CHAPTER II

MHD FLOW OF AN INVISCID FLUID PAST
AN ELLIPTIC CYLINDER
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2.1. Introduction

In this chapter we consider a steady, two-dimensional streaming motion of an incompressible, homogeneous, ideal conducting fluid past a rigid non-conducting elliptic cylinder of small eccentricity held with its axis at right angles to the plane of the motion in the presence of a uniform magnetic field \( \mathbb{H}_0 \). The plane of the motion is taken as the \( x-y \) plane with the origin at the centre of the section of the cylinder. The \( x \)-axis is taken such that at \( x = -\infty \), the streaming motion (with speed \( U \)) is parallel to the \( x \)-axis. The consideration of the problem is restricted to the aligned field case, i.e., the case with the applied field \( \mathbb{H}_0 \) parallel to the \( x \)-axis at \( x = -\infty \); it is also assumed that the \( a \)-axis of the ellipse is along the \( x \)-axis, so that its equation is

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{2.1.1}
\]

An attempt has been made to find, to a first approximation, the interaction of the field and the motion and also the drag force.
experienced by the cylinder when the degree of interaction between the field and the motion is small.

The steady three-dimensional axisymmetric MHD flow of an incompressible viscous fluid past a rigid sphere was first considered by Chester (1957); he assumed both the Reynolds number and the magnetic Reynolds number to be small, and thus the flow considered is essentially Stokes flow with the lines of force slipping freely. The flow was thus treated as a perturbation of the classical Stokes flow. The MHD problem of flow past a sphere was also discussed by Blenko (1960) and Gotoh (1960a) using Oseen-type approximations. Gotoh (1960b) also considered the problem of flow past an obstacle using Stokes approximation with no particular configuration of the flow and the magnetic field. Inviscid flow past a body at low magnetic Reynolds number was treated by Laddford (1960) and others.

The two-dimensional MHD problem of steady flow past a circular cylinder was considered for both aligned - and transverse-field cases by Yosinobu and Kakutani (1959); the investigation by them assumed the same kind of approximation as used by Chester (1957) in dealing with the three-dimensional flow, and accordingly the analysis was carried out by neglecting the inertia terms in the equation of motion and the transport term in the field equation. The general investigation was carried out further by Yosinobu (1960) for finite Reynolds and magnetic Reynolds numbers by using Oseen-type approximations. The most striking feature in the above works is the formation of two wakes instead of one; and the general
nature of these wakes have been dealt with by Hasimoto (1960) and Glauert (1962).

The two-dimensional case of inviscid conducting fluid flow past a circular cylinder was considered by Tamada (1961, '62) and Singh (unpublished). Tamada (1962) considered also the case of a very strong aligned magnetic field; the induced magnetic field was neglected and the discussion was only for the flow pattern. He indicated the existence of a non-diffusive vortical velocity deficiency downstream of the cylinder which was confirmed by Leonard (1962) in the problem of inviscid flow past a non-conducting circular cylinder between two conducting parallel plates, using a numerical technique.

The flow of a viscous conducting fluid past an inclined elliptic cylinder has been studied by Hasimoto (1958), Miyagi (1961), and Takaishi, Takase and Mori (1961).

The approach to the present problem of MHD inviscid flow past an elliptic cylinder in presence of an aligned field is quite different from that of the authors cited above. Here the inertial terms are retained in the equation of motion, and also the transport term is not neglected from the field equation, though the interaction between the field and the motion is assumed small. The features obtained by Tamada and others on wakes are also exhibited by the present discussion for an ideal fluid.

Although the problem of aligned field MHD flow past bodies has received much attention, there has been general lack of
emphasis on cases which can be realised in the laboratory. This situation has been partly due to analytical difficulties and partly due to the limited interest and success of experimenters themselves. The existence of the upstream wake in magnetohydrodynamics was confirmed experimentally by Ahlstrom (1963). Yonas (1967) has measured drag of spheres and disks in slightly viscous fluid (liquid sodium) in presence of a strong aligned magnetic field. Some experiments with transverse magnetic field have been carried out recently, for example, Branover, G.G. et al (1971), Kalis, Kh.E. et al (1966) and others.

2.2. Governing equations and boundary conditions

As stated earlier, the plane of the motion is taken as the x-y plane with origin at the centre of the cylinder and with x-axis along the a-axis of the elliptic section. The streaming motion $\dot{U}$ and the impressed field $H_0$ are assumed to be parallel to the x-axis at $x = -\infty$.

