CHAPTER IV
BIVARIATE BURR TYPE III DISTRIBUTIONS

4.1 Introduction

In the previous chapter we have considered the bivariate Burr system. A detailed study of the distributional properties of each member of the system has not been undertaken in literature. In model building the first choice is on the family and the second choice specific member there. In order to choose the most appropriate member from the family one should have sufficient understanding of the important characteristic of the members. Present chapter is an attempt in this direction. In the present chapter we have discussed the bivariate Burr III distribution. Rodriguez (1980) derived the bivariate Burr III distribution using mixing argument. The form proposed by him is

\[ F(x_1, x_2) = \left[1 + x_1^{-\alpha_1} + x_2^{-\alpha_2} + \theta x_1^{-\alpha_1} x_2^{-\alpha_2}\right]^{-k} \quad 0 < x_i < \infty, \ k, \alpha_i > 0, 0 \leq \theta \leq k + 1, \ i = 1, 2 \quad (4.1.1) \]

But it can be shown that this distribution can be obtained as solution of set of partial differential equations involving distribution function which we have discussed in second chapter. In that unified approach the bivariate Burr III distribution arises by the choice \( G(x_1, x_2) \) as

\[ G(x_1, x_2) = \log\left[\left[1 + x_1^{-\alpha_1} + x_2^{-\alpha_2} + \theta x_1^{-\alpha_1} x_2^{-\alpha_2}\right]^k - 1\right] \quad (4.1.2) \]

in equation (2.2.11)

In view of analytical tractability in the present study we consider the form
Corresponding density function and survival function are

\[ F(x_i, x_j) = \left[1 + x_i^{-c_i} + x_j^{-c_j} \right]^{-k} \quad 0 < x_i < \infty, \ k, c > 0, i = 1, 2 \quad (4.1.3) \]

\[ f(x_i, x_j) = \frac{k(k+1)c_i c_j x_i^{-c_i-1} x_j^{-c_j-1}}{[1 + x_i^{-c_i} + x_j^{-c_j}]^{k+2}} \quad 0 < x_i < \infty, \ k, c > 0, i = 1, 2 \quad (4.1.4) \]

and

\[ R(x_i, x_j) = 1 - [1 + x_i^{-c_i}]^{-k} - [1 + x_j^{-c_j}]^{-k} + [1 + x_i^{-c_i} + x_j^{-c_j}]^{-k} \quad 0 < x_i < \infty, \ k, c > 0, i = 1, 2 \quad (4.1.5) \]

4.2 General Properties of Type III Model (Bismi and Nair, 2005 d)

In this section we consider some general properties of bivariate Burr III distribution specified in equation (4.1.3). It is noted that the marginal distributions are

\[ F_i(x_i) = \left[1 + x_i^{-c_i} \right]^{-k} \quad 0 < x_i < \infty, \ k, c > 0, i = 1, 2 \quad (4.2.1) \]

With a choice of

\[ g_i(x_i) = \frac{kc_i x_i^{-c_i-1}}{[1 + x_i^{-c_i}]^{-k}[1 + x_i^{-c_i}]^{k+1}} \quad 0 < x_i < \infty, \ k, c > 0, i = 1, 2 \quad (4.2.2) \]

equation (4.2.1) satisfies

\[ F_i(x_i) [1 - F_i(x_i)] g_i(x_i) = \frac{kc_i x_i^{-c_i-1}}{[1 + x_i^{-c_i}]^{k+1}} \]

\[ = \frac{dF_i(x_i)}{dx_i} \quad i = 1, 2 \quad (4.2.3) \]

which is univariate Burr type differential equation.

Thus for the bivariate Burr form (4.1.3) marginals are exactly univariate Burr type III.

With the above marginal distributions, conditional densities of \( X_i \) given \( X_j = x_j \)
arise as

\[
f(x_i \mid X_j = x_j) = \frac{(k+1)c_i x_i^{c_i-1}(1 + x_j^{c_j})^{k+1}}{[1 + x_i^{c_i} + x_j^{c_j}]^{k+2}} \quad 0 < x_i < \infty, k, c_i, c_j > 0, i, j = 1, 2 \quad (4.2.4)
\]

Using the transformation

\[
Y_i = \frac{X_i}{(1 + X_j^{c_j})^{-1/c_j}},
\]

it can be seen that \( Y_i \) follows univariate Burr type III with parameters \( c_i \) and \((k+1)\). Hence any property for univariate Burr distribution of \( X_i \) can be extended to the conditional distribution of \( X_i \) given \( X_j = x_j \).

Another type of conditional distribution that of interest especially in reliability modeling is the distribution of \( X_i \) given \( X_j > x_j \).

