5.1 Introduction

The work on the univariate and bivariate Burr distributions were mainly centered on the type XII distribution. In two dimensional case, the type XII introduced by Takahasi (1965) was later studied by Durling (1969, 1974), Johnson and Kotz (1981), Crowder (1985), Crowder and Kimber (1997) and Begum and Khan (1998). In view of the importance of this distribution has enjoyed and the volume of work it has produced in literature we take up a detailed study of the type XII distribution under the new frame work, introduced in chapter II. The logic used in the derivation of bivariate Burr type XII distribution was the mixing argument. But it was pointed out in the second chapter that under a unified frame work the entire Burr system of distributions can be conceived as the solution of a set of partial differential equations, involving distribution function, so that the system contains, besides the generalization of the twelve types in the univariate case, many more absolutely continuous distributions.

It may be noted that apart from specifying the distribution functions much work has not been under taken on the type IX distribution in the univariate set up. As a new probability model with potential for application we discuss this model and bring about some of its salient characteristics. Also in this chapter we have discussed the type II distribution which we have introduced in chapter II.
5.2 General Properties of Type XII Model

In the unified approach the bivariate Burr XII distribution arises by the choice of \( G(x_1, x_2) \) as

\[
G(x_1, x_2) = \log \frac{1 - [1 + x_1^c]^{-k} - [1 + x_2^c]^{-k} + [1 + x_1^c + x_2^c]^{-k}}{[1 + x_1^c]^{-k} + [1 + x_2^c]^{-k} - [1 + x_1^c + x_2^c]^{-k}} \tag{5.2.1}
\]

in equation (2.2.11). We can write the distribution function of bivariate Burr XII distribution in the form

\[
F(x_1, x_2) = 1 - [1 + x_1^c]^{-k} - [1 + x_2^c]^{-k} + [1 + x_1^c + x_2^c]^{-k} \quad 0 < x_i < \infty, k, c_i > 0, i = 1, 2 \tag{5.2.2}
\]

Corresponding density function and survival function are

\[
f(x_1, x_2) = \frac{k(k+1)c_1c_2x_1^{c_1-1}x_2^{c_2-1}}{[1 + x_1^c + x_2^c]^{k+2}} \quad 0 < x_i < \infty, k, c_i > 0, i = 1, 2 \tag{5.2.3}
\]

and

\[
R(x_1, x_2) = [1 + x_1^c + x_2^c]^{-k} \quad 0 < x_i < \infty, k, c_i > 0, i = 1, 2 \tag{5.2.4}
\]

A closely related form of the distribution is discussed in Johnson and Kotz (1972) which is obtained by replacing \( x_i^c \) in equation (5.2.3) by \( \alpha x_i^c \) for \( i = 1, 2 \)

Our derivation of the distribution based on the choice of the functional form \( G(x_1, x_2) \) ensures that the marginals are exactly Burr type XII.

Next we consider some general properties of bivariate Burr XII distribution specified in equation (5.2.3)

It is noted that the marginal distributions are

\[
F_i(x_i) = 1 - [1 + x_i^c]^{-k} \quad 0 < x_i < \infty, k, c_i > 0, i = 1, 2 \tag{5.2.5}
\]

with a choice of
\[ g_i(x_i) = \frac{kc_i x_i^{s_i - 1}}{[1 + x_i^c][1 - [1 + x_i^c]^{-k}]} \quad (5.2.6) \]

Equation (5.2.5) satisfies

\[ F_i(x_i) \left[ 1 - F_i(x_i) \right] = \frac{dF_i(x_i)}{dx_i} \]

\[ = \frac{dF_i(x_i)}{dx_i} g_i(x_i) \quad i = 1, 2 \quad (5.2.7) \]

which is univariate Burr type differential equation in which marginal densities can be written as

\[ \frac{dF_i(x_i)}{dx_i} = f_i(x_i) \]

\[ = \frac{kc_i x_i^{s_i - 1}}{[1 + x_i^c]^{k+1}} \quad 0 < x_i < \infty, k, c > 0 \quad i = 1, 2 \quad (5.2.8) \]

Thus for the bivariate Burr form (5.2.3) marginals are exactly univariate Burr type XII With the above marginal distributions, conditional densities of \( X_i \) given \( X_j = x_j \) arise as

\[ f(x_i | X_j = x_j) = \frac{(k + 1)c_i x_i^{s_i - 1}(1 + x_j^{c_j})^{k+1}}{[1 + x_i^c + x_j^{c_j}]^{k+2}} \quad 0 < x_i < \infty, k, c_i, c_j > 0, i, j = 1, 2 \quad (5.2.9) \]

Using the transformation

\[ Y_i = \frac{X_i}{(1 + X_j^{c_j})^{1/c_j}} \]

it can be seen that \( Y_i \) follows univariate Burr type XII with parameters \( c \) and \( (k+1) \). Hence any property for univariate Burr distribution of \( X_i \) can be extended to the conditional distribution of \( X_i \) given \( X_j = x_j \).
Another type of conditional distribution that of interest especially in reliability modeling is the distribution of $X_1$ given $X_J > x_J$.

Survival function of $X_1$ given $X_2 > x_2$ is

$$R(x_1 | X_2 > x_2) = \frac{P(x_1 | X_2 > x_2)}{P(X_2 > x_2)}$$

$$= \frac{P(X_1 > x_1, X_2 > x_2)}{P(X_2 > x_2)}$$

$$= \frac{[1 + x_1^n + x_2^n]^{-k}}{[1 + x_2^n]^{-k}}$$

$$= [1 + \left(\frac{x_1}{a_2(x_2)}\right)^n]^{-k} \quad (5.2.10)$$

$$= [1 + \left(\frac{x_1}{a_2(x_2)}\right)^n]^{-k} \quad (5.2.11)$$

where

$$a_2(x_2) = [1 + x_2^n]^{1/n} \quad (5.2.12)$$

The corresponding density function is calculated as

$$f(x_1 | X_2 > x_2) = \frac{\partial R(x_1 | X_2 > x_2)}{\partial x_1}$$

$$= \frac{k c_1}{a_2(x_2)} \frac{\left(\frac{x_1}{a_2(x_2)}\right)^{k-1}}{[1 + \left(\frac{x_1}{a_2(x_2)}\right)^n]^{k+1}} \quad (5.2.13)$$

Similarly

$$R(x_2 | X_1 > x_1) = \frac{P(x_2 | X_1 > x_1)}{P(X_1 > x_1)}$$

$$= \frac{P(X_1 > x_1, X_2 > x_2)}{P(X_1 > x_1)}$$
where
\[ a_2(x_i) = [1 + x_i^a]^{\alpha} \]  
(5.2.15)

and
\[ \frac{\partial R(x_2 \mid X_1 > x_j)}{\partial x_2} = \frac{kc_2}{a_2(x_i)} \left[ \frac{x_2}{a_2(x_i)} \right]^{\alpha + 1} \frac{\left[ 1 + \left[ \frac{x_2}{a_2(x_i)} \right]^{\alpha} \right]^{\alpha + 1}}{\left[ 1 + \frac{x_2}{a_2(x_i)} \right]^{\alpha + 1}} \]  
(5.2.16)

An interesting point to be noted is the relationship between the conditional distribution of \( X \) given \( X_j = x_j \) and that of \( X \) given \( X_j > x_j \). The former has Burr form with parameters \((c, k+1)\) while the latter is Burr with parameters \((c, k)\) respectively. This enables us to write variety properties involving \( X \) given \( X_j = x_j \) and \( X \) given \( X_j > x_j \), like the mean, variance, coefficient of variation, skewness etc, some which is independent of the condition involved.

