CHAPTER III
CHARACTERIZATIONS OF BIVARIATE BURR SYSTEM

3.1 Introduction

In modeling problems a common approach adopted is that the investigator initially chooses a family of distributions that have a wide variety of members with different shapes and characteristics and then a member of the family that is consistent with the physical properties of the system is chosen as the final model. When using the families of the distributions as the starting point, often the general properties of the family will be of considerable use in identifying the appropriate member. Accordingly there are several investigations concerning the common characteristics pertaining to various systems of distributions and any attempt at unearthing new properties is worthwhile exercise. It also helps to unify the results in the case of individual distributions that are obtained in separate studies. In view of these facts the present chapter contains the characterizations of bivariate Burr system. In terms of versatility and richness in members, Burr system appears to stand out as the best alternative among various systems of distributions.

3.2 Characterization of bivariate Burr system using reliability concepts

(Bismi and Nair 2005b)

In this section we consider characterizations of bivariate Burr system specified by equations (2.1.3) and (2.1.4) as

\[
\frac{\partial F(x_1, x_2)}{\partial x_1} = F(x_1, x_2) [1 - F(x_1, x_2)] g(x_1, x_2)
\]

and
Theorem 3.2.1

Let \((X_1, X_2)\) be continuous random vector with absolutely continuous distribution function \(F(x_1, x_2)\) in the support of \((a_i, b_i) \times (a_i, b_i), -\infty \leq a_i < b_i \leq \infty, i = 1, 2\).

Then \((X_1, X_2)\) follows bivariate Burr system specified by equations (2.2.11) if and only if

\[ \frac{\partial F(x_1, x_2)}{\partial x_2} = F(x_1, x_2) [1 - F(x_1, x_2)] \frac{g_2(x_1, x_2)}{g_2(x_1, x_2)} \]  

(3.2.1)

Proof

If part is clearly proved in equation (2.3.18).

To prove the only if part we note the form (2.2.11).

Then from equations (2.1.3) and (2.1.4) we have

\[ \lambda_i(x_1, x_2) = g_i(x_1, x_2) [1 - F(x_1, x_2)] \quad i = 1, 2 \]

or

\[ \frac{\partial F(x_1, x_2)}{\partial x_2} \frac{1}{1 - F(x_1, x_2)} = g_i(x_1, x_2) [1 - F(x_1, x_2)] \quad i = 1, 2 \]

whose solution is the Burr distribution given in the theorem.

With the extension of Burr distributions to bivariate case a problem of natural interest is to investigate how far the important properties of univariate Burr distribution can be generalized to appropriate forms in two dimensions. Nair and
Asha (2004) characterize the univariate Burr form using the relationship between hazard rate and reversed hazard rate. Following theorem gives the corresponding results in the bivariate case. In the bivariate case this can lead to four possibilities, the scalar hazard rate $h(x_1, x_2)$, the scalar reversed hazard rate $\lambda(x_1, x_2)$, vector hazard rate $\mathbf{h}(x_1, x_2)$ and vector reversed hazard rate $\mathbf{\lambda}(x_1, x_2)$.

**Theorem 3.2.2**

Let $(X_1, X_2)$ be continuous random vector with absolutely continuous distribution function $F(x_1, x_2)$ in the support of $(a_i, b_i) \times (a_i, b_i), -\infty \leq a_i < b_i \leq \infty, i = 1, 2$. Then $(X_1, X_2)$ belongs to the bivariate Burr system specified by equation (2.2.11) if and only if

\[ \lambda(x_1, x_2) - h(x_1, x_2) \left[ \frac{\lambda_1(x_1)}{\lambda_1(x_1) + \lambda_2(x_2)} + \frac{\lambda_2(x_2)}{\lambda_2(x_2) + \lambda_2(x_2)} \right] = e^{-G(x_1, x_2)} h(x_1, x_2) \left[ \frac{\lambda_1(x_1)}{\lambda_1(x_1) + \lambda_1(x_1)} - \frac{\lambda_2(x_2)}{\lambda_2(x_2) + \lambda_2(x_2)} \right] \]

