p, \ldots, \overline{m}_1, \overline{m}_2, \ldots, p, p, \ldots, \rangle$$

where $t_2, t_3, \ldots, t_p, \ldots, t_2p$, denote the number of occurrences of periods $2, 2p, p, m_j, p_j$ respectively. By (1.4.2) we get

$$g = 1 + pm \left[ 2(\gamma - 1) + \frac{t_2}{2} + t_2p\left(1 - \frac{1}{2p}\right) + t_p\left(1 - \frac{1}{p}\right) + \sum t_{m_j}\left(1 - \frac{1}{m_j}\right) + \sum t_{p_j}\left(1 - \frac{1}{p_j}\right) \right]$$

$$= 1 + 2pm(\gamma - 1) + m\left[ \frac{1}{2}\left(p + (2p - 1)\gamma_{2p}\right) \right] + m(p - 1)\gamma_p + \left\{ \sum_{m_{m_j}}(m_{m_j} - 1)\gamma_{m_j} + \sum_{p_{p_j}}(p_{p_j} - 1)\gamma_{p_j} \right\}. \quad (2.2.1)$$

g being an integer, the expression on the right hand side of (2.2.1) will be an integer if $t_2$ and $t_{2p}$ are of same parity and if one of them is zero, then the other must be even.

Let $x_a, y_b, z_c, u_d$ and $v_e$ be the finite order generators of $\Gamma$ of order $2, 2p, p, \overline{m}_j, p_j$ respectively. As $\phi$ preserves the periods of $\Gamma$, we must have the followings
\[ \phi(x_a) = t^p s^{j_a}, \quad 1 \leq j_a \leq m \text{ and } a = 1, 2, \ldots, t_2. \]

\[ \phi(y_b) = t^{g_b} s^{l_b}, \quad 1 \leq g_b \leq m, \text{ } g_b \neq p \text{ is an odd positive integer } < 2p \]

\[ \text{and } b = 1, 2, \ldots, t_{2p}. \]

\[ \phi(z_c) = t^{2k_c}, \quad k_c \text{ is a positive integer } < p \text{ and } c = 1, 2, \ldots, t_p. \] (2.2.2)

\[ \phi(u_d) = s^{q_d}, \quad 1 \leq q_d \leq m-1 \text{ and } d = 1, 2, \ldots, t_{2p}. \]

\[ \phi(v_e) = t^{h_e}, \quad 1 \leq h_e \leq m-1, \text{ } l_e \text{ is an even positive integer } < 2p \]

\[ \text{and } e = 1, 2, \ldots, t_{2p}. \]

If \( \alpha_i, \beta_i, \quad i = 1, 2, \ldots, \gamma \) are the infinite order generators of \( \Gamma \), then for \( \phi \) to be a homomorphism, we must get

\[ \prod_{a=1}^{t_2} \phi(x_a) \prod_{b=1}^{t_{2p}} \phi(y_b) \prod_{c=1}^{t_p} \phi(z_c) \prod_{d=1}^{t_{2p}} \phi(u_d) \prod_{e=1}^{t_p} \phi(v_e) = 1. \] (2.2.3)

It is observed in section 1, that \( \phi[\alpha_i, \beta_i] = s^{\delta_i} \), where \( \delta_i = m \) or is an even integer, \( 2 \leq \delta_i \leq m \); and \( [\alpha_i, \beta_i] \) denotes the commutator of \( \alpha_i, \beta_i \). If none of the \( t_i \)'s is zero, then (2.2.2) and (2.2.3) gives

\[ \prod_{a=1}^{t_2} t^p s^{j_a} \prod_{b=1}^{t_{2p}} t^{g_b} s^{l_b} \prod_{c=1}^{t_p} t^{2k_c} \prod_{d=1}^{t_{2p}} s^{q_d} \prod_{e=1}^{t_p} t^{h_e} \prod_{i=1}^{\gamma} s^{h_i} = 1 \] (2.2.4)

where \( 1 \leq q_d, h_e \leq m-1, \quad 1 \leq j_a, f_b \leq m \).

When both \( t_2, t_{2p} \) are even, then

\[ \sum_{b=1}^{t_{2p}} g_b + \sum_{c=1}^{t_p} 2k_c + \sum_{e=1}^{t_p} l_e \equiv 0 \pmod{2p} \quad \text{and} \]

\[ \sum_{a=1}^{t_2} (-1)^a j_a + \sum_{b=1}^{t_{2p}} (-1)^{b+1} f_b + \sum_{d=1}^{t_{2p}} q_d + \sum_{e=1}^{t_p} h_e + \sum_{i=1}^{\gamma} \delta_i \equiv 0 \pmod{m} \] (2.2.5)

When \( t_2, t_{2p} \) are odd, then

\[ p + \sum_{b=1}^{t_{2p}} g_b + \sum_{c=1}^{t_p} 2k_c + \sum_{e=1}^{t_p} l_e \equiv 0 \pmod{2p} \quad \text{and} \]

\[ \sum_{a=1}^{t_2} (-1)^a j_a + \sum_{b=1}^{t_{2p}} (-1)^{b+1} f_b + \sum_{d=1}^{t_{2p}} q_d + \sum_{e=1}^{t_p} h_e + \sum_{i=1}^{\gamma} \delta_i \equiv 0 \pmod{m} \] (2.2.6)
For suitable values of $g_b, k_c, l_e, j_a$ etc., the above two congruences have solutions and $\phi: \Gamma \to G$ is a smooth epimorphism with $\gamma \geq 0$.

Here the following cases may arise

Case I: Any one of the $t_i$'s is zero,

Case II: Any two of the $t_i$'s are zero,

Case III: Any three of the $t_i$'s are zero,

Case IV: Any four of the $t_i$'s are zero.

We now examine the above possible cases one by one:

Case I: When any one of the $t_i$'s is zero.

In this case, (2.2.2) and (2.2.3) define a smooth epimorphism $\phi: \Gamma \to G$, for $\gamma \geq 0$.

Case II: When any two of the $t_i$'s are zero.

Here the possible sub-cases are

(a) $t_2 = t_p = 0$,  
(b) $t_2 = t_{\overline{a}_i} = 0$,  
(c) $t_2 = t_{\overline{p}_j} = 0$,  
(d) $t_2 = t_{\overline{a}_i} = t_{\overline{p}_j} = 0$

(e) $t_p = t_{\overline{a}_i} = 0$,  
(f) $t_2 = t_{\overline{a}_i} = t_{\overline{p}_j} = 0$,  
(g) $t_2 = t_{\overline{p}_j} = 0$,  
(h) $t_2 = t_{\overline{a}_i} = t_{\overline{p}_j} = 0$

(i) $t_p = t_{\overline{a}_i} = 0$ and (j) $t_{\overline{a}_i} = t_{\overline{p}_j} = 0$.

We now discuss the sub-cases.

(a) $t_2 = t_p = 0$. Obviously $t_{2p}$ is even. From (2.2.4) we get

$$\sum_{b=1}^{t_b} g_b + \sum_{e=1}^{t_e} l_e \equiv 0 \pmod{2p}$$

$$\sum_{b=1}^{t_b} (-1)^b f_b + \sum_{q=1}^{t_q} q_a + \sum_{e=1}^{t_e} h_x + \sum_{i=1}^{t_i} \delta_i \equiv 0 \pmod{m}$$

For suitable values of $g_b, k_c, l_e, j_a$ etc., the above congruences have solutions and $\phi: \Gamma \to G$ will be a smooth epimorphism for $\gamma \geq 0$ and $t_2 + t_{2p}$ is even. Similarly, it can be shown that $\phi: \Gamma \to G$ will be a smooth epimorphism in cases from (b) through (e) for $\gamma \geq 0$ and $t_2 + t_{2p}$ is even. In case (f), $\phi: \Gamma \to G$ will be a smooth
epimorphism only if \( \gamma \geq 1 \). Because when \( t_2 = t_{2p} = 0 \), we see from (2.2.2), that no finite order element of \( \Gamma \) is mapped to \( t \in G \), and as a result \( \phi(\Gamma) \subset G \). So \( \phi \) is not onto. But if \( \gamma \geq 1 \) we can exhibit a smooth epimorphism by letting \( \phi(\alpha_1) = \phi(\beta_1) = t \) in (2.2.2). In case (g), (2.2.4) gives

\[
\sum_{e=1}^{t_p} l_e \equiv 0 \pmod{2p} \quad \text{and} \quad \sum_{a=1}^{t_p} (-1)^a j_a + \sum_{d=1}^{k_p} q_d + \sum_{e=1}^{t_p} h_e + \sum_{i=1}^{r} \delta_i \equiv 0 \pmod{m}
\]

Obviously the second congruence has solutions, but the first one has a solution only if \( t_p \geq 2 \) as \( l_e < 2p \). So \( \phi : \Gamma \rightarrow G \) is a homomorphism if \( t_p \geq 2 \) and from (2.2.2) it is clear that \( \phi \) will be a smooth epimorphism for \( \gamma \geq 0 \). Similarly in cases (h) and (i), \( \phi : \Gamma \rightarrow G \) will be a smooth epimorphism, if \( t_p \geq 2 \), \( \gamma \geq 0 \) and \( t_{2p} \geq 2 \), \( \gamma \geq 0 \) respectively. Next we consider case (j) \( t_\infty = t_{2p} = 0 \). Here also \( t_2 + t_{2p} \) is even. When \( t_2, t_{2p} \) are even, from (2.2.5) we get the following congruences.

\[
\sum_{b=1}^{t_2} g_b + \sum_{c=1}^{t_2} 2k_c \equiv 0 \pmod{2p} \quad \text{and} \quad \sum_{a=1}^{t_2} (-1)^a j_a + \sum_{b=1}^{t_2} (-1)^{2+b} f_b + \sum_{i=1}^{r} \delta_i \equiv 0 \pmod{m}
\]

and when \( t_2, t_{2p} \) are odd, from (2.2.6) we get

\[
p + \sum_{b=1}^{t_2} g_b + \sum_{c=1}^{t_2} 2k_c \equiv 0 \pmod{2p} \quad \text{and} \quad \sum_{a=1}^{t_2} (-1)^a j_a + \sum_{b=1}^{t_2} (-1)^{2+b} f_b + \sum_{i=1}^{r} \delta_i \equiv 0 \pmod{m}
\]

We claim that when \( \gamma = 0 \), \( \phi \) will be a smooth epimorphism only if \( t_2 + t_{2p} \geq 4 \).