A second useful feature of the above conditional distributions is that they satisfy the differential equations
\[ \frac{\partial R(x_i \mid X_j > x_j)}{\partial x_i} = R(x_i \mid X_j > x_j)[1 - R(x_i \mid X_j > x_j)]g_1(x_i, x_j) \]  
(5.2.17)
\[ \frac{\partial R(x_2 \mid X_1 > x_j)}{\partial x_2} = R(x_2 \mid X_1 > x_j)[1 - R(x_2 \mid X_1 > x_j)]g_2(x_1, x_2) \]  
(5.2.18)

with a choice of
Now we are interested to find the moments and other characteristics.

5.3 Moments and Other Characteristics of Burr Type XII Distribution

Because of the transformation pointed out in the previous section, that induces a relation between the marginal and conditional distributions, many properties of the bivariate distribution can be established without appealing to the bivariate density function. Apart from the mathematical convenience the approach also brings about some results that are useful in reliability context. For example when \((X_1, X_2)\)
represents the random life times of a two component system

\[
m_i(x_j) = E(X_i | X_j > x_j) \quad i, j = 1, 2 \quad i \neq j
\]

(5.3.1)

represents the mean true to failure (MTTF) of the \(i^{th}\) component when the \(j^{th}\) component has survivor time \(x_j\).

The expression for \(m_i(x_2)\) is

\[
m_i(x_2) = \int_0^\infty f(x_1 | X_2 > x_2) \, dx_1
\]

\[
= \int_0^\infty R(x_1 | X_2 > x_2) \, dx_1
\]

\[
= \int_0^\infty \left[ 1 + \frac{x_1^{\xi_1}}{1 + x_2^{\xi_2}} \right]^{-\eta_1} \, dx_1
\]

\[
= \int_0^\infty \left[ 1 + \frac{x_1^{\xi_1}}{1 + x_2^{\xi_2}} \right]^{-\eta_1} \, dx_1
\]

(5.2.19)
which is increasing function of $x_j$.

Means can be directly calculated from $m_i(x_j)$ as

$$E(x_j) = m_i(0^+) = \frac{\Gamma(k - 1/c_i)\Gamma(1 + 1/c_i)}{\Gamma k} \quad c_i > 1/k \quad i = 1, 2 \quad (5.3.3)$$

Further we have following expression for the variance

$$V(x_j) = \frac{\Gamma(k - 2/c_i)\Gamma(1 + 2/c_i)}{\Gamma k} - \left[\frac{\Gamma(k - 1/c_i)\Gamma(1 + 1/c_i)}{\Gamma k}\right]^2 \quad i = 1, 2 \quad (5.3.4)$$

Similarly

$$m_2(x_i) = \int_0^\infty f(x_2 | X_i > x_i) \, dx_2$$

$$= \int_0^\infty R(x_2 | X_i > x_i) \, dx_2$$

$$= \frac{(1 + x_i^{c_2})^{1/c_2}}{c_2} B(1/c_2, k - 1/c_2) \quad (5.3.5)$$

which is increasing function of $x_i$.

This means that the mean life time of component $X_i$ can be increased by increasing the value of component $j$.

The $(r_1, r_2)$th moment of the distribution

$$\mu_{r_1, r_2} = E(X_1^{r_1}X_2^{r_2})$$

$$= \int \int x_1^{r_1} x_2^{r_2} f(x_1, x_2) \, dx_1 \, dx_2$$
In particular the product moment becomes

$$E(X_i X_j) = \frac{1}{c_1 c_2} \frac{1}{\Gamma k} \Gamma(1/c_1) \Gamma(1/c_2) \Gamma(k - 1/c_1 - 1/c_2) \quad 1/c_i > 0, k > 1/c_i + 1/c_2, i = 1, 2 \quad (5.3.7)$$

There is a recurrence relation connecting the moments of the distribution given by

$$\mu_{n-r_i, r_j} = \frac{1}{\Gamma k} \Gamma(1 + (r_i + c_1)/c_1)(1 + (r_j + c_1)/c_1) \Gamma(k - (r_i + c_1)/c_1 - (r_j + c_2)/c_2) = \frac{1}{\Gamma k} \Gamma(2 + r_i/c_1) \Gamma(2 + r_j/c_2) \Gamma(k - r_i/c_1 - r_j/c_2 - 2)$$

$$= \frac{(1 + r_i/c_1)(1 + r_j/c_2)}{(k - r_i/c_1 - r_j/c_2 - 1)(k - r_i/c_1 - r_j/c_2 - 2)/\Gamma k} \Gamma(1 + r_i/c_1) \Gamma(1 + r_j/c_2) \Gamma(k - r_i/c_1 - r_j/c_2) = \frac{(1 + r_i/c_1)(1 + r_j/c_2)}{(k - r_i/c_1 - r_j/c_2 - 1)(k - r_i/c_1 - r_j/c_2 - 2)/\Gamma k} \mu_{n-r_i, r_j} \quad (5.3.8)$$

When $c_1$ and $c_2$ are positive integers, this relation connects the adjacent moments and is useful to calculate all moments of the distribution devoid of gamma functions.

Covariance becomes...
Then the coefficient of correlation has the expression,

\[
\rho = \frac{\Gamma(1+1/c_1)\Gamma(1+1/c_2)}{\Gamma k [\Gamma(k-1/c_1)\Gamma(k-1/c_2)]} \frac{\Gamma(k-1/c_1)\Gamma(k-1/c_2)}{\Gamma k} \quad 1/c_1 > 0, k > 1/c_1 + 1/c_2, i = 1, 2 \quad (5.3.9)
\]

Regression equations are obtained by using the transformation

\[
Y_i = \frac{X_i}{(1 + X_i^{\alpha})^{1/c_i}} \quad \text{discussed earlier.}
\]

\[
E(x_i | X_2 = x_2) = \int_0^{\alpha} x_i f(x_i | X_2 = x_2) \, dx_i
\]

\[
= \int_0^{\alpha} y_1 f(y_1 | X_2 = x_2) \, dy_1
\]

\[
= \int_0^{\alpha} y_1^{\alpha_i} (1 + x_2^{\alpha_i})^{y_1^{\alpha_i}} \, dy_1
\]

\[
= (k + 1)(1 + x_2^{\alpha_i})^{y_1^{\alpha_i}} B(1 + 1/c_1, k + 1 - 1/c_1) \quad (5.3.11)
\]

which is increasing function of \(x_2\).