**Proof**

From marginal hazard rate and marginal reversed hazard rate we can write

\[ F_i(x_i) = \frac{h_i(x_i)}{h_i(x_i) + \lambda_i(x_i)} \quad i = 1, 2 \]

(3.2.3)

Solving $F(x_1, x_2)$ from the scalar hazard rate $h(x_1, x_2)$ and scalar reversed hazard rate $\lambda(x_1, x_2)$ and using equation (3.2.3) we have

\[ F(x_1, x_2) = \frac{h(x_1, x_2) \left[ \frac{\lambda_1(x_1)}{\lambda_1(x_1) + \lambda_1(x_1)} - \frac{\lambda_2(x_2)}{\lambda_2(x_2) + \lambda_2(x_2)} \right]}{\lambda(x_1, x_2) - h(x_1, x_2)} \]

(3.2.4)
Let \((X_1, X_2)\) follows bivariate Burr system specified by equation (2.2.11).

Substituting equation (3.2.4) in equation (2.2.11), we get

\[
\lambda(x_1, x_2) - h(x_1, x_2) \left[ \frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} + \frac{\lambda_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right] = e^{-G(x_1, x_2)} \cdot h(x_1, x_2) \left[ \frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} + \frac{\lambda_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right]
\]

Conversely suppose that equation (3.2.2) holds.

Substituting \(\lambda(x_1, x_2)\) and \(h(x_1, x_2)\) from equation (2.3.19) and (2.3.23) and equation (3.2.3) in equation (3.2.2) we get

\[
F(x_1, x_2) = [1 + e^{-G(x_1, x_2)}]^{-1}
\]

which is the general solution of bivariate Burr system given in the theorem.

**Theorem 3.2.3**

A continuous random vector with absolutely continuous distribution function \(F(x_1, x_2)\) in the support of \((a_i, b_i) \times (a_i, b_i), -\infty \leq a_i < b_i \leq \infty, i=1,2\) belongs to the bivariate Burr system specified by equation (2.2.11) if and only if

\[
\lambda_i(x_1, x_2) + h_i(x_1, x_2) \left[ \frac{\lambda_i(x_1)}{\lambda_i(x_1) + h_i(x_1)} + \frac{\lambda_j(x_2)}{\lambda_j(x_2) + h_j(x_2)} \right] - \lambda_i(x_1)h_i(x_1)
\]

\[
e^{-G(x_1, x_2)} \left[ \frac{\lambda_i(x_1)h_i(x_1)}{\lambda_i(x_1) + h_i(x_1)} - h_i(x_1, x_2) \left[ \frac{h_i(x_1)}{\lambda_i(x_1) + h_i(x_1)} - \frac{\lambda_i(x_1)}{\lambda_i(x_1) + h_i(x_1)} \right] \right] i=1,2 \ (3.2.5)
\]

**Proof**

Solving \(F(x_1, x_2)\) using equations (2.3.13), (2.3.27) and (3.2.3) we find
Assume that \((X_1, X_2)\) follows bivariate Burr system specified by equation (2.2.11).

Substituting equation (3.2.6) in the general solution of bivariate Burr system specified by equation (2.2.11) we get the equation

\[
\frac{\lambda_i(x_i) h_i(x_i)}{\lambda_i(x_i) + h_i(x_i)} - h_i(x_i, x_2) \left[ \frac{\lambda_i(x_i)}{\lambda_i(x_i) + h_i(x_i)} - \frac{h_i(x_i)}{\lambda_i(x_i) + h_i(x_i)} \right] = 1, 2 \quad (3.2.6)
\]

Conversely starting from equations (3.2.5) and substituting equations (2.3.13), (2.3.27) and (3.2.3) that

\[
F(x_1, x_2) = \left[ 1 + e^{-G(x_1, x_2)} \right]^{-1}
\]