Survival function of \( X_i \) given \( X_2 > x_2 \) is

\[
R(x_i \mid X_2 > x_2) = P(x_i \mid X_2 > x_2) = \frac{P(X_1 > x_1, X_2 > x_2)}{P(X_2 > x_2)}
\]

\[
= \frac{1 - [1 + x_1^{-c_1}]^{-k} - [1 + x_2^{-c_2}]^{-k} + [1 + x_1^{-c_1} + x_2^{-c_2}]^{-k}}{1 - [1 + x_2^{-c_2}]^{-k}} \quad (4.2.5)
\]

The corresponding density function is calculated as

\[
f(x_i \mid X_2 > x_2) = \frac{\partial R(x_i \mid X_2 > x_2)}{\partial x_i} = \frac{kc_i x_i^{c_i-1} - kc_i x_i^{c_i-1}}{[1 + x_1^{-c_1} + x_2^{-c_2}]^{k+1} - [1 + x_2^{-c_2}]^{k+1}} \quad (4.2.6)
\]

Similarly

\[
R(x_1 \mid X_i > x_i) = P(x_1 \mid X_i > x_i)
\]
$$\frac{P(X_1 > x_1, X_2 > x_2)}{P(X_1 > x_1)} = 1 - \left[1 + x_1^{-c_1}\right]^{-k} - \left[1 + x_2^{-c_2}\right]^{-k} + \left[1 + x_1^{-c_1} + x_2^{-c_2}\right]^{-k} \quad (4.2.7)$$

and

$$f(x_1 \mid X_1 > x_1) = \frac{\partial R(x_1 \mid X_1 > x_1)}{\partial x_2} = \frac{k c_2 x_2^{-c_2-1} [1 + x_1^{-c_1} + x_2^{-c_2}]^{s+1} - k c_2 x_2^{-c_2-1} [1 + x_2^{-c_2}]^{s+1}}{1 - [1 + x_1^{-c_1}]^{-k}} \quad (4.2.8)$$

Now we are interested to find the moments and other characteristics.

The \((r_1, r_2)^{th}\) moment of the distribution,

$$\mu_{r_1, r_2} = E(X_1^{r_1} X_2^{r_2})$$

$$= \int \int x_1^{r_1} x_2^{r_2} f(x_1, x_2) \, dx_1 \, dx_2$$

$$= k (k + 1) c_1 c_2 \sum_{0}^{m} \frac{x_1^{r_1-1} x_2^{r_2-1}}{[1 + x_1^{-c_1} + x_2^{-c_2}]^{s+2}} \, dx_1 \, dx_2$$

$$= \frac{1}{\Gamma k} \Gamma (1 - r_1 / c_1) \Gamma (1 - r_2 / c_2) \Gamma (k + r_1 / c_1 + r_2 / c_2) \quad r_1 / c_1, r_2 / c_2 < 1, i = 1, 2 \quad (4.2.9)$$

In particular the product moment become

$$E(X_1 X_2) = \frac{1}{\Gamma k} \Gamma (1 - 1 / c_1) \Gamma (1 - 1 / c_2) \Gamma (k + 1 / c_1 + 1 / c_2) \quad 1 / c_i < 1, k + 1 / c_1 + 1 / c_2 > 0, i = 1, 2 \quad (4.2.10)$$

There is a recurrence relation connecting the moments of the distribution given by
\[ \mu_{k-\omega,\lambda-\omega} = \frac{1}{\Gamma(k)} \Gamma(1-(r_1/c_1)) \Gamma(1-(r_2/c_2)) \Gamma(k+(r_1/c_1)/(c_1)+(r_2/c_2)) \]
\[ = \frac{1}{\Gamma(k)} \Gamma(2-r_1/c_1) \Gamma(2-r_2/c_2) \Gamma(k+(r_1/c_1)+(r_2/c_2)-2) \]
\[ = \frac{(1-r_1/c_1)(1-r_2/c_2)}{(k+r_1/c_1+r_2/c_2-1)(k+r_1/c_1+r_2/c_2-2)} \mu_{k-\rho} \] (4.3.11)

when \( c_1 \) and \( c_2 \) are positive integers, this relation connects the adjacent moments and is useful to calculate all moments of the distribution devoid of gamma functions.

Covariance becomes

\[ \text{Cov}(X_1, X_2) = \frac{\Gamma(1-(1/c_1))\Gamma(1-(1/c_2))}{\Gamma(k)} \left[ \Gamma(k+1/c_1+1/c_2) - \frac{\Gamma(k+1/c_1)\Gamma(k+1/c_2)}{\Gamma(k)} \right] \quad \frac{1}{c_1, 1, k+1/c_1+1/c_2 > 0, \ i=1, 2} \] (4.2.12)

Then the coefficient of correlation has the expression

\[ \rho = \frac{\Gamma(1-(1/c_1))\Gamma(1-(1/c_2))}{\Gamma(k)} \frac{\Gamma(k+1/c_1+1/c_2)}{(\Gamma(1)(\Gamma(1))} \frac{\Gamma(k+1/c_1)\Gamma(k+1/c_2)}{\Gamma(k)} \] (4.2.13)

Regression equations are obtained as

\[ E(x_1 | X_2 = x_2) = \int_{0}^{\infty} x_1 f(x_1 | X_2 = x_2) \, dx_1 \]
\[ = (k+1) c_1 \int_{0}^{\infty} x_1^{-1}(1+x_1^{-1})^{k+1} \, dx_1 \]
\[ = (k+1) (1+x_2^{-1})^{-1/c_1} B(1-1/c_1,k+1+1/c_1) \] (4.2.14)
which is decreasing function of $x_2$.