Here \( t_2 + t_{2p} \neq 1 \) or 3, as \( t_2 + t_{2p} \) is always even. When \( t_2 + t_{2p} = 2 \), the possibilities are

\( t_2 = t_{2p} = 1; \quad t_2 = 2, \quad t_2 = 0 \) and \( t_2 = 0, \quad t_{2p} = 2 \).

In all these three cases no finite order element of \( \Gamma \) is mapped to \( s \) and \( t \) simultaneously, satisfying the above two congruences and as a result \( \phi(\Gamma) \subset G \). So \( \phi \) is not onto. Hence if \( \gamma = 0 \), \( \phi : \Gamma \rightarrow G \) will be a smooth epimorphism only if \( t_2 + t_{2p} \geq 4 \).
Case III: When any three of the $t_i$'s are zero.

Here the possible sub-cases are

(a) $t_2 = t_p = t_{ar{m}_i} = 0$
(b) $t_2 = t_p = t_{ar{p}_i} = 0$
(c) $t_2 = t_{ar{m}_i} = t_{ar{p}_i} = 0$
(d) $t_{2p} = t_p = t_{ar{m}_i} = 0$
(e) $t_2 = t_{ar{m}_i} = t_{ar{p}_i} = 0$
(f) $t_2 = t_{2p} = t_p = 0$
(g) $t_2 = t_{2p} = t_{p_i} = 0$
(h) $t_2 = t_{2p} = t_{ar{p}_i} = 0$
(i) $t_{2p} = t_p = t_{ar{p}_i} = 0$
(j) $t_{2p} = t_{ar{m}_i} = t_{p_i} = 0$

and (j) $t_{2p} = t_{ar{m}_i} = t_{p_i} = 0$.

We now discuss the sub-cases.

(a) $t_2 = t_p = t_{ar{m}_i} = 0$. Here $t_{2p}$ is even. From (2.2.4) we get

$$\sum_{b=1}^{l_y} (-1)^b f_b + \sum_{e=1}^{l_y} h_e + \sum_{r=1}^{r_y} \delta_r = 0 \pmod{m} \quad \text{and} \quad \sum_{b=1}^{l_y} g_b + \sum_{e=1}^{l_y} l_e = 0 \pmod{2p}.$$ 

Both the congruences have solutions for suitable values of $g_b$ etc., and in view of (2.2.2), $\phi : \Gamma \to G$ will be a smooth epimorphism for $\gamma \geq 0$. Similarly we have a smooth epimorphism $\phi : \Gamma \to G$ for $\gamma \geq 0$ and $t_{2p}$ is even in cases (b) and (c); for $\gamma \geq 0$, $t_2$ is even, $t_{p_i} \geq 2$ in case (d). In case (e), $\phi : \Gamma \to G$ will be a smooth epimorphism for $\gamma \geq 0$ and $t_2 + t_{2p}$ is even. Here $t_2 + t_{2p} \geq 4$ if $\gamma = 0$, as in sub-case II (j). In case (f), we get from (2.2.4), $\sum_{e=1}^{l_y} t_e = 0 \pmod{m}$. Which implies $t_{p_i} \geq 2$ as $l_e < 2p$. If $\gamma = 0$, $\phi(\Gamma) \subset G$. So $\phi : \Gamma \to G$ will be a smooth epimorphism only if $\gamma \geq 1$. In a similar way $\phi : \Gamma \to G$ will be a smooth epimorphism for $\gamma \geq 1$, $t_p \geq 2$ in the case (g); for $\gamma \geq 1$ in the case (h); for $\gamma \geq 1$, $t_2 \geq 2$ in case (i).

In case (j) $t_{2p} = t_{\bar{m}_i} = t_{p_i} = 0$, $t_2$ is even, $\phi : \Gamma \to G$ will be a smooth homomorphism if $t_p \geq 2$. From (2.2.2) it is clear that when $\gamma = 0$, $\phi : \Gamma \to G$ will be a smooth epimorphism if $t_2 \geq 4$ and when $\gamma \geq 1$, $t_2 \geq 2$.
Case IV: When any four of the periods are zero.

It can be shown that as in Cases II & III, \( \phi : \Gamma \to G \) will be a smooth epimorphism in

(a) \( t_2 = t_p = t_{\overline{m}_j} = t_{p_j} = 0, \ t_{2p} \) is even \( \geq 2 \) and \( \gamma \geq 0 \).

(b) \( t_2 = t_{2p} = t_p = t_{p_j} = 0, \ \gamma \geq 1 \).

(c) \( t_2 = t_{2p} = t_p = t_{\overline{m}_j} = 0, \ t_{p_j} \geq 2 \) and \( \gamma \geq 1 \).

(d) \( t_{2p} = t_p = t_{\overline{m}_j} = t_{p_j} = 0, \ t_2 \geq 2 \) and \( \gamma \geq 1 \).

(e) \( t_2 = t_{2p} = t_{\overline{m}_j} = t_{p_j} = 0, \ t_p \geq 2 \) and \( \gamma \geq 2 \).

Case B: \( m (\geq 2) \) is an even positive integer with \( (p,m) = 1 \).

The following conditions for the periods and genus of \( \Gamma \) are obtained.

If \( \Gamma \) is a surface group, then \( \gamma \geq 2 \) by (1.3.6).

The elements of \( G \) are of orders \( 2,2p,p,\overline{m}_j,p_j \) as stated in Section 1. Since \( \phi : \Gamma \to G \) preserves the periods of \( \Gamma \), so the possible periods of \( \Gamma \) will take the values from \( \{2,2p,p,\overline{m}_j,p_j\} \), \( 1 \leq j \leq m-1 \) where \( \overline{m}_j \) and \( p_j \) are defined as in Section 1.

Suppose \( \Gamma \) has the signature

\[
\Gamma = \left\{ y, 2,2,2, \ldots, 2p,2p,\ldots, p, p, \ldots, \overline{m}_j, \overline{m}_j, \ldots, p_j, p_j, \ldots \right\}.
\]

Let \( t_2, t_{2p}, t_p, t_{\overline{m}_j}, t_{p_j} \) denote the number of occurrences of periods \( 2,2p,p,\overline{m}_j,p_j \) respectively and let \( x_o, y_o, z_c, u_d \) and \( v_e \) be the finite order generators of \( \Gamma \) of orders \( 2,2p,p,\overline{m}_j,p_j \) respectively. As \( \phi \) preserves the periods of \( \Gamma \), from Section 1, we have the following.
\[ \phi(x_a) = t^{n_a} s^{j_a}, \quad n_a = p, \ 1 \leq j_a \leq m \] or \[ n_a = 2p, \ j_a = m/2. \]

\[ \phi(y_b) = t^{g_b} s^{j_b}, \quad g_b (\neq p) \] is an odd positive integer \( < 2p, \ 1 \leq f_b \leq m, \)

or \( g_b (\neq p) \) is any positive integer \( < 2p, \ m/hcf(m, f_b)=2. \)

\[ \phi(z_c) = t^{2k_c}, \quad k_c \] is a positive integer \( < p. \)

\[ \phi(u_d) = s^{q_d}, \quad 1 \leq q_d \leq m-1. \]

\[ \phi(v_e) = t^{l_e} s^{h_e}, \quad l_e \] is an even positive integer \( < 2p, \ 1 \leq h_e \leq m-1. \]

If \( \alpha_i, \beta_i, \ i = 1, 2, \ldots, \gamma \) are the infinite order generators of \( \Gamma, \) then

\[ \phi[\alpha_i, \beta_i] = s^{\delta_i}, \ 2 \leq \delta_i \leq m \] and from (1.3.5) we have

\[ \prod_{a=1}^{l_a} \phi(x_a) \prod_{b=1}^{l_b} \phi(y_b) \prod_{c=1}^{l_c} \phi(z_c) \prod_{d=1}^{l_d} \phi(u_d) \prod_{e=1}^{l_e} \phi(v_e) \prod_{i=1}^{\gamma} \phi[\alpha_i, \beta_i] = 1. \] (2.2.8)

If none of the \( t_i, y_i \)'s is zero, then (1.4.2) and (2.2.7) give

\[ \prod_{a=1}^{l_a} t^{n_a} s^{j_a} \prod_{b=1}^{l_b} t^{g_b} s^{j_b} \prod_{c=1}^{l_c} t^{2k_c} \prod_{d=1}^{l_d} s^{q_d} \prod_{e=1}^{l_e} t^{l_e} s^{h_e} \prod_{i=1}^{\gamma} s^{\delta_i} = 1. \] (2.2.9)

where \( q_d, h_e \) etc. are as stated in (2.2.7). This implies

\[ \sum_{a=1}^{l_a} n_a + \sum_{b=1}^{l_b} g_b + \sum_{c=1}^{l_c} 2k_c + \sum_{d=1}^{l_d} l_d = 0 \pmod{2p} \] and one of the following

\[ \sum_{a=1}^{l_a} (-1)^{j_a} + \sum_{b=1}^{l_b} (-1)^{g_b} f_b + \sum_{d=1}^{l_d} q_d + \sum_{e=1}^{l_e} l_e + \sum_{i=1}^{\gamma} \delta_i = 0 \pmod{m}, \] if \( g_b \) is odd, \( n_a = p \)

\[ \sum_{a=1}^{l_a} (-1)^{j_a} + \sum_{b=1}^{l_b} f_b + \sum_{d=1}^{l_d} q_d + \sum_{e=1}^{l_e} h_e + \sum_{i=1}^{\gamma} \delta_i = 0 \pmod{m}, \] if \( g_b \) is even, \( n_a = p \) (2.2.10)

\[ \sum_{a=1}^{l_a} j_a + \sum_{b=1}^{l_b} (-1)^{g_b} f_b + \sum_{d=1}^{l_d} q_d + \sum_{e=1}^{l_e} h_e + \sum_{i=1}^{\gamma} \delta_i = 0 \pmod{m}, \] if \( g_b \) is odd, \( n_a = 2p \)

\[ \sum_{a=1}^{l_a} j_a + \sum_{b=1}^{l_b} f_b + \sum_{d=1}^{l_d} q_d + \sum_{e=1}^{l_e} h_e + \sum_{i=1}^{\gamma} \delta_i = 0 \pmod{m}, \] if \( g_b \) is even, \( n_a = 2p \)

For suitable values of \( g_b, k_c, l_e, f_a \) etc. the above five congruences have solutions and thus \( \phi: \Gamma \rightarrow G \) is a smooth homomorphism. From (2.2.7) it is clear that \( \phi \) is also an epimorphism with \( \gamma \geq 0. \)
We now examine the following cases:

Case I: Any one of the $t_i$'s is equal to zero.