**Theorem 3.2.4**

A continuous random vector \((X_1, X_2)\) with absolutely continuous distribution function \(F(x_1, x_2)\) in the support of \((a_i, b_i) \times (a_i, b_i), -\infty \leq a_i < b_i \leq \infty, i = 1, 2\) belongs to the bivariate Burr system specified by equation (2.2.11) if and only if

\[
\frac{\lambda_i(x_1) h_i(x_1)}{\lambda_i(x_1) + h_i(x_1)} - h_i(x_1, x_2) \left[ \frac{\lambda_i(x_1)}{\lambda_i(x_1) + h_i(x_1)} - \frac{h_i(x_1)}{\lambda_i(x_1) + h_i(x_1)} \right] = 1, 2
\]

\[(3.2.7)\]

and
\[\lambda_i(x_1, x_2) [\dot{\lambda}_i(x_1, x_2) \dot{\lambda}_i(x_1, x_2) + \dot{\lambda}_j(x_1, x_2)] + \lambda(x_1, x_2) [h_i(x_1, x_2) \left( \frac{-\lambda_i(x_1)}{\lambda_2(x_1) + h_i(x_1)} \frac{\lambda_i(x_1)}{\lambda_4(x_1) + h_i(x_1)} \right) \left( \frac{-\lambda_i(x_1)}{\lambda_2(x_1) + h_i(x_1)} \frac{\lambda_i(x_1)}{\lambda_4(x_1) + h_i(x_1)} \right) \] 

\[= e^{-\frac{\lambda_i(x_1, x_2)}{2}} \lambda(x_1, x_2) \left[ \frac{\lambda_i(x_1) h_i(x_1)}{\lambda_i(x_1)} - h_i(x_1, x_2) \left( \frac{-\lambda_i(x_1)}{\lambda_2(x_1) + h_i(x_1)} \frac{\lambda_i(x_1)}{\lambda_4(x_1) + h_i(x_1)} \right) \right] i = 1, 2 \] (3.2.8)

where

\[\dot{\lambda}_i(x_1, x_2) = \frac{\partial \lambda_i(x_1, x_2)}{\partial x_j} \quad i, j = 1, 2 \quad i \neq j\]

**Proof**

Solving \( F(x_1, x_2) \) from equations (2.3.13), (2.3.19), (2.3.27) and (3.2.3) we find

\[ F(x_1, x_2) \]

\[= \frac{\lambda(x_1, x_2) \left[ \frac{\lambda_i(x_1) h_i(x_1)}{\lambda_i(x_1)} - h_i(x_1, x_2) \left( \frac{-\lambda_i(x_1)}{\lambda_2(x_1) + h_i(x_1)} \frac{\lambda_i(x_1)}{\lambda_4(x_1) + h_i(x_1)} \right) \right]}{\lambda_i(x_1, x_2) \left[ \lambda_i(x_1) \lambda_2(x_1, x_2) + \lambda_4(x_1, x_2) \right] + \lambda(x_1, x_2) h_i(x_1, x_2)} \]

\[= \frac{\lambda(x_1, x_2) \left[ \frac{\lambda_i(x_1) h_i(x_1)}{\lambda_i(x_1)} - h_i(x_1, x_2) \left( \frac{-\lambda_i(x_1)}{\lambda_2(x_1) + h_i(x_1)} \frac{\lambda_i(x_1)}{\lambda_4(x_1) + h_i(x_1)} \right) \right]}{\lambda_i(x_1, x_2) \left[ \lambda_i(x_1) \lambda_2(x_1, x_2) + \lambda_4(x_1, x_2) \right] + \lambda(x_1, x_2) h_i(x_1, x_2)} \]

\[= \frac{\lambda(x_1, x_2) \left[ \frac{\lambda_i(x_1) h_i(x_1)}{\lambda_i(x_1)} - h_i(x_1, x_2) \left( \frac{-\lambda_i(x_1)}{\lambda_2(x_1) + h_i(x_1)} \frac{\lambda_i(x_1)}{\lambda_4(x_1) + h_i(x_1)} \right) \right]}{\lambda_i(x_1, x_2) \left[ \lambda_i(x_1) \lambda_2(x_1, x_2) + \lambda_4(x_1, x_2) \right] + \lambda(x_1, x_2) h_i(x_1, x_2)} \]