Similarly

$$E(x_2 | X_1 = x_1) = (k + 1) (1 + x_1^{-v})^{-1/c_1} B(1-1/c_2, k + 1 + 1/c_2) \quad (4.2.15)$$

which is decreasing function of $x_1$.

$$\sigma(x_i | X_j = x_j) \quad = \quad \left[ (k + 1) \left( 1 + x_j^{-v} \right)^{-2/c_1} B(1-2/c_1, k + 1 + 2/c_1) \right]^{1/2}$$

$$- \left[ (k + 1) \left( 1 + x_j^{-v} \right)^{-1/c_1} B(1-1/c_1, k + 1 + 1/c_1) \right]^2 \quad (4.2.16)$$

The coefficient of variation of $X_i$ given $X_j = x_j$ is

$$\text{cv}(x_i | X_j = x_j) \quad = \quad \frac{\sigma(x_i | X_j = x_j)}{E(x_i | X_j = x_j)} \quad (4.2.17)$$

$$\frac{\left[ (k + 1) \left( 1 + x_j^{-v} \right)^{-2/c_1} B(1-2/c_1, k + 1 + 2/c_1) - \left( (k + 1) \left( 1 + x_j^{-v} \right)^{-1/c_1} B(1-1/c_1, k + 1 + 1/c_1) \right) \right]^{1/2}}{(k + 1) \left( 1 + x_j^{-v} \right)^{-2/c_1} B(1-1/c_1, k + 1 + 1/c_1)}$$

$$= \frac{\Gamma(1+2/c_1)\Gamma(k+1-2/c_1) - \Gamma(1+1/c_1)\Gamma(k+1+1/c_1)}{\Gamma(1+1/c_1)\Gamma(k+1-1/c_1) - \Gamma(1+1/c_1)\Gamma(k+1)} \quad (4.2.18)$$

This is independent of $X_j$ so is the coefficient of skewness of the conditional distributions.

Now we are interested to find some concepts useful in failure time analysis.

The scalar reversed hazard rate is

$$\lambda(x_1, x_2) \quad = \quad \frac{f(x_1, x_2)}{F(x_1, x_2)}$$

$$= \frac{k (k + 1) c_1 c_2 x_1^{-v-1} x_2^{-v-1}}{[1 + x_1^{-v} + x_2^{-v}]^2} \quad (4.2.19)$$
Vector valued reversed hazard rate (Roy(2002)) is

\[
\Delta \left( \log F(x_1, x_2) \right) = (\lambda_1(x_1, x_2), \lambda_2(x_1, x_2))
\]

where

\[
\lambda_1(x_1, x_2) = \frac{\partial}{\partial x_1} \log F(x_1, x_2)
\]

\[
= \frac{k c_1 x_1^{-\alpha_i} x_i^{1-\alpha_i}}{[1 + x_1^{-\alpha_i} + x_2^{-\alpha_i}]} \tag{4.2.20}
\]

and

\[
\lambda_2(x_1, x_2) = \frac{\partial}{\partial x_2} \log F(x_1, x_2)
\]

\[
= \frac{k c_2 x_2^{-\alpha_i} x_i^{1-\alpha_i}}{[1 + x_1^{-\alpha_i} + x_2^{-\alpha_i}]} \tag{4.2.21}
\]

The marginal reverse hazard rate is

\[
\lambda_i(x_i) = \frac{f_i(x_i)}{F_i(x_i)}
\]

\[
= \frac{k c_i x_i^{-\alpha_i}}{[1 + x_i^{-\alpha_i}]} \quad i = 1, 2 \tag{4.2.22}
\]

Basu’s (1971) failure rate is

\[
h(x_1, x_2) = \frac{f(x_1, x_2)}{R(x_1, x_2)}
\]

\[
= \frac{k (k + 1) c_1 c_2 x_1^{-\alpha_i} x_2^{-\alpha_i}}{[1 + x_1^{-\alpha_i} + x_2^{-\alpha_i}]^{k+1}} \tag{4.2.23}
\]

Gradient hazard rate (Johnson and Kotz(1975)) defined in equation (2.3.27) is given by
The marginal failure rate

\[ h_i(x_i, x_2) = -\frac{\partial}{\partial x_i} \log R(x_i, x_2) \quad i = 1, 2 \]

\[
h_i(x_i, x_2) = \frac{k c_1 x_i^{\alpha_i-1}}{[1 + x_i^{-\alpha_i}]^{\alpha_i+1}} - \frac{k c_1 x_2^{\alpha_2-1}}{[1 + x_2^{-\alpha_2}]^{\alpha_2+1}}
\]

\[
1 - [1 + x_i^{-\alpha_i}]^{-k} - [1 + x_2^{-\alpha_2}]^{-k} + [1 + x_i^{-\alpha_i} + x_2^{-\alpha_2}]^{-k}
\]

\[ (4.2.24) \]

and

\[
h_i(x_i, x_2) = \frac{k c_2 x_2^{\alpha_2-1}}{[1 + x_i^{-\alpha_i} + x_2^{-\alpha_2}]^{\alpha_2+1}} - \frac{k c_2 x_i^{\alpha_i-1}}{[1 + x_i^{-\alpha_i}]^{\alpha_i+1}}
\]

\[
1 - [1 + x_i^{-\alpha_i}]^{-k} - [1 + x_2^{-\alpha_2}]^{-k} + [1 + x_i^{-\alpha_i} + x_2^{-\alpha_2}]^{-k}
\]

\[ (4.2.25) \]