Similarly

\[
E(x_2 | X_1 = x_1) = (k + 1)(1 + x_1^{\alpha_i})^{y_1^{\alpha_i}} B(1 + 1/c_2, k + 1 - 1/c_2) \quad (5.3.12)
\]

which is increasing function of \(x_1\).

Further we note that
\[ E(x_1 | X_2 > x_2) = \frac{k}{(k-1/c_1)} E(x_1 | X_2 = x_2) \] (5.3.13)

and similarly

\[ E(x_2 | X_1 > x_1) = \frac{k}{(k-1/c_2)} E(x_2 | X_1 = x_1) \] (5.3.14)

The coefficient of variation of \( X_j \) given \( X_i = x \) is

\[
\text{cv}(x_j | X_i = x_j) = \frac{\sigma(x_j | X_i = x_j)}{E(x_j | X_i = x_j)} = \frac{\sigma(x_j | X_i = x_j)}{E(y_j | X_i = x_j)}
\]

\[ = \frac{1}{\Gamma(k+1)} \left[ \frac{\Gamma(1+2/c_j)\Gamma(k+1-2/c_j)}{\Gamma(1+1/c_j)\Gamma(k+1-1/c_j)} \right]^{1/2} \] (5.3.15)

This is independent of \( X_i \), so is the coefficient of skewness of the conditional distributions.

It has been pointed out earlier that most of the applications of the Burr type XII law is in reliability analysis. Hence we consider the role of bivariate model in explaining the reliability aspect of a two component system.

The scalar reversed hazard rate is

\[ \lambda(x_1, x_2) = \frac{f(x_1, x_2)}{F(x_1, x_2)} \]

\[ = \frac{k(k+1)c_1 c_2 x_1^{c_1-1} x_2^{c_2-1}}{[1 + x_1^{c_1}]^{-k} - [1 + x_2^{c_2}]^{-k} + [1 + x_1^{c_1} + x_2^{c_2}]^{-k}} \] (5.3.16)

Vector valued reversed hazard rate (Roy(2002)) defined in equation (2.3.13) is given by
The marginal reverse hazard rate is
\[ \lambda_i(x_i, x_2) = \frac{\partial}{\partial x_i} \log F(x_i, x_2) \]
\[ = \frac{k c_i x_i^{n-1} - k c_i x_i^{n-1}}{1 - [1 + x_i^n]^k - [1 + x_2^n]^k + [1 + x_i^n + x_2^n]^k} \]  \hspace{1cm} (5.3.17)
and
\[ \lambda_2(x_i, x_2) = \frac{\partial}{\partial x_2} \log F(x_i, x_2) \]
\[ = \frac{k c_2 x_2^{n-1} - k c_2 x_2^{n-1}}{1 - [1 + x_1^n]^k - [1 + x_2^n]^k + [1 + x_1^n + x_2^n]^k} \]  \hspace{1cm} (5.3.18)

The marginal reverse hazard rate is
\[ \lambda_i(x_i) = \frac{f_i(x_i)}{F_i(x_i)} \]
\[ = \frac{k c_i x_i^{n-1}}{[1 + x_i^n]^k [1 + x_i^n]^k} \]  \hspace{1cm} (5.3.19)

Basu's (1971) failure rate is
\[ h(x_i, x_2) = \frac{f(x_i, x_2)}{R(x_i, x_2)} \]
\[ = \frac{k (k + 1) c_1 c_2 x_1^{n-1} x_2^{n-1}}{[1 + x_1^n + x_2^n]^2} \]  \hspace{1cm} (5.3.20)
\[ h(x_i + t, x_2 + t) - h(x_i, x_2) \]
\[ = \frac{k (k + 1) c_1 c_2 (x_i + t)^{n-1} (x_2 + t)^{n-1}}{[1 + (x_i + t)^n + (x_2 + t)^n]^2} - \frac{k (k + 1) c_1 c_2 x_1^{n-1} x_2^{n-1}}{[1 + x_1^n + x_2^n]^2} \]  \hspace{1cm} (5.3.21)

This expression is negative when \( c_i \leq 1 \). Hence bivariate Burr XII has decreasing failure rate when \( c_i \leq 1 \).
When \( c_i > 1 \), small values of \( x_i \) the above expression is positive and large values of \( x_i \) it tends to zero.

Gradient hazard rate (Johnson and Kotz(1975)) is

\[
h(x_1, x_2) = (h_1(x_1, x_2), h_2(x_1, x_2))
\]

where

\[
h_i(x_1, x_2) = \frac{\partial}{\partial x_i} \log R(x_1, x_2) \quad i = 1, 2
\]

\[
h_1(x_1, x_2) = \frac{k c_1 x_1^{\alpha_i-1}}{[1 + x_1^\alpha + x_2^\beta]} \tag{5.3.22}
\]

and

\[
h_2(x_1, x_2) = \frac{k c_2 x_2^{\beta_i-1}}{[1 + x_1^\alpha + x_2^\beta]} \tag{5.3.23}
\]

When \( c_i \leq 1 \),

\[
h_1(x_1 + t_1, x_2 + t_2) = \frac{k c_1 (x_1 + t_1)^{\alpha_i-1}}{[1 + (x_1 + t_1)^\alpha + (x_2 + t_2)^\beta]} \leq h_1(x_1, x_2) \tag{5.3.24}
\]

\[
h_2(x_1 + t_1, x_2 + t_2) = \frac{k c_2 (x_2 + t_2)^{\beta_i-1}}{[1 + (x_1 + t_1)^\alpha + (x_2 + t_2)^\beta]} \leq h_2(x_1, x_2) \tag{5.3.25}
\]

\[
h_1(x_1 + t, x_2 + t) = \frac{k c_1 (x_1 + t)^{\alpha_i-1}}{[1 + (x_1 + t)^\alpha + (x_2 + t)^\beta]} \leq h_1(x_1, x_2) \tag{5.3.26}
\]

\[
h_2(x_1 + t, x_2 + t) = \frac{k c_2 (x_2 + t)^{\beta_i-1}}{[1 + (x_1 + t)^\alpha + (x_2 + t)^\beta]} \leq h_2(x_1, x_2) \tag{5.3.27}
\]

\[
h_1(x_1 + t, x_2) = \frac{k c_1 (x_1 + t)^{\alpha_i-1}}{[1 + (x_1 + t)^\alpha + (x_2)^\beta]} \leq h_1(x_1, x_2) \tag{5.3.28}
\]

\[
h_2(x_1 + t, x_2) = \frac{k c_2 (x_2 + t)^{\alpha_i-1}}{[1 + (x_1)^\alpha + (x_2 + t)^\beta]} \leq h_2(x_1, x_2) \tag{5.3.29}
\]

Hence bivariate Burr XII has decreasing failure rate when \( c_i \leq 1 \).
When $c_1 > 1$, small values of $x_i$ the above expressions is positive and large values of $x_i$ it tends to zero.