Substituting equation (3.2.9) in equation (2.2.11) we get

\[\lambda_i(x_1, x_2) [\dot{\lambda}_i(x_1, x_2) \dot{\lambda}_i(x_1, x_2) + \dot{\lambda}_j(x_1, x_2)] + \lambda(x_1, x_2) [h_i(x_1, x_2) \left( \frac{-\lambda_i(x_1)}{\lambda_2(x_1) + h_i(x_1)} \frac{\lambda_i(x_1)}{\lambda_4(x_1) + h_i(x_1)} \right) \left( \frac{-\lambda_i(x_1)}{\lambda_2(x_1) + h_i(x_1)} \frac{\lambda_i(x_1)}{\lambda_4(x_1) + h_i(x_1)} \right) \]

\[= e^{-\frac{\lambda_i(x_1, x_2)}{2}} \lambda(x_1, x_2) \left[ \frac{\lambda_i(x_1) h_i(x_1)}{\lambda_i(x_1)} - h_i(x_1, x_2) \left( \frac{-\lambda_i(x_1)}{\lambda_2(x_1) + h_i(x_1)} \frac{\lambda_i(x_1)}{\lambda_4(x_1) + h_i(x_1)} \right) \right] i = 1, 2 \]

Similarly equation (3.2.10) in equation (2.2.11)
\begin{align*}
&\lambda_1(x_1, x_2)[\lambda_2(x_1, x_2)\lambda_3(x_1, x_2) + \lambda_4(x_1, x_2)] \\
&+ \lambda(x_1, x_2)[h_1(x_1, x_2)[\frac{\lambda_3(x_1)}{\lambda_1(x_1) + h_1(x_1)} + \frac{\lambda_4(x_1)}{\lambda_2(x_1) + h_2(x_1)}] - \frac{\lambda_1(x_1)h_1(x_1)}{\lambda_1(x_1) + h_1(x_1)}] \\
&= e^{-G(x_1, x_2)} \lambda(x_1, x_2)[\frac{\lambda_3(x_1)h_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - h_1(x_1, x_2)[\frac{\lambda_4(x_1)}{\lambda_2(x_1) + h_2(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_1) + h_2(x_1)}] \quad i = 1, 2 \end{align*}

Conversely starting from equation (3.2.7) and using equations (2.3.13), (2.3.19), (2.3.27) and (3.2.3) gives

\[ F(x_1, x_2) = \left[1 + e^{-G(x_1, x_2)}\right]^{-1} \]

Also starting from equation (3.2.8) and proceeding on same way we get

\[ F(x_1, x_2) = \left[1 + e^{-G(x_1, x_2)}\right]^{-1} \]

Hence the result.

**Theorem 3.2.5**

A continuous random vector \((X_1, X_2)\) with absolutely continuous distribution function \(F(x_1, x_2)\) in the support of \((a_i, b_i) \times (a_j, b_j), -\infty \leq a_i < b_i \leq \infty, i = 1, 2\) belongs to the bivariate Burr system specified by equation (2.2.11) if and only if

\[ \lambda_1(x_1, x_2)\lambda_2(x_1, x_2) + \lambda_3(x_1, x_2) - h_1(x_1, x_2)[\frac{\lambda_4(x_1)}{\lambda_1(x_1) + h_1(x_1)} + \frac{\lambda_2(x_2)}{\lambda_2(x_1) + h_2(x_1)}] \]

\[ = e^{-G(x_1, x_2)} h(x_1, x_2)[\frac{\lambda_3(x_1)}{\lambda_4(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)}] \quad (3.2.11) \]

and

\[ \lambda_1(x_1, x_2)\lambda_2(x_1, x_2) + \lambda_3(x_1, x_2) - h_1(x_1, x_2)[\frac{\lambda_4(x_1)}{\lambda_1(x_1) + h_1(x_1)} + \frac{\lambda_2(x_2)}{\lambda_2(x_1) + h_2(x_1)}] \]

\[ = e^{-G(x_1, x_2)} h(x_1, x_2)[\frac{\lambda_3(x_1)}{\lambda_4(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_1)}] \quad (3.2.12) \]
Proof