The marginal failure rate

\[
h_i(x_i) = \frac{k c_i x_i^{\alpha_i-1}}{[1 + x_i^{-\alpha_i}]^{\alpha_i+1}[1 - [1 + x_i^{-\alpha_i}]^{-k}]}
\]

\[ (4.2.26) \]

4.3 Characterizations of Bivariate Burr Type III Distribution (Bismi and Nair, 2005 d)

In this section we consider some characterization theorems of bivariate Burr type III distribution.

In problem of modeling bivariate data the primary concern is to find an appropriate distribution that explains the data adequately. Partial prior information about the mechanism is some times available in the form of marginal or conditional distributions. The problem is to determine the joint distribution. It is known that the marginal distribution alone is generally insufficient to characterize the joint distribution when the variables are independent. Therefore the specification of the joint distribution through its component densities namely marginals and conditionals have been dealt with many researchers in the past. This include the work of Seshadri.

**Theorem 4.3.1**

Let \((X_1, X_2)\) be a random vector in the support of \(R^2\) having absolutely continuous distribution function with respect to lebesgue measure, with conditional distribution of \(X_1\) given \(X_2 = x_2\) is of the form equation (4.2.4). Then \(X_1\) is Burr type III if and only if \(X_2\) is Burr type III.

**Proof**

The conditional density of \(X_1\) given \(X_2 = x_2\) is of the form equation (4.2.4).

Assume that \(X_1\) follows univariate Burr type III distribution. Then

\[
 f_1(x_1) = \frac{kc_1x_1^{-c_1-1}}{[1 + x_1^{-c_1}]^{c_1+1}} \quad 0 < x_1 < \infty, \; k, c_1 > 0
\]

Also

\[
 f_1(x_1) = \int f(x_1 \mid x_2)f_2(x_2) \, dx_2 \tag{4.3.1}
\]

Hence

\[
 \frac{kc_1x_1^{-c_1-1}}{[1 + x_1^{-c_1}]^{c_1+1}} = (k + 1)c_1 \int_0^\infty \frac{x_1^{-c_1-1}f(x_2) \, dx_2}{[1 + x_1^{-c_1}]^{c_1+2}[1 + \frac{x_2^{-c_1}}{1 + x_1^{-c_1}}]^{k+2}} \tag{4.3.2}
\]

\[
 \frac{k}{k + 1} [1 + x_1^{-c_1}] = \int_0^\infty \frac{[1 + x_2^{-c_1}]^{k+1}f(x_2) \, dx_2}{[1 + \frac{x_2^{-c_1}}{1 + x_1^{-c_1}}]^{k+2}} \tag{4.3.3}
\]

Substituting \(u = x_2^{-c_1}\) in equation (4.3.3) gives
\[
\frac{k}{k+1} [1 + x_i^{-\alpha}] = \int_0^\infty \frac{[1 + u]^{k+1} f(u^{-\frac{1}{\alpha^2}}) u^{\frac{1}{\alpha^2} - 1} du}{[1 + \frac{u}{1 + x_i^{-\alpha^2}}]^{k+2}}
\]

\[
= \int_0^\infty H(u) u_{2}^{\frac{1}{\alpha^2} - 1} du
\] (4.3.4)

Taking inverse Mellin transform ( Ryzhik Pa.1194 )

\[
H(u) = \frac{k c_2 u^{1/\alpha^2}}{[1 + \frac{u}{1 + x_i^{-\alpha^2}}]^{k+2}}
\]

Hence

\[
f_2(x_2) = \frac{k c_2 x_2^{\frac{1}{\alpha^2} - 1}}{[1 + x_2^{-\alpha^2}]^{k+1}} \quad 0 < x_2 < \infty, \ k, c_2 > 0
\]

Thus \(X_2\) is of Burr type III form.

To prove the converse, assume \(X_2\) follows univariate Burr type III.

Then

\[
f_1(x_1) = \int_0^\infty f(x_1 \mid x_2) f_2(x_2) \, dx_2
\]

\[
= k(k+1) c_1 c_2 \int_0^\infty \frac{x_1^{\frac{1}{\alpha^2} - 1} x_2^{\frac{1}{\alpha^2} - 1} dx_2}{[1 + x_1^{-\alpha^2} + x_2^{-\alpha^2}]^{k+2}}
\]

\[
= \frac{k c_1 x_1^{\frac{1}{\alpha^2} - 1}}{[1 + x_1^{-\alpha^2}]^{k+1}} \quad 0 < x_1 < \infty, \ k, c_1 > 0
\]

Hence Proof.