The (dimensional) magnetohydrodynamic equations governing the steady flow of an incompressible, uniform inviscid fluid are (cf. equations (1.3.1), (1.3.7), (1.2.4) and (1.2.10) of chapter 1; in (1.3.7) the viscous term is to be dropped out and use is made of equation (1.2.2))
$$\nabla^2 \vec{H} = -\frac{1}{\gamma_m} \text{Curl} (\vec{v} \times \vec{H}) , \quad (2.2.1)$$

$$\text{Curl} \vec{v} \times \vec{v} = -\text{grad} \vec{\omega} + \frac{\mu}{4\pi \rho} \text{(Curl} \vec{H} \times \vec{H}) , \quad (2.2.2)$$

$$\text{div} \vec{H} = 0, \quad \text{div} \vec{v} = 0, \quad (2.2.3)$$

where

$$\gamma_m = (4\pi \mu \sigma)^{-1} \text{ and } \vec{\omega} = \vec{p}/\rho + \frac{1}{\rho^2} \vec{v}^2 .$$

The symbols have the same meanings as in Chapter I. It may be noted that $\gamma_m$ is written for the magnetic diffusivity (since $\gamma$ will be used for elliptic co-ordinate later).

The boundary conditions to be satisfied by $\vec{v}$ and $\vec{H}$ are as follows:

(i) At $x = -\infty$, $\vec{v} = \vec{1} \vec{u}$, $\vec{H} = \vec{1} \vec{H}_0$,

where $\vec{1}$ is a unit vector parallel to $x$-axis.

(ii) On the surface of the cylinder $V_n$ (the normal velocity) is zero.

(iii) The normal component of $\mu \vec{H}$ and tangential component of $\vec{H}$ are continuous with the field inside the cylinder. Assuming that the permeabilities of the fluid and the cylinder are effectively equal, it follows that both the tangential and the normal components of $\vec{H}$ are continuous with the inside field.

In view of equations (2.2.3) we can define the stream functions $\psi$ and $\left(\frac{4\pi \rho c}{\mu} \right)^{1/2}$ such that
\[ \bar{v} = \text{Curl} \left\{ -\kappa \psi \right\}, \quad \bar{a} = \left( \frac{4\pi e}{\kappa} \right)^{1/2} \text{Curl} \left\{ -\kappa \chi \right\}, \]

where \( \kappa \) is the unit vector in the \( z \)-direction. Substitution in (2.2.1) and (2.2.2) yields the equations

\[ \nabla^2 \chi = \frac{1}{\eta} \bar{k} \cdot \left[ \nabla \psi \times \nabla \chi \right], \tag{2.2.5} \]

and

\[ (\nabla^2 \psi) \nabla \psi = \nabla \omega + \left( \nabla^2 \chi \right) \nabla \chi. \tag{2.2.6} \]

The equation (2.2.5) is obtained by one integration, and the constant which would have appeared is zero since at infinity the field and the streaming motion are parallel and also electric current is zero there.

The equations (2.2.5) and (2.2.6) are now to be solved subject to the boundary conditions as follows:

At \( x = -\infty \), there is no interaction between the motion and the field, and so

at \( x = -\infty \), \( \psi = \bar{\psi} \), \( \chi = \bar{\chi} \) and \( \omega = \bar{\omega} \),

where zero subscripts denote the values when there is no interaction.

On the surface of the cylinder, \( \psi = 0 \), and \( \chi \) is continuous with the corresponding function say \( \chi' \) inside the cylinder and also

\[ \frac{\partial \chi}{\partial n} = \frac{\partial \chi'}{\partial n}, \]

where \( \partial n \) denotes an element of normal to the surface of the cylinder.
2.3. Method of solution

We shall consider the equations (2.2.5) and (2.2.6) for the case when the interaction between the motion and the field is small i.e., when \( \eta \) is large. For such a case we seek the solutions in the form

\[
\begin{align*}
\gamma &= \gamma_0 + \frac{1}{\eta} \gamma_1 + \frac{1}{\eta^2} \gamma_2 + \frac{1}{\eta^3} \gamma_3 + \ldots + \ldots \\
\psi &= \psi_0 + \frac{1}{\eta} \psi_1 + \frac{1}{\eta^2} \psi_2 + \frac{1}{\eta^3} \psi_3 + \ldots + \ldots \\
\omega &= \omega_0 + \frac{1}{\eta} \omega_1 + \frac{1}{\eta^2} \omega_2 + \frac{1}{\eta^3} \omega_3 + \ldots + \ldots
\end{align*}
\]

(2.3.1)