Solving \( F(x_1, x_2) \) using scalar hazard rate in equation (2.3.19) vector valued reversed hazard rate in equation (2.3.23) and equation (3.2.3) we get

\[
F(x_1, x_2) = \frac{h(x_1, x_2)[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{\lambda_2(x_2)}{\lambda_2(x_2) + h_2(x_2)}]}{\lambda_1(x_1, x_2) \lambda_2(x_1, x_2) + \lambda_1(x_1, x_2) - h(x_1, x_2)} \quad (3.2.13)
\]

\[
F(x_1, x_2) = \frac{h(x_1, x_2)[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{\lambda_2(x_2)}{\lambda_2(x_2) + h_2(x_2)}]}{\lambda_1(x_1, x_2) \lambda_2(x_1, x_2) + \lambda_1(x_1, x_2) - h(x_1, x_2)} \quad (3.2.14)
\]

Let \((X_1, X_2)\) follows bivariate Burr system specified by equations (2.2.11). Substituting equation (3.2.13) in the general solution of bivariate Burr system specified by equation (2.2.11) we get the equation

\[
\lambda_1(x_1, x_2) \lambda_2(x_1, x_2) + \lambda_3(x_1, x_2) - h(x_1, x_2) \left[ \frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} + \frac{\lambda_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right]
\]

\[
= e^{-G(x_1, x_2)} h(x_1, x_2) \left[ \frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right]
\]

Substituting equation (3.2.14) in equation (2.2.11) we get

\[
\lambda_1(x_1, x_2) \lambda_2(x_1, x_2) + \lambda_3(x_1, x_2) - h(x_1, x_2) \left[ \frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} + \frac{\lambda_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right]
\]

\[
= e^{-G(x_1, x_2)} h(x_1, x_2) \left[ \frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)} \right]
\]

Conversely suppose that equation (3.2.11) and (3.2.12) holds. Then starting from equation (3.2.11) and using equations (2.3.13), (2.3.23) and (3.2.3) gives

\[
F(x_1, x_2) = [1 + e^{-G(x_1, x_2)}]^{-1}
\]
Starting from equation (3.2.12) and using equations (2.3.13), (2.3.23) and (3.2.3) gives

\[ F(x_1, x_2) = [1 + e^{-g(x_1, x_2)}]^{-1} \]

**Theorem 3.2.6**

Let \((X_1, X_2)\) be a continuous random vector with absolutely continuous distribution function \(F(x_1, x_2)\) in the support of \((a_i, b_i) \times (a_j, b_j), -\infty < a_i < b_i <\infty, i = 1, 2\). Then \((X_1, X_2)\) belongs to the bivariate Burr system specified by equations (2.1.3) and (2.1.4) if and only if

\[ g_i(x_1, x_2) = \frac{\lambda_i(x_1, x_2) [\lambda_i(x_1, x_2) - h(x_1, x_2)]}{\lambda_i(x_1, x_2) - h(x_1, x_2)} \quad i = 1, 2 \quad (3.2.15) \]

**Proof**

Let \((X_1, X_2)\) belongs to the bivariate Burr system specified by equations (2.1.3) and (2.1.4)