Case II : Any two of the $t_i$'s are equal to zero.

Case III : Any three of the $t_i$'s are equal to zero.

Case IV: Any four of the $t_i$'s are equal to zero.

**Case I: Any one of the $t_i$'s is equal to zero.**

Clearly (2.2.7) and (2.2.10) define a smooth epimorphism $\phi: \Gamma \to G$ for $\gamma \geq 0$. From the first congruence relation of (2.2.10), when $t_2 = 0$ we get $\sum g_b = 0(\text{mod} 2p)$, which implies $t_{2p} \geq 2$ and is even, if $g_b$ is odd, for all $b$. And similarly $t_2 \geq 2$ and is even when $t_{2p} = 0$ and $n_a = p$, for all $a$.

**Case II: Any two of the $t_i$'s are equal to zero.**

The possible sub-cases are

(a) $t_2 = t_p = 0$. (b) $t_2 = t_{\bar{m}_i} = 0$. (c) $t_2 = t_{\bar{m}_j} = 0$. (d) $t_p = t_{\bar{m}_i} = 0$.

(e) $t_{2p} = t_{\bar{m}_j} = 0$. (f) $t_2 = t_{2p} = 0$. (g) $t_{2p} = t_p = 0$. (h) $t_{2p} = t_{p_j} = 0$.

(i) $t_p = t_{p_j} = 0$ and (j) $t_{\bar{m}_i} = t_{\bar{m}_j} = 0$.

We first consider sub-case (a)

From (2.2.10) we get

$$\sum_{b=1}^{l_b} g_b + \sum_{e=1}^{l_e} l_e = 0(\text{mod} 2p)$$

and one of the following two

$$\sum_{b=1}^{l_b} (-1)^b f_b + \sum_{d=1}^{l_d} q_d + \sum_{e=1}^{l_e} l_e + \sum_{i=1}^{r} \delta_i = 0(\text{mod} m)$$

if $g_b$ is odd

$$\sum_{b=1}^{l_b} f_b + \sum_{d=1}^{l_d} q_d + \sum_{e=1}^{l_e} l_e + \sum_{i=1}^{r} \delta_i = 0(\text{mod} m)$$

if $g_b$ is even.
Here $g_b = p$ or $2p$ and $l_e$ is even positive integer ($< 2p$) by (2.2.7). So we get separately $\sum_{b=1}^{t_{2p}} g_b \equiv 0 \pmod{2p}$ and $\sum_{e=1}^{l_{2p}} l_e \equiv 0 \pmod{2p}$, which implies $t_{2p}$ is even when $g_b = p$ for all $b$ and $l_e$ is always even. We can choose $g_b, l_e, f_a, h_e, q_d$ etc. satisfying the above congruences. But when $g_b$ is even for all $b$, then no finite order element of $\Gamma$ is mapped to the generating element $t$ and so $\phi : \Gamma \to G$ will be a smooth epimorphism only if $\gamma \geq 1$.

With the similar conditions $\phi : \Gamma \to G$ will be a smooth epimorphism in sub-cases (b), (c) and (d). In sub-case (e) $t_{2p} = t_{p'} = 0$, (2.2.10) gives

$$\sum_{a=1}^{t_p} n_a + \sum_{e=1}^{l_e} 2k_e + \sum_{e=1}^{l_{p'}} l_e \equiv 0 \pmod{2p}$$

and one of the following two

$$\sum_{a=1}^{t_p} (-1)^a j_a + \sum_{e=1}^{l_e} h_e + \sum_{i=1}^{\gamma} \delta_i \equiv 0 \pmod{m}$$

or

$$\sum_{a=1}^{t_p} j_a + \sum_{e=1}^{l_e} h_e + \sum_{i=1}^{\gamma} \delta_i \equiv 0 \pmod{m}.$$

Here also $t_2$ is even when $n_a = p$, for all $a$. The above congruences have solution. But $l_e$ is an even positive integer, therefore if $n_a = 2p$, for all $a$, no finite order element of $\Gamma$ is mapped to the generating element $t$ and so $\phi : \Gamma \to G$ will be a smooth epimorphism only if $\gamma \geq 1$. In sub-case (f) also, $\phi : \Gamma \to G$ will be a smooth epimorphism only if $\gamma \geq 1$. Because from (2.2.7), it is clear that no element of order $t_p$, $t_{p'}$ and $t_{p''}$ of $\Gamma$ or any finite product of these elements is mapped to $t \in G$.

Therefore no finite order elements of $\Gamma$ is mapped to $t \in G$ and as a result $\phi(\Gamma) \subset G$, that is $\phi$ is not onto. In sub-case (g), (2.2.10) gives

$$\sum_{a=1}^{t_p} n_a + \sum_{e=1}^{l_e} l_e \equiv 0 \pmod{2p}.$$

Which implies $\sum_{a=1}^{t_p} n_a \equiv 0 \pmod{2p}$ and $\sum_{e=1}^{l_e} l_e \equiv 0 \pmod{2p}$. It gives $t_2$ is even when
\( n_a = p, \) for all \( a \) and \( t_a \geq 2. \) Moreover, if \( n_a = 2p, \) for all \( a, \) then \( \phi(\Gamma) \subset G \) for \( \gamma = 0, \) since no finite order element of \( \Gamma \) is mapped to \( t \in G. \) So in this case, \( \phi : \Gamma \rightarrow G \) will be a smooth epimorphism only if \( \gamma \geq 1. \) Similar conditions will be held for sub-case (h). In sub-case (i), (2.2.10) gives

\[
\sum_{a=1}^{t_a} n_a + \sum_{b=1}^{t_b} g_b = 0 \pmod{2p} \quad \text{and one of the following two}
\]

\[
\sum_{a=1}^{t_a} (-1)^a j_a + \sum_{b=1}^{t_b} (-1)^{t_a+t_b} f_b + \sum_{d=1}^{t_d} q_d + \sum_{i=1}^{\gamma} \delta_i = 0 \pmod{m} \quad \text{when } n_a = p
\]

\[
\sum_{a=1}^{t_a} j_a + \sum_{b=1}^{t_b} (-1)^{t_a+t_b} f_b + \sum_{d=1}^{t_d} q_d + \sum_{i=1}^{\gamma} \delta_i = 0 \pmod{m} \quad \text{when } n_a = 2p
\]

The first congruence has solutions if \( t_{2p} \geq 2 \) and is even if \( g_b \) is odd, for all \( b. \) Here \( \phi \) will be a smooth epimorphism for \( \gamma \geq 0. \) In sub-case (j), we have

\[
\sum_{a=1}^{t_a} n_a + \sum_{b=1}^{t_b} g_b + \sum_{c=1}^{t_c} 2k_c = 0 \pmod{2p}
\]

\[
\sum_{a=1}^{t_a} (-1)^a j_a + \sum_{b=1}^{t_b} (-1)^{t_a+t_b} f_b + \sum_{i=1}^{\gamma} \delta_i = 0 \pmod{m}
\]

The above congruences have solutions. But when \( \gamma = 0, \) \( \phi \) will be a smooth epimorphism only if \( t_{2} + t_{2p} \geq 4, \) because otherwise no finite order element of \( \Gamma \) is mapped to \( s \) and \( t \) simultaneously satisfying this two congruences.

**Case III : Any three of the \( t_i \)'s are equal to zero.**

The possible sub-cases are

(a) \( t_2 = t_p = t_{\bar{m}} = 0, \) (b) \( t_2 = t_p = t_{\bar{p}} = 0, \) (c) \( t_2 = t_{\bar{m}} = t_{\bar{p}} = 0, \)

(d) \( t_{2p} = t_p = t_{\bar{m}} = 0, \) (e) \( t_p = t_{\bar{m}} = t_{\bar{p}} = 0, \) (f) \( t_2 = t_{2p} = t_p = 0, \)

(g) \( t_2 = t_{2p} = t_{\bar{p}} = 0, \) (h) \( t_2 = t_{2p} = t_{\bar{m}} = 0, \) (i) \( t_{2p} = t_p = t_{\bar{p}} = 0, \)

and (j) \( t_{2p} = t_{\bar{m}} = t_{\bar{p}} = 0. \)

We first consider sub-case (a) \( t_2 = t_p = t_{\bar{m}} = 0. \) Here (2.2.10) gives
Both have solutions. But the first relation gives \( \phi: \Gamma \to G \) is a smooth homomorphism if \( t_{2p} \geq 2 \) and \( t_{2p} \) is even if \( g_b \) is odd, for all \( b \), because \( l_e \) is even. Here \( \phi \) will be a smooth epimorphism for \( \gamma \geq 0 \). With the similar condition, \( \phi \) will be a smooth epimorphism for \( \gamma \geq 1 \) in sub-cases

(b) \( t_2 = t_p = t_{p_1} = 0, \quad t_{2p} \geq 2 \) and is even if \( g_b \) is odd, for all \( b \).

(c) \( t_2 = t_{\overline{a}} = t_{p_1} = 0, \quad t_{2p} \geq 2 \) and is even if \( g_b \) is odd, for all \( b \).

(d) \( t_{2p} = t_p = t_{\overline{a}} = 0, \quad t_2 \geq 2 \) and is even if \( n_a = p \), for all \( a \) and \( t_{p_1} \geq 2 \).

(e) \( t_p = t_{\overline{a}} = t_{p_1} = 0, \quad t_2 \geq 2 \) and is even if \( n_a = p \), for all \( a \) and \( t_{2p} \geq 2 \).