5.4 Characterizations of Bivariate Burr Type XII Distribution

In this section we consider some characterization theorems of bivariate Burr type XII distribution.

Theorem 5.4.1

Let $(X_1, X_2)$ be a random vector in the support of $R_x$ having absolutely continuous distribution function with respect to lebesgue measure, with conditional distribution of $X_1$ given $X_2 = x_2$ is of the form equation (5.2.9). Then $X_1$ is Burr type XII if and only if $X_2$ is Burr type XII.

Proof

The conditional density of $X_1$ given $X_2 = x_2$ is of the form equation (5.2.9).

Assume that $X_1$ follows univariate Burr type XII distribution. Then

$$f_1(x_1) = \frac{kc_1x_1^{\alpha-1}}{[1 + x_1^{\alpha}]^{\gamma+1}} \quad 0 < x_1 < \infty \quad k, c_1 > 0$$

Also

$$f_1(x_1) = \int f(x_1 | x_2)f_2(x_2) \, dx_2 \quad (5.4.1)$$

Hence

$$\frac{kc_1x_1^{\alpha-1}}{[1 + x_1^{\alpha}]^{\gamma+1}} = (k + 1)c_1 \int_0^{\infty} [1 + x_2^{\alpha}]^{-(\gamma+1)} [1 + x_2^{\alpha}]^{\gamma+2} f(x_2) \, dx_2$$

(5.4.2)
\[
\frac{k}{k+1} \left[ 1 + x_1^{c_1} \right] = \int_0^\infty \frac{[1+x_2^{c_1}]^{k+2} f(x_2) \, dx_2}{[1 + \frac{x_2^{c_1}}{1+x_1^{c_1}}]^{k+2}} \tag{5.4.3}
\]

Substituting \( u = x_2^{c_1} \) in equation (5.4.3) gives

\[
\frac{k}{k+1} \left[ 1 + x_1^{c_1} \right] = \int_0^\infty \frac{[1+u]^{k+1} f(u^{1/c_1}) u^{(l/c_1)-1} \, du}{[1 + \frac{u}{1+x_1^{c_1}}]^{k+2}}
\]

\[
= \int_0^\infty H(u) u^{(l/c_1)-1} \, du \tag{5.4.4}
\]

Taking inverse Mellin transform (Ryzik P.1194)

\[
H(u) = \frac{k c_2 u^{-l/c_1}}{[1+\frac{u}{1+x_1^{c_1}}]^{k+2}}
\]

Hence

\[
f_2(x_2) = \frac{k c_2 x_2^{l-1}}{[1+x_2^{c_1}]^{k+1}} \quad 0 < x_2 < \infty \quad k, c_2 > 0
\]

Thus \( X_2 \) is of Burr type XII form.

To prove the converse, assume \( X_2 \) follows univariate Burr type XII.

Then

\[
f_1(x_1) = \int_0^\infty f(x_1 \mid x_2) f_2(x_2) \, dx_2
\]

\[
= k(k+1)c_1 c_2 \int_0^\infty \frac{x_1^{c_1-1} x_2^{l-1} \, dx_2}{[1+x_1^{c_1} + x_2^{c_1}]^{k+2}}
\]

\[
= \frac{k c_1 x_1^{c_1-1}}{[1+x_1^{c_1}]^{k+1}} \quad 0 < x_1 < \infty \quad k, c_1 > 0
\]
Apart from the marginal distribution of $X_i$ and the conditional distribution of $X_j$
given $X_i = x_i, i = 1, 2, i \neq j$ from which the joint distribution can always found, the
other quantity that are relevance to the problem is marginal and conditional
distribution of the same component. In the corollary 5.4.1 we consider a
characterization on the marginal and conditional distribution of the same component
which incidentally also provides a characterization of univariate Burr type XII
distribution using bivariate Burr type XII.

**Corollary 5.4.1**

Let $(X_1, X_2)$ be a random vector in the support of $R^*_2$ having absolutely
continuous distribution function with respect to lebesgue measure, with conditional
distribution of $X_1$ given $X_2 = x_2$ is of the form equation (5.2.9). Then $(X_1, X_2)$ is
Burr type XII if and only if $X_2$ is Burr type XII.

**Theorem 5.4.2**

Let $(X_1, X_2)$ be continuous random vector in the support of $R^*_2$ having
absolutely continuous distribution function with respect to lebesgue measure.
Then $(X_1, X_2)$ follows bivariate Burr type XII distribution if and only if conditional
densities are of the form equation (5.2.9).

**Proof**

Let $(X_1, X_2)$ follows bivariate Burr type XII distribution.

Then $f(x_i | x_j) i = 1, 2, i \neq j$ is of the form (5.2.9)

Conversely
\[
\frac{f(x_1 | x_2)}{f(x_2 | x_1)} = \frac{c_1 x_1^{n-1} [1 + x_1^\alpha]^s + 1}{c_2 x_2^{s-1} [1 + x_2^\alpha]^s + 1} = \frac{A_1(x_1)}{A_2(x_2)}
\]

where

\[
A_i(x_i) = \frac{c_i x_i^{s-1}}{[1 + x_i^\alpha]^{s+1}} \quad i = 1, 2
\]

(5.4.6)

\[
\int_0^\infty A_i(x_i) \, dx_i = \int_0^\infty A_i(x_i) \, dx_i = 1/k
\]

(5.4.7)

Abraham and Thomas condition for unique determination of the joint density using conditional density is satisfied. Hence proof.

Next we consider some characterization theorems using the relationship between scalar hazard rate, scalar reversed hazard rate, gradient hazard rate and gradient reversed hazard rate.

**Theorem 5.4.3**

Let \((X_1, X_2)\) be continuous random vector in the support of \(R^2_+\) having absolutely continuous distribution function with respect to lebesgue measure.

Then \((X_1, X_2)\) belongs to the bivariate Burr type XII distribution if and only if

\[
\lambda(x_1, x_2) - h(x_1, x_2) = [1 + x_1^{c_1} + x_2^{c_2}] \lambda(x_1, x_2)[\frac{\lambda_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{h_2(x_2)}{\lambda_2(x_2) + h_2(x_2)}]
\]

(5.4.8)

**Proof**

Let \((X_1, X_2)\) follows to the bivariate Burr type XII distribution.
Solving $R(x_1, x_2)$ from equations (2.3.19), (2.3.23) and (3.2.3) we have

$$R(x_1, x_2) = \frac{\lambda(x_1, x_2) \left[ \frac{\lambda_1(x_1)}{h_1(x_1) + \lambda_1(x_1)} - \frac{h_2(x_2)}{h_2(x_2) + \lambda_2(x_2)} \right]}{\lambda(x_1, x_2) - h(x_1, x_2)} \quad (5.4.9)$$

Then using equation (5.4.9) in equation (5.2.4) we have equation (5.4.8).