Using the identity (3.2.4) and (2.3.13) in equations (2.1.3) and (2.1.4) gives

\[ 1 - \frac{\lambda_i(x_1, x_2)}{g_i(x_1, x_2)} = \frac{h(x_1, x_2) [\lambda_i(x_1, x_2) - h(x_1, x_2)]}{\lambda_i(x_1, x_2) - h(x_1, x_2)} \quad i = 1, 2 \quad (3.2.16) \]

which on simplification gives

\[ g_i(x_1, x_2) = \frac{\lambda_i(x_1, x_2) [\lambda_i(x_1, x_2) - h(x_1, x_2)]}{\lambda_i(x_1, x_2) - h(x_1, x_2) \left[ \frac{\lambda_i(x_1)}{\lambda_i(x_1) + h_i(x_1)} + \frac{\lambda_i(x_2)}{\lambda_i(x_2) + h_i(x_2)} \right]} \quad i = 1, 2 \]

Conversely suppose that equation (3.2.15) holds

Then using equations (2.3.13), (2.3.19), (2.3.23) and (3.2.3) we get
\[ g_i(x_1, x_2) = \frac{\frac{\partial F(x_1, x_2)}{\partial x_i}}{F(x_1, x_2)[1 - F(x_1, x_2)]} \quad i = 1, 2 \]

which proves the result.

**Theorem 3.2.7**

Let \((X_1, X_2)\) be a continuous random vector with absolutely continuous distribution function \(F(x_1, x_2)\) in the support of \((a_i, b_i) \times (a_i, b_i), -\infty \leq a_i < b_i \leq \infty, i = 1, 2\). Then \((X_1, X_2)\) belongs to the bivariate Burr system specified by equations (2.1.3) and (2.1.4) if and only if

\[ g_i(x_1, x_2) \]

\[ \frac{\lambda_i(x_1, x_2)[\lambda_i(x_1, x_2) + h_i(x_1, x_2)]}{\lambda_i(x_1, x_2) + h_i(x_1, x_2)[\frac{\lambda_i(x_1)}{\lambda_i(x_1) + h_i(x_1)} + \frac{\lambda_i(x_2)}{\lambda_i(x_2) + h_i(x_2)}] - \frac{\lambda_i(x_1)h_i(x_1)}{\lambda_i(x_1) + h_i(x_1)}} \quad i = 1, 2 \]

**Proof**

Suppose that \((X_1, X_2)\) belongs to the bivariate Burr system specified by equations (2.1.3) and (2.1.4).

Using the identity (3.2.6) and (2.3.15) we find

\[ 1 - \frac{\lambda_i(x_1, x_2)}{g_i(x_1, x_2)} = \frac{\frac{\lambda_i(x_1)h_i(x_1)}{\lambda_i(x_1) + h_i(x_1)} - h_i(x_1, x_2)[\frac{\lambda_i(x_1)}{\lambda_i(x_1) + h_i(x_1)} - \frac{h_i(x_2)}{\lambda_i(x_2) + h_i(x_2)}]}{[\lambda_i(x_1, x_2) + h_i(x_1, x_2)]} \quad i = 1, 2 \]

which gives

\[ g_i(x_1, x_2) \]

\[ = \frac{\lambda_i(x_1, x_2)[\lambda_i(x_1, x_2) + h_i(x_1, x_2)]}{\lambda_i(x_1, x_2) + h_i(x_1, x_2)[\frac{\lambda_i(x_1)}{\lambda_i(x_1) + h_i(x_1)} + \frac{\lambda_i(x_2)}{\lambda_i(x_2) + h_i(x_2)}] - \frac{\lambda_i(x_1)h_i(x_1)}{\lambda_i(x_1) + h_i(x_1)}} \quad i = 1, 2 \]
Conversely starting from equation (3.2.17) and using equations (2.3.13), (2.3.27) and (3.2.3) we have

\[ g_i(x_1, x_2) = \frac{\partial F(x_1, x_2)}{\partial x_i} F(x_1, x_2)[1 - F(x_1, x_2)] \quad i = 1, 2 \]

which proves the result.