In sub-case (f) \( t_2 = t_{2p} = t_p = 0 \), we get

\[
\sum_{e=1}^{t_{p_1}} l_e = 0 \pmod{2p}
\]

\[
\sum_{d=1}^{t_{\overline{a}}} d_a + \sum_{e=1}^{t_{p_1}} l_e + \sum_{i=1}^{\gamma} \delta_i = 0 \pmod{m}.
\]

The first congruence gives \( t_{p_1} \geq 2 \). But \( \phi(\Gamma) \subseteq G \) for \( \gamma \geq 0 \), because the elements of orders \( t_{\overline{a}} \) and \( t_{p_1} \) of \( \Gamma \) and any finite product of them do not map to the generating element \( t \) of \( G \). Hence \( \phi \) will be a smooth epimorphism for \( \gamma \geq 1 \). Similarly \( \phi \) will be a smooth epimorphism for \( \gamma \geq 1 \) in sub-cases

(g) \( t_2 = t_{2p} = t_{p_1} = 0, \quad \text{with } t_p \geq 2, \gamma \geq 1. \)

(h) \( t_2 = t_{2p} = t_{\overline{a}} = 0, \quad \text{with } t_{p_1} \geq 2, \gamma \geq 1. \)

(i) \( t_{2p} = t_p = t_{p_1} = 0, \quad \text{with } t_2 \geq 2 \) and is even if \( n_a = p \) for all \( a \), \( \gamma \geq 1. \)

In sub-case (j) \( t_{2p} = t_{\overline{a}} = t_{p_1} = 0 \), we get

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\[ \sum_{a=1}^{n_a} n_a + \sum_{a=1}^{2k_c} 2k_c \equiv 0 \pmod{2p} \]
\[ \sum_{a=1}^{n_a} (-1)^a j_a + \sum_{i=1}^{\gamma} \delta_i \equiv 0 \pmod{m} \]

Therefore, \( \phi \) will be a smooth homomorphism if \( t_2 \geq 2 \), \( t_p \geq 2 \) and \( t_2 \) is even if \( n_a = p \), for all \( a \). From (2.2.10), it is clear that when \( \gamma = 0 \), \( \phi \) will be a smooth epimorphism for \( t_2 \geq 4 \) and when \( \gamma \geq 1 \), \( \phi \) will be a smooth epimorphism for \( t_2 \geq 2 \).

**Case IV: Any four of the \( t_i \)'s are equal to zero.**

As in above, here also we can show that, there will be a smooth epimorphism in the following sub-cases.

(a) \( t_2 = t_p = t\bar{w}_j = t\bar{w}_i = 0 \), with \( t_{2p} \geq 4 \).

(b) \( t_2 = t_{2p} = t_p = t_{\bar{w}_i} = 0 \), with \( \gamma \geq 1 \).

(c) \( t_2 = t_{2p} = t_p = t\bar{w}_j = 0 \), with \( t_{p_i} \geq 2 \) and \( \gamma \geq 1 \).

(d) \( t_{2p} = t_p = t\bar{w}_j = t_{p_i} = 0 \), with \( t_2 \geq 2 \) and \( \gamma \geq 1 \).

(e) \( t_2 = t_{2p} = t_w\bar{j}_i = t_p = 0 \), with \( t_p \geq 2 \) and \( \gamma \geq 2 \).

The conditions, discussed above are necessary for the existence of a smooth epimorphism \( \phi : \Gamma \to G \) and we now show that these conditions are also sufficient.

Suppose \( x_a, y_h, z_c, u_d \) and \( v_e \) be the elements of \( \Gamma \) of orders 2, 2p, p, \( m_j \) and \( p_j \) respectively.

Here we exhibit epimorphisms \( \phi : \Gamma \to G \) satisfying the necessary conditions and this will confirm that the above conditions are also sufficient for the existence of such an epimorphism. We examine the possible cases as follows.
Case 1: \( \sum t_i = 0 \) i.e. \( \Gamma \) is a surface group.

We may define a smooth epimorphism as
\[
\phi(\alpha_i) = \phi(\beta_i) = s, \ \phi(\alpha_2) = \phi(\beta_2) = t, \ \phi(\alpha_i) = \phi(\beta_i) = 1, \ i = 3,4,\ldots
\]

Case 2: \( \Gamma \) be a Fuchsian group with a single period.

In this case \( \gamma \geq 1 \) and the epimorphism can be defined as follows.

(i) \( t_2 \neq 0 \). Here \( \gamma \geq 2 \) and \( t_2 \) (even) \( \geq 2 \).
\[
\phi(x_a) = t^a, \ a = 1,\ldots,t_2; \ \phi(\alpha_i) = \phi(\beta_i) = t, \ \phi(\alpha_i) = \phi(\beta_i) = 1, \ i = 2,3,\ldots
\]

(ii) \( t_{2p} \neq 0 \). Here \( \gamma \geq 0 \) and \( t_{2p} \) (even) \( \geq 2 \). If \( \gamma = 0 \), then \( t_{2p} \geq 4 \).
\[
\phi(y_i) = \phi(y_i) = \ldots = t, \ \phi(y_3) = \phi(y_4) = \ldots = t^{2p-1},
\phi(\alpha_1) = \phi(\beta_1) = s, \ \phi(\alpha_i) = \phi(\beta_i) = 1, \ i = 3,4,\ldots
\]

for \( \gamma \geq 0 \), and
\[
\phi(y_1) = \phi(y_2) = t, \ \phi(y_3) = ts, \ \phi(y_4) = t^{2p-1}s,
\phi(y_5) = \phi(y_7) = \ldots = t, \ \phi(y_6) = \phi(y_8) = \ldots = t^{2p-1}.
\]

for \( \gamma = 0 \).

(iii) \( t_p \neq 0 \). Here \( \gamma \geq 2 \), \( t_p \geq 2 \).
\[
\phi(z_1) = \phi(z_2) = \ldots = t, \ \phi(z_3) = \phi(z_4) = \ldots = t^{2p-2},
\phi(\alpha_1) = \phi(\beta_1) = s, \ \phi(\alpha_2) = \phi(\beta_2) = t, \ \phi(\alpha_i) = \phi(\beta_i) = t, \ i = 3,4,\ldots
\]

(iv) \( t_{m_p} \neq 0 \). Here \( \gamma \geq 1 \).
\[
\phi(u_1) = \phi(u_2) = \ldots = s, \ \phi(u_3) = \phi(u_4) = \ldots = s^{m-2}, \ \text{if} \ t_{m_p} \ \text{is odd},
\phi(u_1) = \phi(u_2) = s, \ \phi(u_3) = \phi(u_4) = \ldots = s = \phi(u_6) = \ldots = s^{m-1}, \ \text{if} \ t_{m_p} \ \text{is even},
\phi(\alpha_1) = s, \ \phi(\beta_1) = t, \ \phi(\alpha_i) = \phi(\beta_i) = t, \ i = 2,3,4,\ldots
\]

(v) \( t_{p_j} \neq 0 \). Here \( \gamma \geq 1 \), \( t_{p_j} \geq 2 \).
\[
\phi(v_1) = \phi(v_2) = \ldots = t^{2p} s, \ \phi(v_3) = \phi(v_4) = \ldots = t^{2p-2} s^{m-1},
\phi(\alpha_1) = \phi(\beta_1) = t, \ \phi(\alpha_i) = \phi(\beta_i) = t, \ i = 2,3,4,\ldots
\]
Case 3: Let $\Gamma$ be a Fuchsian group with two distinct periods.

In this case $\gamma \geq 0$ and the epimorphism can be defined as follows.

(i) $t_2 \neq 0$, $t_{2p} \neq 0$. Here $\gamma \geq 0$, $t_{2p} \geq 2$ and $t_2 + t_{2p} \geq 4$.

\[
\phi(x_a) = t^p s, \quad a = 1, 2, \ldots, t_2,
\]

\[
\phi(y_1) = \phi(y_3) = \ldots = t, \quad \phi(y_2) = \phi(y_4) = \ldots = t^{2p-1}, \quad \text{if} \ t_{2p} \text{ is even},
\]

\[
\phi(y_1) = \phi(y_2) = t, \quad \phi(y_3) = t^{p-2} s, \phi(y_4) = \phi(y_5) = \ldots = t, \phi(y_6) = \phi(y_7) = t^{2p-1}, \quad \text{if} \ t_{2p} \text{ is odd},
\]

\[
\phi(\alpha_i) = \phi(\beta) = 1, \quad i = 1, 2, \ldots, \gamma.
\]

(ii) $t_2 \neq 0$, $t_p \neq 0$. Here $\gamma \geq 0$, $t_2 \ (\text{even}) \geq 2$ and $t_p \geq 2$.

\[
\phi(x_a) = t^p, \quad a = 1, 2, \ldots, t_2,
\]

\[
\phi(z_1) = \phi(z_3) = \ldots = t^2, \quad \phi(z_2) = \phi(z_4) = \ldots = t^{2p-2},
\]

\[
\phi(\alpha_i) = \phi(\beta) = s, \phi(\alpha_i) = \phi(\beta) = 1, \quad i = 2, 3, \ldots, \gamma.
\]

\[
\phi(x_1) = \phi(x_2) = t^p, \phi(x_3) = t^p, \quad a = 3, 4, \ldots, t_2,
\]

\[
\phi(z_1) = \phi(z_3) = \ldots = t^2, \quad \phi(z_2) = \phi(z_4) = \ldots = t^{2p-2},
\]

\[
\phi(\alpha_i) = \phi(\beta) = s, \phi(\alpha_i) = \phi(\beta) = 1, \quad i = 2, 3, \ldots, \gamma.
\]

(iii) $t_2 \neq 0$, $t_{2j} \neq 0$. Here $\gamma \geq 1$, $t_2 \ (\text{even}) \geq 2$.

\[
\phi(x_a) = t^p, \quad a = 1, 2, \ldots, t_2,
\]

\[
\phi(u_1) = \phi(u_3) = \ldots = s^2, \quad \phi(u_2) = \phi(u_4) = \ldots = s^{2p-2}, \quad \text{if} \ t_{2j} \text{ is odd}
\]

\[
\phi(u_1) = \phi(u_2) = s, \phi(u_3) = \phi(u_4) = \ldots = s, \phi(u_5) = \phi(u_6) = \ldots = s^{2p-1}, \quad \text{if} \ t_{2j} \text{ is even}
\]

\[
\phi(\alpha_1) = s, \phi(\beta_1) = t, \phi(\alpha_i) = \phi(\beta) = 1, \quad i = 2, 3, \ldots, \gamma.
\]

(iv) $t_2 \neq 0$, $t_{pj} \neq 0$. Here $\gamma \geq 0$, $t_2 \ (\text{even}) \geq 2$, $t_{pj} \geq 2$.