Apart from the marginal distribution of \(X_i\), and the conditional distribution of \(X_j\), given \(X_i = x_i, i = 1, 2 \ i \neq j\) from which the joint distribution can always found, the other quantity that are relevance to the problem is marginal and conditional
distribution of the same component. In the corollary 4.3.1 we consider a
characterization on the marginal and conditional distribution of the same component
which incidentally also provides a characterization of univariate Burr type III
distribution using bivariate Burr type III.

**Corollary 4.3.1**

Let \((X_1, X_2)\) be a random vector in the support of \(R\) having absolutely
continuous distribution function with respect to Lebesgue measure, with conditional
distribution of \(X_1\) given \(X_2 = x_2\) is of the form equation (4.2.4). Then \((X_1, X_2)\) is
Burr type III if and only if \(X_2\) is Burr type III.

It is well known that a bivariate distribution is not always determined by
marginal densities. Many researchers considered the problem of determination of joint
density when the conditional distributions are known. Abraham and Thomas (1984),
Gouriorex and Monfort (1979) have developed the condition under which the densities
\(f(x_1 | x_2)\) and \(f(x_2 | x_1)\) determine the joint density uniquely. According to Abraham
and Thomas (1984) if the ratio of the conditional density can be written as

\[
\frac{f(x_1 | x_2)}{f(x_2 | x_1)} = \frac{A_1(x_1)}{A_2(x_2)}
\]

where

\[
\int A_1(x_1) \, dx_1 = \int A_2(x_2) \, dx_2
\]

then it will uniquely determine the joint density.

Next theorem shows that bivariate Burr III distribution satisfies compatibility of
conditional densities.
Theorem 4.3.2

Let \((X_1, X_2)\) be continuous random vector in the support of \(R_2^+\) having absolutely continuous distribution function with respect to lebesgue measure. Then \((X_1, X_2)\) follows bivariate Burr type III distribution if and only if conditional densities are of the form equation (4.2.4).

Proof

Let \((X_1, X_2)\) follows bivariate Burr type III distribution. Then \(f(x_i | x_j) \ i = 1, 2 \ i \neq j\) is of the form (4.2.4)

Conversely

\[
\frac{f(x_i | x_j)}{f(x_j | x_i)} = \frac{c_1 x_1^{-\xi_1-1} [1 + x_2^{-\xi_2}]^{\beta+1}}{c_2 x_2^{-\xi_2-1} [1 + x_1^{-\xi_1}]^{\beta+1}}
\]

\[
= \frac{A_i(x_i)}{A_j(x_j)}
\]

(4.3.5)

where

\[
A_i(x_i) = \frac{c_i x_i^{-\xi_i-1}}{[1 + x_i^{-\xi_i}]^{\beta+1}} \ i = 1, 2
\]

(4.3.6)

\[
\int_0^\infty A_i(x_i) \, dx_i = \int_0^\infty A_j(x_j) \, dx_j = 1/k
\]

(4.3.7)

Hence Abraham and Thomas (1984) condition for unique determination of the joint density using conditional density is satisfied.

Hence proof.
Next we consider some characterization theorems using the relationship between scalar hazard rate, scalar reversed hazard rate, gradient hazard rate and gradient reversed hazard rate.

**Theorem 4.3.3**

Let \((X_1', X_2')\) be continuous random vector in the support of \(R^+_2\) having absolutely continuous distribution function with respect to Lebesgue measure. Then \((X_1', X_2')\) belongs to the bivariate Burr type III distribution if and only

\[
\lambda(x_1, x_2) - h(x_1, x_2) = [1 + x_1^{-c_1} + x_2^{-c_2}]^k \lambda(x_1, x_2)[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)}] \quad (4.3.8)
\]

**Proof**

Let \((X_1', X_2')\) follows to the bivariate Burr type III distribution.

Then using equation (3.2.4) in equation (4.1.3) we have equation (4.3.8).

Conversely starting from (4.3.8) and using (2.3.19), (2.3.23) and (3.2.3) we get

\[
F(x_1, x_2) = [1 + x_1^{-c_1} + x_2^{-c_2}]^k \quad 0 < x_i < \infty, k, c_i > 0 \quad i = 1, 2
\]

**Theorem 4.3.4**

Let \((X_1, X_2)\) be continuous random vector in the support of \(R^+_2\) having absolutely continuous distribution function with respect to Lebesgue measure. Then \((X_1, X_2)\) belongs to the bivariate Burr type III distribution if and only

\[
\lambda_i(x_1, x_2) + h_i(x_1, x_2) = [1 + x_1^{-c_1} + x_2^{-c_2}]^k [\frac{\lambda_i(x_1) h_i(x_1)}{\lambda_i(x_1) + h_i(x_1)} - h_i(x_1, x_2)[\frac{\lambda_1(x_1)}{\lambda_i(x_1) + h_i(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)}]] i = 1, 2 \quad (4.3.9)
\]
Proof

Let \((X_1, X_2)\) follows to the bivariate Burr type III distribution.

Then using equation (3.2.6) in equation (4.1.3) we have equation (4.3.9).