**Theorem 3.2.8**

Let \((X_1, X_2)\) be continuous random vector with absolutely continuous distribution function \(F(x_1, x_2)\) in the support of \((a_i, b_i) \times (a_2, b_2), -\infty \leq a_i < b_i \leq \infty, i = 1, 2\). Then \((X_1, X_2)\) belongs to the bivariate Burr system specified by equations (2.1.3) and (2.1.4) if and only if

\[
\begin{align*}
\lambda_i(x_1, x_2) &\left[ A_i(x_1) A_i(x_2) + \lambda_i(x_1) \lambda_i(x_2) h_i(x_1, x_2) \right] - i(X_1, x_2) + \lambda_i(x_1) \lambda_i(x_2) h_i(x_1, x_2) \\
\lambda_i(x_1, x_2) &\left[ A_i(x_1) A_i(x_2) + \lambda_i(x_1) \lambda_i(x_2) h_i(x_1, x_2) \right] - i(X_1, x_2) + \lambda_i(x_1) \lambda_i(x_2) h_i(x_1, x_2)
\end{align*}
\]

and

\[
\begin{align*}
\lambda_i(x_1, x_2) &\left[ A_i(x_1) A_i(x_2) + \lambda_i(x_1) \lambda_i(x_2) h_i(x_1, x_2) \right] - i(X_1, x_2) + \lambda_i(x_1) \lambda_i(x_2) h_i(x_1, x_2) \\
\lambda_i(x_1, x_2) &\left[ A_i(x_1) A_i(x_2) + \lambda_i(x_1) \lambda_i(x_2) h_i(x_1, x_2) \right] - i(X_1, x_2) + \lambda_i(x_1) \lambda_i(x_2) h_i(x_1, x_2)
\end{align*}
\]

**Proof**

Let \((X_1, X_2)\) belongs to the bivariate Burr system specified by equations (2.1.3) and (2.1.4)

Using equations (3.2.9) and (2.3.13) in equations (2.1.3) and (2.1.4) gives
\[ 1 - \frac{\hat{\lambda}(x_1, x_2)}{g_i(x_1, x_2)} \]
\[ = \frac{\lambda(x_1, x_2)[\frac{\hat{\lambda}(x_1) h(x_1) - h_i(x_1, x_2) \left( \frac{\hat{\lambda}(x_1)}{\hat{\lambda}(x_1) + h(x_1)} - \frac{h_i(x_1, x_2)}{\hat{\lambda}(x_1) + h_i(x_1, x_2)} \right)]}{\lambda_i(x_1, x_2)[\lambda_i(x_1, x_2) + \hat{\lambda}_2(x_1, x_2) - h(x_1, x_2)\left( \frac{\hat{\lambda}(x_1)}{h_1(x_1) + \hat{\lambda}(x_1)} + \frac{\hat{\lambda}_2(x_1)}{h_2(x_1) + \hat{\lambda}_2(x_1)} \right)]} \]
\[ i = 1, 2 \quad (3.2.21) \]

which on simplification gives equation (3.2.19).

Similarly using equation (3.2.10) and (2.3.13) in equation (2.1.3) and (2.1.4) gives equation (3.2.20).

Conversely suppose that equations (3.2.19) and (3.2.20) hold.

Then using equations (2.3.13), (2.3.19), (2.3.27) and (3.2.3) we get

\[ g_i(x_1, x_2) = \frac{\partial F(x_1, x_2)}{F(x_1, x_2)[1 - F(x_1, x_2)]} \quad i = 1, 2 \]

which proves the result.