\[
\phi(x_1) = \phi(x_3) = \ldots = t^p, \phi(x_2) = \phi(x_4) = \ldots = t^p s,
\]

\[
\phi(v_1) = \phi(v_3) = \ldots = t^2, \quad \phi(v_2) = \phi(v_4) = \ldots = t^{2p-2} s^{p-1},
\]

\[
\phi(\alpha_i) = \phi(\beta) = 1, \quad i = 1, 2, \ldots, \gamma.
\]
(v) \( t_{2p} \neq 0, \ t_p \neq 0. \) Here \( \gamma \geq 0, \ t_{2p} \) (even) \( \geq 4. \)

\[
\phi(y_1) = \phi(y_2) = \phi(y_3) = t_s, \ \phi(y_4) = t^{2p-3}s, \quad \text{if} \ t_p \text{ is even,}
\]

\[
\phi(y_1) = \phi(y_2) = \phi(y_3) = t_s, \ \phi(y_4) = t^{2p-5}s, \quad \text{if} \ t_p \text{ is odd,}
\]

\[
\phi(y_5) = \phi(y_6) = \ldots = t, \ \phi(y_6) = \phi(y_7) = \ldots = t^{2p-1},
\]

\[
\phi(z_1) = \phi(z_2) = \ldots = t^2, \ \phi(z_4) = \phi(z_4) = \ldots = t^{2p-2},
\]

\[
\phi(\alpha_i) = \phi(\beta_i) = 1, \ i = 1,2,\ldots,\gamma.
\]

(vi) \( t_{2p} \neq 0, \ t_{\bar{m}_j} \neq 0. \) Here \( \gamma \geq 0, \ t_{2p} \) (even) \( \geq 2. \)

\[
\phi(b_1) = \phi(b_2) = \ldots = t, \ \phi(b_2) = \phi(b_4) = \ldots = t^{2p-1}, \quad \text{if} \ t_{\bar{m}_j} \text{ is even,}
\]

\[
\phi(b_1) = \phi(b_2) = \ldots = t, \ \phi(b_2) = \phi(b_4) = \ldots = t^{2p-1}s^{-1}, \quad \text{if} \ t_{\bar{m}_j} \text{ is odd,}
\]

\[
\phi(d_1) = \phi(d_2) = \ldots = s, \ \phi(d_2) = \phi(d_4) = \ldots = s^{-1},
\]

\[
\phi(\alpha_i) = \phi(\beta_i) = 1, \ i = 1,2,\ldots,\gamma.
\]

(vii) \( t_{2p} \neq 0, \ t_{\bar{m}_i} \neq 0. \) Here \( \gamma \geq 0, \ t_{2p} \) (even) \( \geq 2. \)

\[
\phi(y_1) = \phi(y_3) = \ldots = t, \ \phi(y_2) = \phi(y_4) = \ldots = t^{2p-1}, \quad \text{if} \ t_{\bar{m}_i} \text{ is even,}
\]

\[
\phi(y_1) = t, \ \phi(y_2) = t^{2p-3}s, \ \phi(y_3) = \phi(y_4) = \ldots = t^{2p-1}s^{-1}, \quad \text{if} \ t_{\bar{m}_i} \text{ is odd,}
\]

\[
\phi(v_1) = \phi(v_3) = \ldots = t^2s^{-1}, \ \phi(v_2) = \phi(v_4) = \ldots = t^{2p-2}s,
\]

\[
\phi(\alpha_i) = \phi(\beta_i) = 1, \ i = 1,2,\ldots,\gamma.
\]

(viii) \( t_p \neq 0, \ t_{\bar{m}_i} \neq 0. \) Here \( \gamma \geq 1, \ t_p \) (even) \( \geq 2. \)

\[
\phi(z_1) = \phi(z_3) = \ldots = t^2, \ \phi(z_2) = \phi(z_4) = \ldots = t^{2p-2},
\]

\[
\phi(u_1) = \phi(u_3) = \ldots = s, \ \phi(u_2) = \phi(u_4) = \ldots = s^{-1} \text{ and}
\]

\[
\phi(\alpha_1) = \phi(\beta_1) = t, \ \phi(\alpha_i) = \phi(\beta_i) = 1, \ i = 2,3,\ldots,\gamma; \quad \text{if} \ t_{\bar{m}_i} \text{ is even.}
\]

\[
\phi(u_1) = \phi(u_3) = \ldots = s^2, \ \phi(u_2) = \phi(u_4) = \ldots = s^{-2} \text{ and}
\]

\[
\phi(\alpha_1) = s, \ \phi(\beta_1) = t, \ \phi(\alpha_i) = \phi(\beta_i) = 1, \ i = 2,3,\ldots,\gamma; \quad \text{if} \ t_{\bar{m}_i} \text{ is odd.}
\]
(ix) \( t_\rho \neq 0, \ t_{\rho j} \neq 0 \). Here \( \gamma \geq 1 \).

\[
\phi(z_1) = \phi(z_2) = \ldots = t^2, \ \phi(z_3) = \phi(z_4) = \ldots = t^{2p-2},
\]
\[
\phi(v_1) = \phi(v_2) = \ldots = t^{2p-2}s^2, \ \phi(v_3) = \phi(v_4) = \ldots = t^2s^{\alpha-2}, \quad \text{if } \ t_\rho \text{ and } t_{\rho j} \text{ are odd}
\]
\[
\phi(v_1) = t^{2p-2}s^4, \ \phi(v_3) = \phi(v_4) = \ldots = t^{2p-2}s^2, \ \phi(v_2) = \phi(v_5) = \ldots = t^4s^{\alpha-2}, \quad \text{if } t_\rho \text{ and } t_{\rho j} \text{ are even}
\]
\[
\phi(z_1) = \phi(z_2) = t^2, \ \phi(z_3) = \phi(z_4) = \ldots = t^2s^{\alpha-2}, \quad \text{if } \ t_\rho \text{ is odd, } t_{\rho j} \text{ is even}
\]
\[
\phi(v_1) = \phi(v_2) = \ldots = t^{2p-2}s^2, \ \phi(v_3) = \phi(v_4) = \ldots = t^4s^{\alpha-2}, \quad \text{if } \ t_\rho \text{ is even, } t_{\rho j} \text{ is odd}
\]
\[
\phi(x_1) = s, \ \phi(x_2) = t, \ \phi(x_3) = \phi(x_4) = 1, \ i = 2,3,\ldots \gamma; \quad \text{if } t_{\rho j} \text{ is odd.}
\]

(x) \( t_{\rho j} \neq 0, \ t_{\rho j} \neq 0 \). Here \( \gamma \geq 1 \), \( t_{\rho j} \) (even) \( \geq 2 \).

\[
\phi(u_1) = s^2, \ \phi(u_3) = \phi(u_4) = \ldots = s, \ \phi(u_2) = \phi(u_5) = \ldots = s^{-\alpha-1} \text{ if is odd}
\]
\[
\phi(u_1) = \phi(u_2) = s, \ \phi(u_3) = \phi(u_5) = \ldots = s, \ \phi(u_4) = \phi(u_6) = \ldots = s^{-\alpha-1} \text{ if is even}
\]
\[
\phi(v_1) = \phi(v_3) = \ldots = t^2s^2, \ \phi(v_4) = \phi(v_5) = \ldots = t^{2p-2}s^{\alpha-1}
\]
\[
\phi(x_1) = s, \ \phi(x_2) = t, \ \phi(x_3) = \phi(x_4) = 1, \ i = 2,3,\ldots \gamma; \quad \text{if } t_{\rho j} \text{ is odd.}
\]

Case 4 : \( \Gamma \) be a Fuchsian group with three different periods.

In this case \( \gamma \geq 0 \) and the epimorphism can be defined as follows.

(i) \( t_2 \neq 0, \ t_{2p} \neq 0, \ t_{\rho j} \neq 0 \). Here \( t_2 + t_{2p} \geq 4 \).

\[
\phi(x_a) = t^\alpha s^\alpha, \ a = 1, t_2,
\]
\[
\phi(y_1) = \phi(y_3) = \ldots = t, \ \phi(y_2) = \phi(y_4) = \ldots = t^{2p-1} \text{ if } t_2 \text{ and } t_{2p} \text{ are even},
\]
\[
\phi(y_1) = \phi(y_2) = t, \phi(y_3) = t^{p-2}s, \ \phi(y_4) = \phi(y_5) = \ldots = t,
\]
\[
\phi(y_5) = \phi(y_4) = \ldots = t^{2p-1} \text{ if } t_2 \text{ and } t_{2p} (\text{assuming } t_{2p} \geq 3) \text{ are odd},
\]
\[
\phi(z_1) = \phi(z_3) = \ldots = t^2, \ \phi(z_2) = \phi(z_4) = \ldots = t^{2p-2}, \quad \text{assuming } t_\rho \text{ as even,}
\]
\[
\phi(x_1) = \phi(x_2) = 1, \ i = 1,2,3,\ldots \gamma.
\]

(ii) \( t_2 \neq 0, \ t_{2p} \neq 0, \ t_{\rho j} \neq 0 \). Here \( t_{2p} \geq 2 \) and \( t_2, t_{2p} \) are of same parity.

\[
t_2 \geq 2 \quad \text{if } \ t_{2p} = 2.
\]

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\[ \phi(x_a) = t^p s, \ a = 1, 2, \ldots, t_2, \]
\[ \phi(y_1) = \phi(y_3) = \ldots = t, \ \phi(y_2) = \phi(y_4) = \ldots = t^{2p-1} \text{ if } t_2 \text{ and } t_{2p} \text{ are even,} \]
\[ \phi(y_1) = \phi(y_2) = t, \ \phi(y_3) = \phi(y_4) = t^{p-2} s, \ \phi(y_5) = \phi(y_6) = \ldots = t, \]
\[ \phi(y_3) = \phi(y_7) = \ldots = t^{2p-1} \text{ if } t_2 \text{ and } t_{2p} (\text{assuming } t_{2p} \geq 3) \text{ are odd,} \]
\[ \phi(u_1) = \phi(u_3) = \ldots = s, \ \phi(u_2) = \phi(u_4) = \ldots = s^{m-1}, \text{ assuming } t_{2i} \text{ as even,} \]
\[ \phi(\alpha_i) = \phi(\beta_i) = 1, \ i = 1, 2, 3, \ldots, \gamma. \]