Conversely starting from (5.4.8) and using (2.3.19), (2.3.23) and (3.2.3) we get

$$R(x_1, x_2) = [1 + x_1^i + x_2^j]^t \quad 0 < x_i < \infty, k_i c_i > 0, i = 1, 2$$

**Theorem 5.4.4**

Let $(X_1', X_2')$ be continuous random vector in the support of $R^*$

having absolutely continuous distribution function with respect to lebesgue measure.

Then $(X_1', X_2')$ belongs to the bivariate Burr type XII distribution if and only

$$\lambda_i(x_1, x_2) + h_i(x_1, x_2)$$

$$= [1 + x_1^i + x_2^j] \left[ \frac{\lambda_i(x_1) h_i(x_2)}{\lambda_i(x_i) + h_i(x_1)} + \frac{\lambda_i(x_1, x_2)}{\lambda_i(x_1) + h_i(x_1)} - \frac{h_2(x_2)}{h_2(x_2) + \lambda_2(x_2)} \right] i = 1, 2 \quad (5.4.10)$$

**Proof**

Let $(X_1, X_2)$ follows to the bivariate Burr type XII distribution.

Solving $R(x_1, x_2)$ from equations (2.3.13), (2.3.27) and (3.2.3) we find

$$R(x_1, x_2) = \frac{\lambda_1(x_1) h_1(x_1) - \lambda_2(x_1, x_2) h_1(x_1)}{\lambda_1(x_1) + h_1(x_1)} - \frac{\lambda_2(x_1) h_2(x_2)}{\lambda_2(x_1) + h_2(x_2)} \quad i = 1, 2 \quad (5.4.11)$$

Then using equation (5.4.11) in equation (5.2.4) we have equation (5.4.10).

Conversely starting from (5.4.10) and using (2.3.13), (2.3.29) and (3.2.3) we get
\[ R(x_i, x_j) = [1 + x_i^i + x_j^j]^{-k} \quad 0 < x_i < \infty, k, c > 0 \quad i = 1, 2 \]

**Theorem 5.4.5**

Let \((X_1', X_2')\) be continuous random vector in the support of \(R_2^+\) having absolutely continuous distribution function with respect to lebesgue measure. Then \((X_1, X_2)\) belongs to the bivariate Burr type XII distribution if

\[ h(x_1, x_2) = \frac{k+1}{k} h_1(x_1, x_2) h_2(x_1, x_2) \] (5.4.12)

**Proof**

Let \((X_1, X_2)\) follows to the bivariate Burr type XII distribution.

Then by equation (5.3.20), (5.3.22) and (5.3.23) we have equation (5.4.12).

**5.5 Relation Between Burr Type XII and Other Distributions**

Let \((X_1, X_2)\) follows to the bivariate Burr type XII distribution. Table 5.5.1 gives relation between this distribution and other distributions.

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Distribution function</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Y_i = [1 + X_i^i]^{-k})</td>
<td>([y_1^{-1/k} + y_2^{-1/k} - 1]^{-k} 0 &lt; y_i &lt; 1, k &gt; 0 \quad i = 1, 2)</td>
</tr>
<tr>
<td>(U_i = -\log X_i^c)</td>
<td>([1 + e^{-w_1} + e^{-w_2}]^{-k} \quad 0 &lt; u_i &lt; \infty, k &gt; 0) (Burr II)</td>
</tr>
<tr>
<td>(V_i = \frac{1}{X_i})</td>
<td>([1 + w_1^{c_1} + w_2^{c_2}]^{-k} 0 &lt; v_i &lt; \infty, k, c &gt; 0 \quad i = 1, 2) (Burr III)</td>
</tr>
<tr>
<td>(W = \frac{X_i^c}{a_i})</td>
<td>([1 + w_1^{c_1} + w_2^{c_2}]^{-k}) (Nayak (1987))</td>
</tr>
<tr>
<td>(c_i = 1)</td>
<td>([1 + x_i + x_j]^{-k} 0 &lt; x_i &lt; \infty, c_i &gt; 0, k &gt; 0, i = 1, 2) (Mardia (1960))</td>
</tr>
</tbody>
</table>
5.6 Bivariate Burr IX Distribution (Bismi and Nair, 2005 f)

In this section we consider general properties, characterizations of bivariate Burr IX distribution. The distribution arises by the choice of $G(x_1, x_2)$ as

$$
G(x_1, x_2) = \log \frac{1 + \frac{2}{2 + c_1[(1 + e^{x_1})^k - 1]} + \frac{2}{2 + c_2[(1 + e^{x_2})^k - 1]} + \frac{2}{2 + c_1[(1 + e^{x_1})^k - 1] + c_2[(1 + e^{x_2})^k - 1]}}{2 + c_1[(1 + e^{x_1})^k - 1] + 2 + c_2[(1 + e^{x_2})^k - 1] + \frac{2}{2 + c_1[(1 + e^{x_1})^k - 1] + c_2[(1 + e^{x_2})^k - 1]}}
$$

(5.6.1)

in equation (2.2.11)

The distribution function is

$$
F(x_1, x_2) = 1 - \frac{2}{2 + c_1[(1 + e^{x_1})^k - 1]} - \frac{2}{2 + c_2[(1 + e^{x_2})^k - 1]} + \frac{2}{2 + c_1[(1 + e^{x_1})^k - 1] + c_2[(1 + e^{x_2})^k - 1]}
$$

(5.6.2)

$$
f(x_1, x_2) = \frac{4k^2c_1c_2e^{x_1}e^{x_2}(1 + e^{x_1})^{k-1}(1 + e^{x_2})^{k-1}}{[2 + c_1[(1 + e^{x_1})^k - 1] + c_2[(1 + e^{x_2})^k - 1]]^2} - \infty < x_1 < \infty, k > 0, c_i > 0, i = 1, 2
$$

(5.6.3)

and

$$
R(x_1, x_2) = \frac{2}{2 + c_1[(1 + e^{x_1})^k - 1] + c_2[(1 + e^{x_2})^k - 1]} - \infty < x_1, x_2 < \infty, k > 0, c_i > 0, i = 1, 2
$$

(5.6.4)

The marginal distributions are specified by

$$
F_i(x_i) = 1 - \frac{2}{2 + c_i[(1 + e^{x_i})^k - 1]} - \infty < x_i < \infty, k > 0, c_i > 0, i = 1, 2
$$

(5.6.5)

and

$$
F_i(x_i)[1 - F_i(x_i)] g_i(x_i) = \frac{2c_i k e^{x_i}(1 + e^{x_i})^{k-1}}{[2 + c_i[(1 + e^{x_i})^k - 1]]^2} - \infty < x_i < \infty, k > 0, c_i > 0, i = 1, 2
$$

(5.6.6)
which is univariate Burr type differential equation where

\[ g_i(x_i) = \frac{ke^\alpha (1 + e^\alpha)^{k+1}}{[(1 + e^\alpha)^k - 1]} \quad i = 1, 2 \]  (5.6.8)

Thus the marginals are exactly univariate Burr type IX distribution.