**Theorem 3.2.9**

Let \((X_1, X_2)\) be continuous random vector with absolutely continuous distribution function \(F(x_1, x_2)\) in the support of \((a_1, b_1) \times (a_2, b_2), -\infty \leq a_i < b_i \leq \infty, i = 1, 2\). Then \((X_1, X_2)\) belongs to the bivariate Burr system specified by equations (2.1.3) and (2.1.4) if and only if

\[ g_i(x_1, x_2) \]
\[ = \frac{\lambda_i(x_1, x_2)[\lambda_i(x_1, x_2)\lambda_i(x_1, x_2) + \lambda_i(x_1, x_2) - h(x_1, x_2)]}{\lambda_i(x_1, x_2) + \lambda_i(x_1, x_2) - h(x_1, x_2)\left( \frac{\lambda_i(x_1)}{h_1(x_1) + \lambda_i(x_1)} + \frac{\lambda_i(x_2)}{h_2(x_1) + \lambda_i(x_2)} \right)} \quad i = 1, 2 \quad (3.2.22) \]

and
Proof

Let \((X_1, X_2)\) belongs to the bivariate Burr system specified by equations (2.1.3) and (2.1.4).

Using equations (3.2.13) and (2.3.13) in equations (2.1.3) and (2.1.4) gives

\[
g_i(x_1, x_2) = \frac{\lambda_i(x_1, x_2)[\lambda_1(x_1, x_2)\lambda_2(x_1, x_2) + \lambda_1(x_1, x_2) - h(x_1, x_2)]}{\lambda_1(x_1, x_2)\lambda_2(x_1, x_2) + \lambda_1(x_1, x_2) - h(x_1, x_2)[\frac{\lambda_1(x_1)}{h_1(x_1)} + \frac{\lambda_2(x_1)}{h_2(x_1)}]} \quad i = 1, 2 \quad (3.2.23)
\]

which on simplification results equation(3.2.22).

Similarly using equation (3.2.14) and (2.3.13) in equation (2.1.3) and (2.1.4) gives the result in equation (3.2.23).

Conversly suppose that equations (3.2.22) and (3.2.23) hold.

Then using equations (2.3.13), (2.3.19) and (3.2.3) we get

\[
g_i(x_1, x_2) = \frac{\partial F(x_1, x_2)}{\partial x_i} \quad i = 1, 2
\]

which proves the result.

Next theorem shows that bivariate Burr system satisfies compatibility of conditional densities.
3.3 Characterization using conditional densities

**Theorem 3.3.1**

Let \((X_1, X_2)\) be continuous random vector with absolutely continuous distribution function \(F(x_1, x_2)\) in the support of \((a_i, b_i) \times (a_j, b_j), -\infty < a_i < b_i \leq \infty, i = 1, 2\). Then \((X_1, X_2)\) belongs to the bivariate Burr system specified by equations (2.2.11) if and only if the conditional densities are of the form equation (2.3.8) and (2.3.9).

**Proof**

Let the random vector \((X_1, X_2)\) belongs to the bivariate Burr family specified by equations (2.2.11). Then conditional densities are of the form (2.3.8) and (2.3.9). Conversely suppose that conditional densities are of the form (2.3.8) and (2.3.9).

Then

\[
\frac{f(x_1/x_2)}{f(x_1/x_1)} = \frac{e^{-G(x_1,b_1)} g_1(x_1,b_1) [1 + e^{-G(x_1,b_1)}]^{\frac{1}{x_1}}}{[1 + e^{-G(x_1,b_1)}]^{\frac{1}{x_1}}} e^{-G(x_1,b_1)} g_1(b_1,x_1)
\]

\[
= \frac{A_1(x_1)}{A_2(x_2)} \quad (3.3.1)
\]

where

\[
A_1(x_1) = \frac{e^{-G(x_1,b_1)} g_1(x_1,b_1)}{[1 + e^{-G(x_1,b_1)}]^{\frac{1}{x_1}}} \quad (3.3.2)
\]

and

\[
A_2(x_2) = \frac{e^{-G(x_2,b_2)} g_2(b_1,x_2)}{[1 + e^{-G(x_2,b_2)}]^{\frac{1}{x_2}}} \quad (3.3.3)
\]
\[ \int_{a_1}^{b_1} A_i(x_1) \, dx_1 = \int_{a_2}^{b_2} A_i(x_2) \, dx_2 \quad (3.3.4) \]

Hence Abraham and Thomas (1984) conditions are satisfied and therefore the bivariate distribution has bivariate Burr form.