(iii) \( t_2 \neq 0, \ t_{2p} \neq 0, \ t_{2p} \neq 0. \) Here \( t_2 \) (even) \( \geq 2, \) \( t_{2p} \geq 2. \)

\[ \phi(x_0) = t^p s, \ a = 1, 2, \ldots, t_2, \]
\[ \phi(z_1) = \phi(z_2) = \ldots = t^2, \ \phi(z_2) = \phi(z_4) = \ldots = t^{2p-2} \]
\[ \phi(u_1) = \phi(u_3) = \ldots = s, \ \phi(u_2) = \phi(u_4) = \ldots = s^{m-1}, \text{ assuming all of } t_2, t_{2p}, t_{2p} \text{ as even,} \]
\[ \phi(\alpha_i) = \phi(\beta_i) = 1, \ i = 1, 2, 3, \ldots, \gamma. \]

(iv) \( t_2 \neq 0, \ t_p \neq 0, \ t_{p_j} \neq 0. \) Here \( t_2 \) (even) \( \geq 2, \) \( t_p \geq 2. \)

\[ \phi(x_0) = t^p s, \ a = 1, 2, \ldots, t_2, \]
\[ \phi(z_1) = \phi(z_2) = \ldots = t^2, \ \phi(z_3) = \phi(z_4) = \ldots = t^{2p-2} \]
\[ \phi(v_1) = \phi(v_2) = \ldots = t^{p-2} s, \ \phi(v_2) = \phi(v_4) = \ldots = t^p s^{m-1}, \text{ assuming all } t_2, t_{2p}, t_{p_j} \text{ as even}, \]
\[ \phi(\alpha_i) = \phi(\beta_i) = 1, \ i = 1, 2, 3, \ldots, \gamma. \]

(v) \( t_2 \neq 0, \ t_p \neq 0, \ t_{p_j} \neq 0. \) Here \( t_2 \) (even) \( \geq 2, \) \( t_{2p} \geq 2. \)

\[ \phi(x_0) = t^p s, \ a = 1, 2, \ldots, t_2, \]
\[ \phi(z_1) = \phi(z_2) = \ldots = t^2, \ \phi(z_2) = \phi(z_4) = \ldots = t^{2p-2} \]
\[ \phi(v_1) = \phi(v_2) = \ldots = t^{p-2} s, \ \phi(v_2) = \phi(v_4) = \ldots = t^p s^{m-1}, \text{ assuming all } t_2, t_{2p}, t_{p_j} \text{ as even,} \]
\[ \phi(\alpha_i) = \phi(\beta_i) = 1, \ i = 1, 2, 3, \ldots, \gamma. \]

(vi) \( t_2 \neq 0, \ t_{2p} \neq 0, \ t_{p_j} \neq 0. \) Here \( t_2 \) (even) \( \geq 2, \) \( t_{p_j} \geq 2. \)

\[ \phi(x_0) = t^p s, \ a = 1, 2, \ldots, t_2, \]
\[ \phi(u_1) = \phi(u_3) = \ldots = s, \ \phi(u_2) = \phi(u_4) = \ldots = s^{m-1}, \]
\[ \phi(v_1) = \phi(v_3) = \ldots = t^2 s, \ \phi(v_2) = \phi(v_4) = \ldots = t^{2p-2} s^{m-1}, \text{ if } t_{2p} \text{ is even,} \]
\[ \phi(v_1) = \phi(v_3) = \ldots = t^2 s, \ \phi(v_2) = t^{2p-2} s^{m-2}, \ \phi(v_4) = \phi(v_6) = \ldots = t^{2p-2} s^{m-1}, \text{ if } t_{2p} \text{ is odd,} \]
\[ \phi(\alpha_i) = \phi(\beta_i) = 1, \ i = 1, 2, 3, \ldots, \gamma. \]

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(vii) \( t_{2p} \neq 0, \ t_{p} \neq 0, \ t_{\bar{m}_j} \neq 0 \). Here \( t_{2p} \) (even) \( \geq 2 \).

\[
\phi(y_1) = \phi(y_3) = \ldots = t, \ \phi(y_2) = \phi(y_4) = \ldots = t^{2p-1}, \ \text{assuming all of } t_p, t_{\bar{m}_j} \text{ as even},
\]

\[
\phi(y_1) = t, \phi(y_2) = t_s, \phi(y_3) = \phi(y_5) = \ldots = t, \ \phi(y_4) = \phi(y_6) = \ldots = t^{2p-1}, \ \text{assuming } t_p, t_{\bar{m}_j} \text{ as odd},
\]

\[
\phi(z_1) = \phi(z_3) = \ldots = t^{2p-2}, \ \phi(z_2) = \phi(z_4) = \ldots = t^2,
\]

\[
\phi(u_1) = \phi(u_3) = \ldots = s^{m-1}, \ \phi(u_2) = \phi(u_4) = \ldots = s,
\]

\[
\phi(\alpha_i) = \phi(\beta_i) = 1, \ i = 1, 2, 3, \ldots \gamma.
\]

(viii) \( t_{2p} \neq 0, \ t_{p} \neq 0, \ t_{\bar{m}_j} \neq 0 \). Here \( t_{2p} \) (even) \( \geq 2 \).

\[
\phi(y_1) = \phi(y_3) = \ldots = t, \ \phi(y_2) = \phi(y_4) = \ldots = t^{2p-1}, \ \text{assuming all } t_p, t_{\bar{m}_j} \text{ as even},
\]

\[
\phi(y_1) = t, \phi(y_2) = t^{2p-1}s, \phi(y_3) = \phi(y_5) = \ldots = t, \ \phi(y_4) = \phi(y_6) = \ldots = t^{2p-1},
\]

\[
\text{assuming } t_p, t_{\bar{m}_j} \text{ as odd},
\]

\[
\phi(z_1) = \phi(z_3) = \ldots = t^2, \ \phi(z_2) = \phi(z_4) = \ldots = t^{2p-2},
\]

\[
\phi(v_1) = \phi(v_3) = \ldots = t^{2p-2}s^{m-1}, \ \phi(v_2) = \phi(v_4) = \ldots = t^2s,
\]

\[
\phi(\alpha_i) = \phi(\beta_i) = 1, \ i = 1, 2, 3, \ldots \gamma.
\]

(ix) \( t_{2p} \neq 0, \ t_{\bar{m}_j} \neq 0, \ t_{p} \neq 0 \). Here \( t_{2p} \) (even) \( \geq 2 \).

\[
\phi(y_1) = \phi(y_3) = \ldots = t, \ \phi(y_2) = \phi(y_4) = \ldots = t^{2p-1}, \ \text{assuming all } t_{\bar{m}_j}, t_{\bar{m}_j} \text{ as even},
\]

\[
\phi(y_1) = \phi(y_2) = t, \phi(y_3) = \phi(y_5) = \ldots = t, \ \phi(y_4) = \phi(y_6) = \ldots = t^{2p-1}, \ \text{assuming } t_{\bar{m}_j}, t_{\bar{m}_j} \text{ as odd},
\]

\[
\phi(u_1) = \phi(u_3) = \ldots = s, \ \phi(u_2) = \phi(u_4) = \ldots = s^{m-1},
\]

\[
\phi(v_1) = \phi(v_3) = \ldots = t^{2p-2}s^{m-1}, \ \phi(v_2) = \phi(v_4) = \ldots = t^2s,
\]

\[
\phi(\alpha_i) = \phi(\beta_i) = 1, \ i = 1, 2, 3, \ldots \gamma.
\]

(x) \( t_p \neq 0, \ t_{\bar{m}_j} \neq 0, \ t_{p} \neq 0 \).

\[
\phi(z_1) = \phi(z_3) = \ldots = t^2, \ \phi(z_2) = \phi(z_4) = \ldots = t^{2p-2},
\]

\[
\phi(u_1) = \phi(u_3) = \ldots = s, \ \phi(u_2) = \phi(u_4) = \ldots = s^{m-1},
\]

\[
\phi(v_1) = \phi(v_3) = \ldots = t^{2p-2}s^{m-1}, \ \phi(v_2) = \phi(v_4) = \ldots = t^2s,
\]

\[
\phi(\alpha_i) = \phi(\beta_i) = t, \phi(\alpha_i) = \phi(\beta_i) = 1, \ i = 2, 3, \ldots \gamma.
\]

assuming all of \( t_p, t_{\bar{m}_j}, t_{\bar{m}_j} \) as even or odd together.

Case 4: \( \Gamma \) be a Fuchsian group with four different periods.

In this case \( \gamma \geq 0 \) and the epimorphism can be defined as follows.
(i) \( t_2 \neq 0, \ t_{2p} \neq 0, \ t_p \neq 0, \ t_{\bar{m}} \neq 0. \)
\[
\phi(x_a) = t^a, \ a = 1,2,\ldots,t_2,
\phi(y_1) = \phi(y_3) = \ldots = t, \ \phi(y_2) = \phi(y_4) = \ldots = t^{2p-1},
\phi(z_1) = \phi(z_3) = \ldots = t^2, \ \phi(z_2) = \phi(z_4) = \ldots = t^{2p-2},
\phi(u_1) = \phi(u_3) = \ldots = s, \ \phi(u_2) = \phi(u_4) = \ldots = s^{m-1},
\phi(\alpha_i) = \phi(\beta_i) = 1, \ i = 1,2,3,\ldots.
\]
assuming all of \( t_i \)'s are even.

(ii) \( t_2 \neq 0, \ t_{2p} \neq 0, \ t_p \neq 0, \ t_{\bar{m}} \neq 0. \)
\[
\phi(x_a) = t^a s, \ a = 1,2,\ldots,t_2,
\phi(y_1) = \phi(y_3) = \ldots = t, \ \phi(y_2) = \phi(y_4) = \ldots = t^{2p-1},
\phi(z_1) = \phi(z_3) = \ldots = t^2, \ \phi(z_2) = \phi(z_4) = \ldots = t^{2p-2},
\phi(u_1) = \phi(u_3) = \ldots = s, \ \phi(u_2) = \phi(u_4) = \ldots = s^{m-1},
\phi(v_1) = \phi(v_3) = \ldots = t^{2p-2}s^{m-1}, \ \phi(v_2) = \phi(v_4) = \ldots = t^2 s,
\phi(\alpha_i) = \phi(\beta_i) = 1, \ i = 1,2,3,\ldots.
\]
assuming all of \( t_i \)'s are even.