Conditional density of \( X_i \) given \( X_j = x_j \) is

\[ f(X_i | X_j = x_j) = \frac{2kc_i e^\alpha (1 + e^\alpha)^{k+1}[2 + c_i[(1 + e^\alpha)^k - 1]]}{[2 + c_i[(1 + e^\alpha)^k - 1] + c_j[(1 + e^\alpha)^k - 1]]} \]  (5.6.9)

\[ i, j = 1, 2, i \neq j \]

In view of the closed form expression for the survival function of the distribution, it is handy to compute the reliability characteristics such as failure rate, reversed failure rate etc.

The Basu's (1971) failure rate is given by

\[ h(x_1, x_2) = \frac{f(x_1, x_2)}{R(x_1, x_2)} \]

\[ = \frac{2k^2 c_c e^\alpha e^{\alpha_2} (1 + e^\alpha)^{k+1}(1 + e^\alpha)^{k+1}}{[2 + c_i[(1 + e^\alpha)^k - 1] + c_j[(1 + e^\alpha)^k - 1]]} \]  (5.6.10)

The vector valued failure rate (Johnson and Kotz (1975)) is given by

\[ h(x_1, x_2) = (h_1(x_1, x_2), h_2(x_1, x_2)) \]

\[ h_i(x_i, x_j) = -\frac{\partial \log R(x_i, x_j)}{\partial x_i} \]

\[ = \frac{ke^\alpha (1 + e^\alpha)^{k+1}}{2 + c_i[(1 + e^\alpha)^k - 1] + c_j[(1 + e^\alpha)^k - 1]} \]  (5.6.11)
\[ \lambda(x_1,x_2) = \frac{f(x_1,x_2)}{F(x_1,x_2)} \]

\[ = \frac{4k^2c_1e^{x_1}e^{x_2}(1+e^{x_1})^{k-1}(1+e^{x_2})^{k-1}}{[2+c_1(1+e^{x_1})^k-1]+c_2(1+e^{x_2})^k-1]^2} \]

\[ \frac{2}{2+c_1[1+e^{x_1})^k-1]} - \frac{2}{2+c_2[(1+e^{x_2})^k-1]+c_2[(1+e^{x_2})^k-1]} \]

The vector valued reversed hazardrate (Johnson and Kotz (1975)) is given by

\[ h(x_1,x_2) = \left( \lambda_1(x_1,x_2), \lambda_2(x_1,x_2) \right) \]

\[ \lambda_i(x_1,x_2) = \frac{\partial \log F(x_1,x_2)}{\partial x_i} \]

\[ = \frac{2kc_i e^{x_i}(1+e^{x_i})^{k-1}}{[2+c_i(1+e^{x_i})^k-1]^2} - \frac{2kc_j e^{x_j}(1+e^{x_j})^{k-1}}{[2+c_j[(1+e^{x_j})^k-1]+c_j[(1+e^{x_j})^k-1]]^2} \]

\[ 1 - \frac{2}{2+c_i[(1+e^{x_i})^k-1]} - \frac{2}{2+c_j[(1+e^{x_j})^k-1]+c_j[(1+e^{x_j})^k-1]} \]

\[ i,j = 1,2 \quad i \neq j \]

### 5.7 Characterizations of Burr Type IX Distribution (Bismi and Nair, 2005 f)

In this section we consider some characterizations of Burr type IX distribution.

The following theorem characterizes the Burr type IX distribution using conditional densities.

**Theorem 5.7.1**

Let \((X_1,X_2)\) be a random vector in the support of \(R_2\) having absolutely continuous distribution function with respect to lebesgue measure, with conditional distribution of \(X_1\) given \(X_2 = x_2\) is of the form equation (5.6.9). Then \(X_1\) is Burr type IX if and only if \(X_2\) is Burr type IX.
Proof

The conditional density of $X_1$ given $X_2 = x_2$ is of the form equation (5.6.9).

Assume that $X_1$ follows univariate Burr type IX distribution. Then

$$f_i(x_i) = \frac{2c_1e^{a_1}(1+e^{a_2})^{k-1}}{[2 + c_1[(1 + e^{a_3})^k - 1]]} \quad -\infty < x_i < \infty, k > 0, c_1 > 0$$

Then using the relation

$$f_i(x_i) = \int f(x_i \mid x_2)f_2(x_2) \, dx_2$$

we have

$$\frac{2kc_1e^{a_1}(1+e^{a_2})^{k-1}}{[2 + c_1[(1 + e^{a_3})^k - 1]]^2} = 2k c_1\int_{-\infty}^{\infty} \frac{2c_1e^{a_1}(1+e^{a_2})^{k-1}[2 + c_1[(1 + e^{a_3})^k - 1]]^2 f_2(x_2) \, dx_2} {[2 + c_1[(1 + e^{a_3})^k - 1]+ c_1[(1 + e^{a_3})^k - 1]]^2} \quad (5.7.1)$$

Substituting $u = c_1[(1 + e^{a_3})^k - 1]$ in equation (5.7.1) gives

$$2 + c_1[(1 + e^{a_3})^k - 1] = \int_{0}^{\infty} \frac{[2 + u]^2 f_2(\log(\frac{u}{c_2} + 1)^{1/c} - 1) du}{u^2 + \frac{u}{2 + c_1[(1 + e^{a_3})^k - 1]]^{1/c - 1}} \quad (5.7.2)$$

$$= \int_{0}^{\infty} H(u) u_{c_2 - 1} \, du \quad (5.7.3)$$

Taking inverse Mellin transform (Rhyzik P.1194)

$$H(u) = \frac{u^{-1}}{\left[\frac{u}{2 + c_1[(1 + e^{a_3})^k - 1]]^{1/c - 1}}\right]}$$

Hence

$$f_2(x_2) = \frac{2c_2e^{a_2}(1+e^{a_2})^{k-1}}{[2 + c_2[(1 + e^{a_2})^k - 1]]^2} \quad -\infty < x_2 < \infty, k > 0, c_2 > 0$$

Thus $X_2$ is of Burr type IX form.

To prove the converse, assume $X_2$ follows univariate Burr type IX.
Then

\[ f_i(x_i) = \int_{-\infty}^{x_i} f(x_i | x_2) f_2(x_2) \, dx_2 \]

\[ = 2k c_i \int_{-\infty}^{x_i} \frac{2e^h (1 + e^h)^{k-1} [2 + c_2 ((1 + e^h)^k - 1)]^2 \, dx_2}{[2 + c_2 ((1 + e^h)^k - 1)]^2} \]

\[ = \frac{2e^h (1 + e^h)^{k-1}}{[2 + c_2 ((1 + e^h)^k - 1)]^2} \quad -\infty < x_i < \infty, k > 0, c_i > 0 \]

**Corollary 5.7.1**

Let \((X_1, X_2)\) be a random vector in the support of \(R\) having absolutely continuous distribution function with respect to Lebesgue measure, with conditional distribution of \(X_i\) given \(X_2 = x_2\) is of the form equation (5.6.9). Then \((X_1, X_2)\) is Burr type IX if and only if \(X_2\) is Burr type IX.