Conversely starting from (4.3.9) and using (2.3.13), (2.3.27) and (3.2.3) we get

\[
F(x_1, x_2) = [1 + x_1^{-c_1} + x_2^{-c_2}]^k \quad 0 < x_i < \infty, k, c_i > 0 \quad i = 1, 2
\]

**Theorem 4.3.5**

A continuous random vector \((X_1, X_2)\) in the support of \(R_2^c\) with distribution function \(F(x_1, x_2)\) belongs to the bivariate Burr type III distribution if

\[
\lambda(x_1, x_2) = \frac{k+1}{k} \lambda_1(x_1, x_2) \lambda_2(x_1, x_2) \quad (4.3.10)
\]

Proof

Let \((X_1, X_2)\) follows to the bivariate Burr type III distribution.

Then by equation (4.2.19), (4.2.20) and (4.2.21) we have equation (4.3.10).

**4.4 Relation between Burr Type III and Other Distributions**

Let \((X_1, X_2)\) follows to the bivariate Burr type XII distribution. Table 4.5.1 gives relation between this distribution and other distributions.

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Distribution function</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Y_i = [1 + X_i^c]^{-k})</td>
<td>([y_1^{-1/k} + y_2^{-1/k} - 1]^{-k} \quad 0 &lt; y_i &lt; 1, k &gt; 0 \quad i = 1, 2) (Cook and Johnson (1986))</td>
</tr>
<tr>
<td>(U_i = -\log X_i^c)</td>
<td>([1 + e^{-w_i} + e^{-w_i}]^{-k} \quad -\infty &lt; u_i &lt; \infty, k &gt; 0) (Burr II)</td>
</tr>
<tr>
<td>(V_i = \frac{1}{X_i})</td>
<td>([1 + v_i^{-c_i} + v_i^c]^{-k} \quad 0 &lt; v_i &lt; \infty, k, c_i &gt; 0 \quad i = 1, 2) (Burr XII)</td>
</tr>
<tr>
<td>(W_i = X_i^{-c_i})</td>
<td>([1 + w_i + w_2]^k \quad 0 &lt; w_i &lt; \infty, k &gt; 0, i = 1, 2) (Mardia (1960))</td>
</tr>
</tbody>
</table>
4.4 Bivariate Burr Type III Distribution form II (Bismi and Nair, 2005 e)

We can develop another bivariate form for the type III distribution using mixing argument.

Suppose the variables $X_i$'s $i = 1,2$ have conditional upon a common scale parameter $\theta$, independent transformed gamma distribution and $\theta$ follows Weibull distribution. Then

$$f(x_i | \theta) = \frac{c\theta^{ck_i}x_i^{ck_i-1}e^{-\theta x_i}}{\Gamma k_i} \quad 0 < x_i < \infty, \theta, c, k_i > 0 \quad i = 1,2 \quad (4.4.1)$$

and

$$f(\theta) = c\theta^{-c}e^{-\theta c} \quad \theta, c > 0 \quad (4.4.2)$$

Then the unconditional density is of the form

$$f(x_1, x_2) = \int_0^\infty f(x_1 | x_2) f(\theta) d\theta$$

$$= \frac{c^3}{\Gamma k_1\Gamma k_2} \int_0^\infty \theta^{(k_1+k_2+1-c)} e^{-\theta(1+x_1^{ck_1}+x_2^{ck_2})^\frac{1}{c}}x_1^{ck_1-1}x_2^{ck_2-1} d\theta$$

$$= \frac{c^2\Gamma(k_1+k_2+1)}{\Gamma k_1\Gamma k_2} \frac{x_1^{ck_1-1}x_2^{ck_2-1}}{[1+x_1^{ck_1}+x_2^{ck_2}]^{(k_1+k_2+1-c)}} \quad 0 < x_i < \infty, c, k_i > 0 \quad i = 1,2 \quad (4.4.3)$$

We define the distribution as bivariate Burr type III distribution.

Corresponding distribution function is

$$F(x_1, x_2)$$

$$= \frac{c^2\Gamma(k_1+k_2+1)}{\Gamma k_1\Gamma k_2} \int_0^{x_1} \int_0^{x_2} \frac{t_1^{ck_1-1}t_2^{ck_2-1}}{[1+t_1^{ck_1}+t_2^{ck_2}]^{(k_1+k_2+1-c)}} dt_1 dt_2 \quad 0 < x_i < \infty, c, k_i > 0 \quad i = 1,2 \quad (4.4.4)$$

Also this distribution can be derived under the unified approach which we have considered in chapter II by choosing
Marginal densities are

\[ f(x_i) = \frac{ck_i x_i^{c_k-1}}{(1 + x_i^{c_k})^{k+1}} \quad 0 < x_i < \infty, c, k_i > 0 \quad i = 1, 2 \quad (4.4.6) \]

Hence \( X_i, i = 1, 2 \) follows univariate Burr type III with parameters \( c \) and \( k_i \).