(iii) \( t_2 \neq 0, \ t_p \neq 0, \ t_{\bar{m}} \neq 0, \ t_{\bar{p}} \neq 0. \ \ t_2 \ \text{(even)} \geq 2. \)
\[
\phi(x_a) = t^a s, \ a = 1,2,\ldots,t_2,
\phi(z_1) = \phi(z_3) = \ldots = t^2, \ \phi(z_2) = \phi(z_4) = \ldots = t^{2p-2},
\phi(u_1) = \phi(u_3) = \ldots = s, \ \phi(u_2) = \phi(u_4) = \ldots = s^{m-1},
\phi(v_1) = \phi(v_3) = \ldots = t^{2p-2}s^{m-1}, \ \phi(v_2) = \phi(v_4) = \ldots = t^2 s,
\phi(\alpha_i) = \phi(\beta_i) = 1, \ i = 1,2,3,\ldots.
\]
assuming all of \( t_i \)'s are even.

(iv) \( t_2 \neq 0, \ t_{2p} \neq 0, \ t_{\bar{m}} \neq 0, \ t_{\bar{p}} \neq 0. \)
\[
\phi(x_a) = t^a s, \ a = 1,2,\ldots,t_2,
\phi(y_1) = \phi(y_3) = \ldots = t, \ \phi(y_2) = \phi(y_4) = \ldots = t^{2p-1},
\phi(u_1) = \phi(u_3) = \ldots = s, \ \phi(u_2) = \phi(u_4) = \ldots = s^{m-1},
\phi(v_1) = \phi(v_3) = \ldots = t^{p-1}s^{m-1}, \ \phi(v_2) = \phi(v_4) = \ldots = t^p s,
\phi(\alpha_i) = \phi(\beta_i) = 1, \ i = 1,2,3,\ldots.
\]
assuming all of \( t_i \)'s are even or odd together.
\( \phi(y_1) = \phi(y_3) = \ldots = t, \ \phi(y_2) = \phi(y_4) = \ldots = t_{p-1}, \)
\( \phi(z_1) = \phi(z_3) = \ldots = t^2, \ \phi(z_2) = \phi(z_4) = \ldots = t^{2p-2}, \)
\( \phi(u_1) = \phi(u_3) = \ldots = s, \ \phi(u_2) = \phi(u_4) = \ldots = s^{m-1}, \)
\( \phi(v_1) = \phi(v_3) = \ldots = t^{2p-2}s^{m-1}, \ \phi(v_2) = \phi(v_4) = \ldots = t^2s, \)
\( \phi(a_i) = \phi(\beta_i) = 1, \ i = 1,2,3,\ldots, \gamma. \)

assuming \( t_p,t_{\pi_j},t_{\pi_j} \) as even or odd together.

**Case 5:** \( \Gamma \) be a Fuchsian group with all the five periods.

In this case \( \gamma \geq 0 \) and the epimorphism can be defined as follows.

\( \phi(x_a) = t^a, \ a = 1,2,\ldots, t_2, \)
\( \phi(y_1) = \phi(y_3) = \ldots = t^{p+2}, \ \phi(y_2) = \phi(y_4) = \ldots = t^{p-2}, \)
\( \phi(z_1) = \phi(z_3) = \ldots = t^{2p-4}, \ \phi(z_2) = \phi(z_4) = \ldots = t^4, \)
\( \phi(u_1) = \phi(u_3) = \ldots = s, \ \phi(u_2) = \phi(u_4) = \ldots = s^{m-1}, \)
\( \phi(v_1) = \phi(v_3) = \ldots = t^{2p-2}s^{m-1}, \ \phi(v_2) = \phi(v_4) = \ldots = t^{2p-2}s, \)
\( \phi(a_i) = \phi(\beta_i) = 1, \ i = 1,2,3,\ldots, \gamma. \)

assuming all of same parity.

The results discussed so far, can be summarized as follows.

**Theorem 2.1**

Let \( \Gamma \) be a Fuchsian group with signature \( \Delta(\gamma;m_1,m_2,\ldots,m_r) \) and let \( G = C_p \times D_m, \) where \( p \) is an odd prime and \( m \) is an integer greater than one with \( \text{hcf}(p,m) = 1. \)

Let \( \overline{m}_j = \frac{m}{d_j}, \) where \( d_j = \text{hcf}(m,j), \ 1 \leq j \leq m-1 \) and \( p_j = \text{lcm}[p,\overline{m}_j]. \)

Then there exists a smooth epimorphism \( \phi : \Gamma \rightarrow G \) if and only if

1. The periods of \( \Gamma \) (if any) take the values from the set \( \{2,2p,p,\overline{m}_j,p_j\}, \)
2. If \( t_2,t_{2p},t_p,t_{\overline{m}_j},t_{p_j} \) denote the number of occurrences of periods \( 2,2p,p,\overline{m}_j,p_j \) respectively in \( \Gamma, \) then
(a) $t_2 + t_{2p}$ is even when $m$ is odd.

(b) If one of $t_2$ and $t_{2p}$ is zero, then other must be even.

(c) $\gamma = 0$ is possible in the following cases only
   
   (i) when $t_{2p} \geq 2$, 
   
   or (ii) when $t_2 \geq 2$. In case $m$ is odd, at least one of $t_p$ and $t_{p_j}$ should be nonzero and $t_2 \geq 4$ if $t_{p_j} = 0$.

(d) (i) If any two of $t_{2p}$, $t_p$, and $t_{p_j}$ are zero, then the third must be greater than or equal to 2.

(ii) $t_2 + t_{2p} \geq 4$, if $t_{p_j} = t_{\overline{m}_j} = 0$.

(e) When $\Gamma$ is a surface group and also when $\Gamma$ contains $p$ as its only period, we have $\gamma \geq 2$.

If $(p, m) = p$, then $p = q_i$, where $q_i$ is a factor of $m$. Therefore we get $p_j = \overline{m}_j$ and in this case the periods of $\Gamma$ will take the values from the set \{2, 2p, \overline{m}_j\} and we have the following Theorem.

**Theorem 2.2**

Let $\Gamma$ be a Fuchsian group of signature $\Delta(\gamma; m_1, m_2, \ldots, m_r)$ and $G = C_p \times D_m$, where $p$ is an odd prime and $m$ is an integer greater than one with $(p, m) = p$. Then there exists a smooth epimorphism $\phi: \Gamma \to G$ if and only if

1. The periods of $\Gamma$ (if any) take the values from the set \{2, 2p, \overline{m}_j\}.

2. If $t_2, t_{2p}, t_{p_j}$ denote the number of occurrences of periods $2, 2p, \overline{m}_j$ respectively in $\Gamma$ then

   (a) $t_2 + t_{2p}$ is even when $m$ is odd.

   (b) If one of $t_2$ and $t_{2p}$ is zero, other must be even.

   (c) $\gamma = 0$ is possible in the following cases only

   (i) when $t_{2p} \neq 0$. Here $t_{2p} \geq 2$ if $t_{\overline{m}_j} = 0$.

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or (ii) when $t_2 \geq 4$ and $t_{\bar{m}_j} \neq 0$.

2.3 Determination of minimum genera

In Section 2.2, we have proved that the Fuchsian group $\Gamma$ has the signatures

\begin{align*}
\Delta \{ \gamma; 2,2,\ldots, 2p,2p,\ldots, p,p,\ldots, \bar{m}_{j_1},\bar{m}_{j_2},\ldots, p_{j_1},p_{j_2},\ldots \}, \quad & \text{if } (p,m) = 1 \\
\Delta \{ \gamma; 2,2,\ldots, 2p,2p,\ldots, \bar{m}_{j_1},\bar{m}_{j_2},\ldots \}, \quad & \text{if } (p,m) = p
\end{align*}

and

\begin{align*}
\Delta \{ \gamma; 2,2,\ldots, 2p,2p,\ldots, \bar{m}_{j_1},\bar{m}_{j_2},\ldots \}, \quad & \text{if } (p,m) = 1 \\
\Delta \{ \gamma; 2,2,\ldots, 2p,2p,\ldots, \bar{m}_{j_1},\bar{m}_{j_2},\ldots \}, \quad & \text{if } (p,m) = p
\end{align*}

where $\gamma, t, \bar{m}_j, p_j$ s are as defined in Section 2.1 and Section 2.2 and satisfying the conditions of Theorem 2.1 and Theorem 2.2.

We can write

\begin{align*}
&g = 1 + pm \left[ 2(\gamma - 1) + \frac{1}{p} \right] + \sum_{i=1}^{t_{\bar{m}_j}} \left( 1 - \frac{1}{p} \right) \frac{t_{j_i}}{2} + \frac{t_{\bar{m}_j}}{2} \\
&\text{when } (p,m) = 1 \\
&g = 1 + pm \left[ 2(\gamma - 1) + \frac{t_2}{2} + \frac{2p - 1}{2} \right] + \sum_{i=1}^{t_{\bar{m}_j}} \left( 1 - \frac{1}{p} \right) \frac{t_{j_i}}{2} \\
&\text{when } (p,m) = p.
\end{align*}

We now determine the sets of values of $\gamma$ and $t_i$s for which we may obtain the minimum values of $g$. We discuss the possible cases on the periods of $\Gamma$ as follows.

Case 1 \quad $\sum t_i = 0$.

Here $\gamma \geq 2$. Therefore $g \geq 1 + 2pm$.

Case 2 \quad $\sum t_i = 1$.

Here $\gamma \geq 1$ and only possible period is $\bar{m}_j$, for a suitable $j$ and $t_{\bar{m}_j} = 1$.

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Hence \( g \geq 1 + \frac{pm - 1}{m_j} = 1 + m \left( p - \frac{p}{m_j} \right) \).

The right hand expression will be minimum when \( m_j \) is minimum. But \( m_j = \frac{m}{d_j} \), 
\( d_j = \text{h.c.f.} \,(j, m) \). Here \( m_j \) is minimum when \( d_j \) is maximum.

Case 3 \( \sum t_i = 2 \).

Here \( \gamma \geq 1 \) and it can be shown that \( g \) will be minimum when \( t_2 = 2 \), and 
\( g \geq 1 + pm \cdot 1 = 1 + pm \).

Case 4 \( \sum t_i \geq 3 \).

Here \( \gamma \geq 0 \). From (2.3.1) we have
\[
g = 1 + pm \left[ 2(\gamma - 1) + \sum t_i - \left( \frac{t_2}{2} + \frac{t_2p}{2p} + \frac{t_p}{p} + \sum \frac{t_{m_j}}{m_j} + \sum \frac{t_{p_{j_i}}}{p_{j_i}} \right) \right].
\]
Putting \( \gamma \geq 1 \) and \( \sum t_i = 3 \), we get
\[
g \geq 1 + pm \left[ 3 - \left( \frac{t_2}{2} + \frac{t_2p}{2p} + \frac{t_p}{p} + \sum \frac{t_{m_j}}{m_j} + \sum \frac{t_{p_{j_i}}}{p_{j_i}} \right) \right].
\]
As \( \sum t_i = 3 \) and \( m, p \geq 3 \), it can be verified that \( \frac{t_2}{2} + \frac{t_2p}{2p} + \frac{t_p}{p} + \sum \frac{t_{m_j}}{m_j} + \sum \frac{t_{p_{j_i}}}{p_{j_i}} < 2 \)
and therefore \( g > 1 + pm \).