**Theorem 5.7.2**

Let \((X_1, X_2)\) be random vector in the support of \(R\) having absolutely continuous distribution function with respect to Lebesgue measure. Then \((X_1, X_2)\) follows bivariate Burr type IX distribution if and only if conditional densities are of the form equation (5.6.9).

**Proof**

Let \((X_1, X_2)\) follows bivariate Burr type IX distribution.

Then \(f(x_i | x_j) \quad i=1,2 \quad i \neq j\) is of the form (5.6.9)
Conversely

\[
\frac{f(x_1 | x_2)}{f(x_2 | x_1)} = \frac{c_i e^{x_i}(1 + e^{x_i})^{k-1}[2 + c_i(1 + e^{x_i})^k - 1]}{c_j e^{x_j}(1 + e^{x_j})^{k-1}[2 + c_j(1 + e^{x_j})^k - 1]}
= \frac{A_i(x_1)}{A_j(x_2)}
\]

where

\[
A_i(x_i) = \frac{c_i e^{x_i}(1 + e^{x_i})^{k-1}}{[2 + c_i(1 + e^{x_i})^k - 1]} \quad i = 1, 2 \quad (5.7.4)
\]

\[
\int_{-\infty}^{\infty} A_i(x_i) \, dx_i = \int_{-\infty}^{\infty} A_j(x_2) \, dx_2 = 2k \quad (5.7.5)
\]

Abraham and Thomas condition for unique determination of the joint density using conditional density is satisfied. Hence proof.

Next we consider some characterization theorems using the relationship between scalar hazard rate, scalar reversed hazard rate, gradient hazard rate and gradient reversed hazard rate

**Theorem 5.7.3**

Let \((X_1, X_2)\) be continuous random vector in the support of \(R_2\) having absolutely continuous distribution function with respect to lebesgue measure. Then \((X_1, X_2)\) belongs to the bivariate Burr type IX distribution if and only

\[
2[\lambda(x_1, x_2) - h(x_1, x_2) = [2 + c_1[(1 + e^{x_1})^k - 1] + c_2[(1 + e^{x_2})^k - 1]]\lambda(x_1, x_2)[\frac{\lambda_i(x_1)}{\lambda_i(x_1) + \lambda_2(x_1)} - \frac{h_i(x_2)}{\lambda_2(x_2) + h_2(x_2)}] \quad (5.7.6)
\]
Proof

Let \((X_1, X_2)\) follows to the bivariate Burr type IX distribution.

Then using equation (5.4.9) in equation (5.6.4) we have equation (5.7.6).

Conversely starting from (5.7.6) and using (2.3.19), (2.3.23) and (3.2.3) we get

\[
R(x_1, x_2) = \frac{2}{2 + c_1[(1 + e^{x_1})^k - 1] + c_2[(1 + e^{x_2})^k - 1]} \quad -\infty < x_i < \infty, k, c_i > 0 \quad i = 1, 2
\]

Theorem 5.7.4

Let \((X_1, X_2)\) be continuous random vector in the support of \(R_2\) having absolutely continuous distribution function with respect to lebesgue measure. Then \((X_1, X_2)\) belongs to the bivariate Burr type XII distribution if and only if

\[
2[ \lambda_i(x_i, x_2) + h_i(x_i, x_2) ] = [2 + c_1((1 + e^{x_1})^k - 1) + c_2((1 + e^{x_2})^k - 1)][ \frac{\lambda_i(x_i) h_i(x_i)}{\lambda_i(x_i) + h_i(x_i)} + \lambda_i(x_i, x_2)[ \frac{\lambda_i(x_2)}{\lambda_i(x_i) + h_i(x_2)} - \frac{h_i(x_2)}{\lambda_i(x_i) + h_i(x_2)} ] ] \quad i = 1, 2 \tag{5.7.7}
\]

Proof

Let \((X_1, X_2)\) follows to the bivariate Burr type IX distribution.

Then using equation (5.4.11) in equation (5.6.4) we have equation (5.7.7).

Conversely starting from (5.7.7) and using (2.3.13), (2.3.27) and (3.2.3) we get

\[
R(x_1, x_2) = \frac{2}{2 + c_1[(1 + e^{x_1})^k - 1] + c_2[(1 + e^{x_2})^k - 1]} \quad -\infty < x_i < \infty, k, c_i > 0 \quad i = 1, 2
\]

Theorem 5.7.5

Let \((X_1, X_2)\) be continuous random vector in the support of \(R_2\) having absolutely continuous distribution function with respect to lebesgue measure. Then \((X_1, X_2)\) belongs to the bivariate Burr type IX distribution if
\[ h(x_1, x_2) = 2h_1(x_1, x_1)h_2(x_1, x_1) \]  \hspace{1cm} (5.7.8)

**Proof**

Let \((X_1, X_2)\) follows to the bivariate Burr type IX distribution.

Then by equation (5.6.10) and (5.6.11) we have equation (5.7.8).

**5.8 Bivariate Burr II Distribution (Bismi and Nair, 2005)**

In this section we consider general properties, characterizations of bivariate Burr II distribution. The distribution arises by the choice of \(G(x_1, x_2)\) as

\[
G(x_1, x_2) = -\log\left[\frac{1}{1 + e^{-x_1} + e^{-x_2} + \theta e^{-x_1-x_2}}\right] + 1 \hspace{1cm} (5.8.1)
\]

in equation (2.2.11)

The distribution function of bivariate Burr type II distribution is

\[
F(x_1, x_2) = \left[1 + e^{-x_1} + e^{-x_2} + \theta e^{-x_1-x_2}\right]^{-k} - \infty < x_i < \infty, k > 0, 0 \leq \theta \leq k + 1 \hspace{1cm} i = 1, 2
\]

Corresponding density function and survival function is

\[
f(x_1, x_2) = \frac{k(k+1)e^{-x_1}e^{-x_2}(1+\theta e^{-x_1})(1+\theta e^{-x_2})}{[1+e^{-x_1}+e^{-x_2}+\theta e^{-x_1-x_2}]^{k+2}} \cdot \frac{k\theta e^{-x_1}e^{-x_2}}{[1+e^{-x_1}+e^{-x_2}+\theta e^{-x_1-x_2}]^{k+1}} \hspace{1cm} (5.8.5)
\]

\[-\infty < x_i < \infty, k > 0, 0 \leq \theta \leq k + 1 \hspace{1cm} i = 1, 2\]

and

\[
R(x_1, x_2) = 1 - \left[1 + e^{-x_1}\right]^{-k} - \left[1 + e^{-x_2}\right]^{-k} + \left[1 + e^{-x_1} + e^{-x_2} + \theta e^{-x_1-x_2}\right]^{-k} \hspace{1cm} (5.8.4)
\]

\[-\infty < x_i < \infty, k > 0, 0 \leq \theta \leq k + 1 \hspace{1cm} i = 1, 2\]