Conditional density of \( X_i \) given \( X_j = x_j \), \( i, j = 1, 2 \) is

\[ f(x_i | X_j = x_j) = \frac{c \Gamma(k_i + k_j + 1)}{\Gamma k_i \Gamma(k_j + 1)} \left[ \frac{x_i^{c_k}}{1 + x_i^{c_k}} \right]^{k_j-1} \quad 0 < x_i < \infty, c, k_i > 0 \quad i, j = 1, 2 \quad (4.4.7) \]

Conditional moments are given by

\[ E(x_i' | X_j = x_j) = \int_0^{\infty} x_i' f(x_i / X_j = x_j) \, dx_i \]

\[ = \frac{c \Gamma(k_i + k_j + 1)}{\Gamma k_i \Gamma(k_j + 1)} \int_0^{\infty} x_i' \left[ \frac{x_i^{c_k}}{1 + x_i^{c_k}} \right]^{k_j-1} \, dx_i \]

\[ = \left[ 1 + x_j^{c_k} \right]^{r/c} \frac{\Gamma(k_i + r/c) \Gamma(k_j + 1 - r/c)}{\Gamma k_i \Gamma(k_j + 1)} \quad i, j = 1, 2 \quad (4.4.8) \]

Regression function of \( X_i \) given \( X_j = x_j \), \( i, j = 1, 2 \) is

\[ E(x_i | X_j = x_j) = \left[ 1 + x_j^{c_k} \right]^{1/c} \frac{\Gamma(k_i + 1/c) \Gamma(k_j + 1 - 1/c)}{\Gamma k_i \Gamma(k_j + 1)} \quad i, j = 1, 2 \quad (4.4.9) \]

which is increasing in \( x_j \).

Point of intersection of two regression lines is \( (c_1(x_1, x_2), c_2(x_1, x_2)) \) where
\[ c_1(x_1, x_2) = \frac{\Gamma(k_1 + 1/c)\Gamma(k_1 + 1 - 1/c)}{\Gamma(k_1 + 1)^{1/c}} \left[ 1 + \frac{\Gamma(k_2 + 1/c)\Gamma(k_1 + 1 - 1/c)}{\Gamma(k_2 + 1/c)\Gamma(k_1 + 1)} \right]^{1/c} \]

\[ c_2(x_1, x_2) = \frac{\Gamma(k_2 + 1/c)\Gamma(k_1 + 1 - 1/c)}{\Gamma(k_2 + 1/c)^{1/c}} \left[ 1 + \frac{\Gamma(k_1 + 1/c)\Gamma(k_1 + 1 - 1/c)}{\Gamma(k_2 + 1/c)\Gamma(k_1 + 1)} \right]^{1/c} \]

and the product moment

\[ E(X_1X_2) = \frac{\Gamma(k_1 + 1/c)\Gamma(k_2 + 1/c)\Gamma(1 - 2/c)}{\Gamma(k_1)\Gamma(k_2)} \]

correlation coefficient

\[ \rho = \frac{\Gamma(k_1 + 1/c)\Gamma(k_2 + 1/c)^{1/2}}{\Gamma(k_1)^{1/2}\Gamma(k_2)^{1/2}} \left[ \Gamma(1 - 2/c) - \left( \Gamma(1 - 1/c) \right)^2 \right] \]

\[ E(X_1X_2) = \frac{\Gamma(k_1 + 1/c)\Gamma(k_2 + 1/c)\Gamma(1 - 2/c)}{\Gamma(k_1)\Gamma(k_2)} \]

Correlation tends to zero as \( c \) tends to \( \infty \).

Let \((X_1, X_2)\) has the form \((4.4.3)\). Then table \(4.4.1\) gives relation between this distribution and other distribution.

**Table 4.4.1**

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Corresponding density</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c = 1 )</td>
<td>( \frac{\Gamma(k_1 + k_2 + 1)}{\Gamma(k_1)\Gamma(k_2)} \frac{x_1^{k_1}x_2^{k_2}}{[1 + x_1 + x_2]^{k_1 + k_2 + 1}} ) 0 &lt; ( k_i &gt; 0 ) ( i = 1, 2 )</td>
</tr>
<tr>
<td></td>
<td>(Inverted Dirichlet(Tio and Guttman(1965)))</td>
</tr>
<tr>
<td>( f_i = \frac{X_i}{k_i} )</td>
<td>( \frac{\Gamma(k_1 + k_2 + 1) 2(2k_1)^{k_1} (2k_2)^{k_2} f_1^{k_1} f_2^{k_2}}{\Gamma(k_1)\Gamma(k_2)} ) 0 &lt; ( f_i &lt; \infty, k_i &gt; 0 ) ( i = 1, 2 )</td>
</tr>
<tr>
<td></td>
<td>(Bivariate F)</td>
</tr>
<tr>
<td>( k_1 = 1, k_2 = 1, Y_i = -\log X_i^c )</td>
<td>( \frac{2e^{-y_i}e^{-y_2}}{[1 + e^{-y_1} + e^{-y_2}]^3} ) 0 &lt; ( y_i &lt; \infty, k_i &gt; 0 ) ( i = 1, 2 )</td>
</tr>
<tr>
<td></td>
<td>(Bivariate logistic (Gumbel(1961)))</td>
</tr>
</tbody>
</table>
Next we consider some characterizations using conditional densities.