Now we consider the cases for \( \gamma = 0 \).

We have \( g = 1 + pm \left[ 2(\gamma - 1) + \sum t_i - \left( \frac{t_2}{2} + \frac{t_2p}{2p} + \frac{t_p}{p} + \sum \frac{t_{m_j}}{m_j} + \sum \frac{t_{p_{j_i}}}{p_{j_i}} \right) \right]. \)

Putting \( \gamma = 0 \) and \( \sum t_i = 3 \) we get
\[
g = 1 + pm \left[ -2 + 3 - \left( \frac{t_2}{2} + \frac{t_2p}{2p} + \frac{t_p}{p} + \sum \frac{t_{m_j}}{m_j} + \sum \frac{t_{p_{j_i}}}{p_{j_i}} \right) \right].
\]
For $\sum t_i = 3$, the possible sub-cases are,

4a. $t_2 = 1, t_{2p} = 1, t_{p_j} = 1, 1 \leq j \leq m-1, (p, m) = 1$.

4b. $t_{2p} = 2, t_{p_j} = 1, 1 \leq j \leq m-1, (p, m) = 1$.

4c. $t_{2p} = 2, t_{m_{j'}} = 1, 1 \leq j \leq m-1, (p, m) = 1$.

4d. $t_2 = 1, t_{2p} = 1, t_{m_{j'}} = 1, 1 \leq j \leq m-1, (p, m) = p$.

4e. $t_{2p} = 2, t_{m_{j'}} = 1, 1 \leq j \leq m-1, (p, m) = p$.

It is observed that, when $(p, m) = 1$, the minimum genus is obtained when the periods of $\Gamma$ satisfy the conditions of case (4a) and when $(p, m) = p$, it is obtained when the periods satisfy the conditions of case (4d). We determine the values of genus $g$ for these two cases.

4a. $t_2 = 1, t_{2p} = 1, t_{p_j} = 1, 1 \leq j \leq m-1, (p, m) = 1$.

$$g = 1 + pm \left[ 1 - \left( \frac{1}{2} + \frac{1}{2p} + \frac{1}{p_j} \right) \right] = 1 - pm \left( \frac{p+1}{2p} + pm \left( \frac{1}{p_j} \right) \right) = 1 - m \frac{p+1}{2} + m \left( \frac{p}{p_j} \right)$$

(2.3.3)

Here $g$ will be minimum if $p_j$ is minimum.

When $p = 3, g = 1 + m \left( 1 - \frac{3}{3m} \right) = m$  (2.3.4)

4d. $t_2 = 1, t_{2p} = 1, t_{m_{j'}} = 1, 1 \leq j \leq m-1, (p, m) \neq 1$.

Here $g = 1 + pm \left[ 1 - \left( \frac{1}{2} + \frac{1}{2p} + \frac{1}{m_j} \right) \right] = 1 - m \frac{p+1}{2} + m \left( \frac{p}{m_j} \right)$.  

(2.3.5)

When $p = m = 3$, the case (4d) is untenable for $\gamma = 0$, by (1.3.2). With this condition, the minimum genus is obtained in the case (4e) when $t_{2p} = 2, t_{m_{j'}} = 1$. 

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Here \( g = 1 + pm \left( 1 - \left( \frac{2}{2p} + \frac{1}{m_j} \right) \right) = 1 + 3.3 \left( 1 - \left( \frac{2}{6} + \frac{1}{3} \right) \right) = 4. \)

Therefore we get the minimum genus \( g \) for \( \Sigma t_i = 3 \), as follows

\[
g = 1 - m \frac{p+1}{2} + m \left( p - \frac{p}{p_j} \right), \quad (p, m) = 1;
\]

\[
g = 1 - m \frac{p+1}{2} + m \left( p - \frac{p}{m_j} \right), \quad (p, m) = p
\]

and \( g = 4 \), when \( p = m = 3 \).

The signatures of the corresponding Fuchsian groups are \( \Delta (0; 2, 2p, p_j) \), \( \Delta (0; 2, 2p, m_j) \) and \( \Delta (0; 3, 6, 6) \) respectively.

We now show that \( g \) cannot have a lower minimum for \( \Sigma t_i \geq 4 \). It can be verified that if \( g \) takes a lower minimum value for \( \Sigma t_i \geq 4 \), then it is obtained for \( \gamma = 0 \) only. Putting \( \gamma = 0 \) in (2.3.1) we have

\[
g = 1 + pm \left[ -2 + \sum t_i - \left( \frac{t_2}{2} + \frac{t_{2p}}{2p} + \frac{t_p}{p} + \sum \frac{t_{m_j}}{m_j} + \sum \frac{t_{p_j}}{p_j} \right) \right].
\]

Considering \( \Sigma t_i = 4 \), we get the possible values of the periods of \( \Gamma \) are

(i) \( t_2 = t_{2p} = 2 \); (ii) \( t_2 = t_{2p} = 1, t_{p_j} = 2 \); (iii) \( t_{2p} = 2, t_{p_j} = 2 \) and (iv) \( t_{2p} = 4 \).

It is observed that the minimum \( g \) is obtained when \( t_2 = t_{2p} = 2 \) and then

\[
g = 1 + pm \left[ 1 - \frac{1}{p} \right]. \quad (2.3.6)
\]

Now if possible, suppose

\[
1 + pm \left[ 1 - \frac{1}{p} \right] < 1 - \frac{p+1}{2}m + m \left( p - \frac{p}{p_j} \right), \quad (p, m) = 1.
\]

We have

\[
1 + pm \left[ 1 - \frac{1}{p} \right] - \left[ 1 - \frac{p+1}{2}m + m \left( p - \frac{p}{p_j} \right) \right] = m \left[ \frac{p+1}{2} - 1 + \frac{p}{p_j} \right] > 0,
\]

which is a contradiction.
Therefore the value of \( g \) obtained in (2.3.6) is not smaller than that obtained in (2.3.3) for \((p,m)=1\). Similarly it can be shown that when \((p,m)=p\), it is not smaller than \( g = 1 - m \frac{p+1}{2} + m \left( p - \frac{p}{m_j} \right) \).

If \( p = m = 3 \), (2.3.6) gives, \( g = 7 \) when \( t_2 = t_{2p} = 2 \) and it is also greater than 4.

The case \( \sum t_i \geq 5 \) can be disposed of similarly. The above discussions may be concluded as

**Theorem 2.3**

The minimum value of the genus \( g \) of a compact Riemann surface having \( G \), as it's group of automorphisms is

(i) \( g = 1 - m \frac{p+1}{2} + m \left( p - \frac{p}{p_j} \right) \), if \((p,m) = 1\);

(ii) \( g = 1 - m \frac{p+1}{2} + m \left( p - \frac{p}{m_j} \right) \), if \((p,m) = p\);

(iii) \( g = 4 \), if \( p = m = 3 \);

The results of Harvey and of Chutiya on the minimum genus can be deduced as a corollary of our Theorem 2.1, Theorem 2.2 and Theorem 2.3.

**Cor. 2.1:**

Considering a cyclic group \( C_p \), \( p \) is an odd prime, as \( C_p \cong \overline{G} \), \( \overline{G} = I_1 \times C_p \).

where \( I_1 \) is the identity group of \( D_m \), a dihedral group, we obtain the result as follows.

A smooth epimorphism \( \phi : \Gamma \rightarrow C_p \), \( p \) is an odd prime, will exist if and only if

1. \( \gamma \geq 2 \) in \( \Gamma \), if \( \Gamma \) is a surface group.
2. \( p \) is the only period of \( \Gamma \).
2. \( p \) is the only period of \( \Gamma \).

3. if \( t_p \) denotes the number of occurrences of period \( p \), then

\[
\begin{align*}
(a) & \quad \gamma = 0 \text{ is possible if } t_p \geq 3 \text{ for } p \neq 3, \\
(b) & \quad t_p \geq 4 \text{ when } \gamma = 0 \text{ and } p = 3.
\end{align*}
\]

The minimum genus \( g \) of the compact Riemann surface on which \( C_p \) acts as a group of automorphisms, is

\[
\begin{align*}
(i) & \quad g = 2 \text{ when } p = 3. \\
(ii) & \quad g = 1 + \frac{p - 3}{2} \text{ when } p > 3.
\end{align*}
\]

\( \Delta (0; 3,3,3,3) \) and \( \Delta (0; p, p, p) \) are the signatures of the respective Fuchsian groups. This result was obtained by Harvey [82].

**Cor. 2.2.**

Considering a dihedral group \( D_m \), \( m \) is an integer greater than one, as \( D_m \cong \overline{D_m}, \overline{D_m} = D_m \times I_2, \) \( I_2 \) is the identity group of \( C_p \), we obtain a result as follows.

A smooth epimorphism \( \phi : \Gamma \rightarrow D_m \), \( m \) is odd integer greater than one, exists if and only if

1. \( \gamma \geq 2 \), if \( \Gamma \) is a surface group,

2. \( \Gamma \) takes periods (if any) from \( \{ 2, \overline{m}_j \} \), \( \overline{m}_j = \frac{m}{d_j} \), \( d_j = \text{hcf} ( m, j ) \), \( 1 \leq j \leq m \).

3. if \( t_2 \) denotes the number of occurrences of the period 2, then

\[
\begin{align*}
(i) & \quad \gamma = 0 \text{ is possible if } t_2 \neq 0 \text{ otherwise } \gamma \geq 1. \\
(ii) & \quad t_2 \geq 2, \text{ and always even.}
\end{align*}
\]

The minimum genus \( g \) of the compact Riemann surface on which \( D_m \) acts as a group of automorphisms is \( g = 1 + m \left( 1 - \frac{2}{q_1} \right) \), \( q_1 \) is the least odd prime factor of \( m \).
in the prime decomposition of \( m \) and the corresponding Fuchsian group has the signature \( \Delta(0; 2, 2, q_x, q_x) \), which was obtained by Chutiya [42].

***