The marginal distributions are specified by

\[
F_i(x_i) = \left[1 + e^{-x_i}\right]^{-k} - \infty < x_i < \infty, k > 0, i = 1, 2 \hspace{1cm} (5.8.5)
\]
\[ F(x_i)[1 - F(x_j)] g_j(x_i) = \frac{ke^{-x_i}}{[1 + e^{-x_i}]^{i+1}} - \infty < x_i < \infty, k > 0, i = 1,2 \quad (5.8.6) \]

\[ \frac{dF(x_j)}{dx_i} i = 1,2 \quad (5.8.7) \]

which is univariate Burr type differential equation where

\[ g_j(x_i) = \frac{ke^{-x_i}[1 + e^{-x_i}]^{i+1}}{[1 + e^{-x_i}]^{i+1} - 1} i = 1,2 \quad (5.8.8) \]

Hence marginals are exactly univariate Burr type II distribution.

Conditional density of \( X_i \) given \( X_j = x_j \) is

\[ f(X_i \mid X_j = x_j) = \frac{k(k + 1)e^{-x_i}(1 + \theta e^{-x_i})(1 + \theta e^{-x_j})}{[1 + e^{-x_i}(1 + \theta e^{-x_j})]^{i+2}} - \frac{k\theta e^{-x_i}}{[1 + e^{-x_i}(1 + \theta e^{-x_j})]^{i+1}} \quad (5.8.9) \]

\[ -\infty < x_i < \infty, k > 0, 0 \leq \theta \leq k + 1, i, j = 1,2, i \neq j \]

Now we are interested to find concepts useful in failure time analysis.

The Basu's (1971) failure rate is given by

\[ h(x_1, x_2) = \frac{f(x_1, x_2)}{R(x_1, x_2)} \]

\[ \frac{k(k + 1)e^{-x_1}e^{-x_2}(1 + \theta e^{-x_1})(1 + \theta e^{-x_2})}{[1 + e^{-x_1} + e^{-x_2} + \theta e^{-x_1} + \theta e^{-x_2}]^{i+2}} - \frac{k\theta e^{-x_1}e^{-x_2}}{[1 + e^{-x_1} + e^{-x_2} + \theta e^{-x_1} + \theta e^{-x_2}]^{i+1}} \quad (5.8.10) \]

The vector valued failure rate (Johnson and Kots (1975)) is given by

\[ h(x_1, x_2) = (h_1(x_1, x_2), h_2(x_1, x_2)) \]

\[ h_i(x_i, x_j) = \frac{-\partial \log R(x_i, x_j)}{\partial x_i} \]
The scalar reversed hazard rate is given by

$$\lambda(x_1, x_2) = \frac{f(x_1, x_2)}{F(x_1, x_2)}$$

$$= \frac{2k(k+1)e^{-x_1}e^{-x_2}(1+\theta e^{-x_1})(1+\theta e^{-x_2})}{[1+e^{-x_1}+e^{-x_2}+\theta e^{-x_1}e^{-x_2}]^2} - \frac{2k\theta e^{-x_1}e^{-x_2}}{[1+e^{-x_1}+e^{-x_2}+\theta e^{-x_1}e^{-x_2}]} \quad (5.8.12)$$

The vector valued reverse hazard rate (Johnson and Kotz (1975)) is given by

$$\Delta \log F(x_1, x_2) = (\lambda_1(x_1, x_2), \lambda_2(x_1, x_2))$$

$$= \frac{\partial \log F(x_1, x_2)}{\partial x_i}$$

$$= \frac{ke^{-x_i}(1+\theta e^{-x_j})}{[1+e^{-x_1}+e^{-x_2}+\theta e^{-x_1}e^{-x_2}]} \quad i, j = 1, 2 \quad i \neq j \quad (5.8.13)$$

5.9 Characterizations of Burr Type II Distribution (Bisimi and Nair, 2005)

Theorem 5.9.1

Let \((X_1, X_2)\) be continuous random vector in the support of \(R_2\) having absolutely continuous distribution function with respect to Lebesgue measure. Then

\((X_1, X_2)\) belongs to the bivariate Burr type II distribution if and only if

$$\lambda(x_1, x_2) - h(x_1, x_2)$$

$$= [1+e^{-x_1}+\theta e^{-x_1}e^{-x_2}]h(x_1, x_2)\left[\frac{\lambda_1(x_1)}{\lambda_1(x_1)+h_1(x_1)} - \frac{h_1(x_1)}{\lambda_1(x_1)+h_1(x_1)}\right] \quad (5.9.1)$$
Proof

Let \((X_1, X_2)\) follows to the bivariate Burr type II distribution.

Then using equation (3.2.4) in equation (5.8.2) we have equation (5.9.1).

Conversely starting from (5.9.1) and using (2.3.19), (2.3.23), and (3.2.3) we get

\[
F(x_1, x_2) = [1 + e^{-x_1} + e^{-x_2} + \theta e^{-x_1} e^{-x_2}]^{-\frac{1}{k}} - \infty < x_i < \infty, k > 0, 0 \leq \theta \leq k + 1 \quad i = 1, 2
\]

Theorem 5.9.2

Let \((X_1, X_2)\) be continuous random vector in the support of \(R_2\) having absolutely continuous distribution function with respect to lebesgue measure. Then \((X_1, X_2)\) belongs to the bivariate Burr type II distribution if and only

\[
\lambda_i(x_1, x_2) + h_i(x_1, x_2)
\]

\[
= [1 + e^{-x_1} + e^{-x_2} + \theta e^{-x_1} e^{-x_2}]^{-\frac{1}{k}} \left[ \frac{\lambda_i(x_1)}{\lambda_i(x_1) + h_i(x_1)} + \lambda_i(x_1, x_2) \left[ \frac{\lambda_i(x_1)}{\lambda_i(x_1) + h_i(x_1)} - \frac{h_i(x_1)}{\lambda_i(x_2) + h_i(x_2)} \right] \right] i = 1, 2 \tag{5.9.2}
\]

Proof

Let \((X_1, X_2)\) follows to the bivariate Burr type II distribution.

Then using equation (3.2.6) in equation (5.8.2) we have equation (5.9.2).

Conversely starting from (5.9.2) and using (2.3.13), (2.3.27) and (3.2.3) we get

\[
F(x_1, x_2) = [1 + e^{-x_1} + e^{-x_2} + \theta e^{-x_1} e^{-x_2}]^{-\frac{1}{k}} - \infty < x_i < \infty, k > 0, 0 \leq \theta \leq k + 1 \quad i = 1, 2
\]