**Theorem 4.4.1**

Let \((X_1, X_2)\) be a random vector in the support of \(R^*_2\) having absolutely continuous distribution function with respect to lebesgue measure, with conditional distribution of \(X_1\) given \(X_2 = x_2\) is of the form equation (4.4.7). Then \(X_1\) is Burr type III if and only if \(X_2\) is Burr type III.

**Proof**

The conditional density of \(X_1\) given \(X_2 = x_2\) is of the form equation (4.4.7). Assume that \(X_1\) follows univariate Burr type III distribution. Then

\[
f_1(x_1) = \frac{k_1 c x_1^{c_k-1}}{[1 + x_1^{\alpha}]^{\beta+1}} \quad 0 < x_1 < \infty \quad k_1, c > 0
\]

Also

\[
f_1(x_1) = \int f(x_1 | x_2) f_2(x_2) \, dx_2
\]

Hence

\[
\frac{k_1 c x_1^{c_k-1}}{[1 + x_1^{\alpha}]^{\beta+1}} = \frac{\Gamma(k_1 + k_2 + 1) c^2}{\Gamma(k_1 + 1)} \int_0^{x_1^{\alpha}} \frac{f(x_1) \, dx_1}{[1 + x_1^{\alpha} + x_2^{\alpha}]^{k_1 \beta + k_2 + 1}} \quad (4.4.14)
\]

Substituting \(u = x_1^{\alpha}\) in equation (4.4.14) gives

\[
\frac{\Gamma(k_1 + 1) \Gamma k_2}{\Gamma(k_1 + k_2 + 1)} [1 + x_1^{\alpha}]^{\beta+1} = \frac{1}{k_1 c} \int_0^{x_1^{\alpha}} \frac{f(u^{\alpha/c}) \, du}{[1 + u^{\alpha}]^{k_1 \beta + k_2 + 1}}
\]

\[
= \frac{1}{k_1 c} \int_0^{x_1^{\alpha}} H(u) u^{\alpha-1} \, du \quad (4.4.15)
\]
Taking inverse Mellin transform (Ryzik Pa. 1194)

\[ H(u) = \frac{k_2 c u^{k_2-1/c}}{[1 + \frac{u}{1 + x_2}]^{k_2 + k_1 + 1}} \]

Hence

\[ f_2(x_2) = \frac{k_2 c x_2^{c k_2-1}}{[1 + x_2^{c}]^{k_2 + 1}} \quad 0 < x_2 < \infty \quad k_2, c > 0 \]

Thus \( X_2 \) is of Burr type III form.

To prove the converse, assume \( X_2 \) follows univariate Burr type III.

Then

\[ f_i(x_i) = \int_0^\infty f(x_1 | x_2) f_2(x_2) \, dx_2 \]

\[ = \frac{\Gamma(k_1 + k_2 + 1) c^2}{\Gamma k_1 \Gamma k_2 + 1} \int_0^\infty \frac{x_1^{c k_1-1} x_2^{c k_2-1}}{[1 + x_1^{c} + x_2^{c}]^{k_1 + k_2+1}} \, dx_2 \]

\[ = \frac{k c x_1^{c k_1-1}}{[1 + x_1^{c}]^{k_1+1}} \quad 0 < x_1 < \infty \quad c, k_1 > 0 \]

**Corollary 4.4.1**

Let \((X_1, X_2)\) be a random vector in the support of \( R \) having absolutely continuous distribution function with respect to lebesgue measure, with conditional distribution of \( X_1 \) given \( X_2 = x_2 \) is of the form equation (4.4.7). Then \((X_1, X_2)\) is Burr type III if and only if \( X_1 \) is Burr type III.

Next we show that the bivariate Burr III with form (4.4.1) satisfies compatibility of conditional densities.
**Theorem 4.4.2**

Let \((X_1, X_2)\) be continuous random vector in the support of \(R^2\) having absolutely continuous distribution function with respect to Lebesgue measure. Then \((X_1, X_2)\) follows bivariate Burr type III specified by equation (4.4.1) if and only if the conditional densities are of the form in equation (4.4.7)

**Proof**

Let \((X_1, X_2)\) follows bivariate Burr type III distribution. Then \(f(x_i | x_j) i = 1, 2 \ i \neq j\) is of the form (4.4.7)

Conversely

\[
\frac{f(x_i | x_j)}{f(x_j | x_i)} = \frac{k_1 x_i^{k_1-1}[1 + x_i] x_i^{k_1-1}[1 + x_i] x_i^{k_1-1}}{k_2 x_2^{k_2-1}}
\]

\[
= \frac{A_1(x_i)}{A_2(x_2)}
\]

(4.4.16)

where

\[
A_i(x_i) = \frac{k_1 x_i^{k_1-1}}{[1 + x_i] x_i^{k_1-1}} i = 1, 2
\]

(4.4.17)

\[
\int_0^\infty A_i(x_i) dx_i = \int_0^\infty A_2(x_2) dx_2
\]

\[
= 1/c
\]

(4.4.18)

Hence Abraham and Thomas (1984) condition for unique determination of the joint density using conditional density is satisfied.

Hence proof.