Substituting in equations (2.2.5) and (2.2.6) and then equating like powers of \( \eta \) we get the following set of equations:

\[
\begin{align*}
\nabla^2 \gamma_0 &= 0 \\
(\nabla^2 \psi_0) \nabla \psi_0 &= \nabla \omega_0, \\
\nabla^2 \gamma_1 &= \mathbf{F} \cdot (\nabla \psi_0 \times \nabla \gamma_0) \\
(\nabla^2 \gamma_1) \nabla \psi_0 &= \nabla \omega_1 + (\nabla^2 \gamma_1) \nabla \chi_0, \\
\nabla^2 \gamma_2 &= \mathbf{F} \cdot \left\{ \nabla \psi_0 \times \nabla \gamma_1 + \nabla \psi_1 \times \nabla \chi_0 \right\} \\
(\nabla^2 \psi_2) \nabla \psi_0 + (\nabla^2 \psi_1) \nabla \psi_1 &= \nabla \omega_2 + (\nabla^2 \psi_2) \nabla \chi_0 + (\nabla^2 \psi_1) \nabla \chi_1.
\end{align*}
\]
The functions \( \gamma_0, \varphi_0 \) and \( \omega_0 \) (as stated earlier) define the field and the motion when \( \gamma = 0 \), when there is no interaction between the motion and the field. These functions satisfying (2.3.2a) are (from classical hydrodynamics and electromagnetic theory)

\[
\begin{align*}
\gamma_0 &= -V_A \cosh \xi \sin \eta \\
\varphi_0 &= -U(a+b) \sinh(\xi - \xi_0) \sin \eta \\
\omega_0 &= C_0 \text{(constant)}
\end{align*}
\]

where \( V_A = (\frac{\mu}{4\pi})^{1/2} H_0 \) (Alfvén velocity), and \( \xi, \eta \) are the elliptic co-ordinates such that

\[
x = c \cosh \xi \cos \eta, \quad y = c \sinh \xi \sin \eta
\]

and that \( \xi = \xi_0 \) gives the boundary of the ellipse with

\[
a = c \cosh \xi_0, \quad b = c \sinh \xi_0 \text{ and } a^2 - b^2 = c^2.
\]

The equations (2.3.2b), (2.3.2c) etc. are all linear, and they are to be solved subject to the appropriate boundary conditions. The equations except for \( \gamma_1 \) are very difficult to get solved analytically. However, in the following, we determine first \( \gamma_1 \), and then using the known value of \( \gamma_1 \), we proceed to determine \( \omega_1 \) and \( \varphi_1 \).
To determine $\gamma_1$, $\omega_1$ and $\varphi_1$:

On substituting the values of $\psi_0$ and $\gamma_0$ in the first equation of (2.3.2b), and noting that

$$\nabla^2 = h^{-2} \nabla_1^2,$$

where

$$h^2 = e^2 (\cosh^2 \varphi - \cos^2 \gamma) \quad \text{and} \quad \nabla_1^2 = \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial \gamma^2},$$

the equation for $\gamma_1$ is

$$\nabla_1^2 \gamma_1 = A \sin 2\gamma, \quad (2.3.6)$$

where

$$\lambda = \frac{1}{2} \psi A U (a+b)b. \quad (2.3.7)$$

On solving,

$$\gamma_1 = (A_1 \cosh 2\varphi + B_1 \sinh 2\varphi - \frac{A}{4}) \sin 2\gamma,$$

where $A_1$ and $B_1$ are two constants to be determined.

Since $\gamma_1$ cannot be infinite at $x = -\infty$ i.e., $\varphi = \infty$, $A_1 = -B_1$.

Again, inside the cylinder (i.e., $\varphi < \varphi_0$), the equation corresponding to (2.3.6) is

$$\nabla_1^2 \gamma_1 = 0,$$

whence $\gamma_1$ is of the form

$$A_2 \sinh 2\varphi \cdot \sin 2\gamma.$$

Hence applying the boundary conditions for $\varphi = \varphi_0$ one finds
\[ A_1 = \frac{1}{4} \cosh 2 \xi_0. \]

\[ \therefore \quad \chi_1 = \frac{1}{4} \left[ \cosh 2 \xi_0 \left( \cosh 2 \xi - \sinh 2 \xi \right) - 1 \right] \sin 2 \eta. \]  

(2.3.8)

Next, taking the vector product of the second equation of (2.3.8b) with \( \nabla \psi_0 \), we get for \( \omega_1 \) the equation

\[ q_0 \frac{\partial \omega_1}{\partial s} = - (\nabla \cdot \chi_1)^2 = - \frac{A^2 \sin^2 2 \eta}{c^2 \left( \cosh^2 \xi - \cos^2 \eta \right)^2}, \]  

(2.3.9)

where \( \delta s \) is an element on the line of flow \( \psi_0 \) and \( q_0 \) the corresponding velocity along that line. This equation can be integrated along the stream line \( \psi_0 = \text{constant} \).

Put

\[ \sinh \left( \xi - \xi_0 \right) \sin \eta = K, \]  

(2.3.10)

so that \( K = \text{constant} \) gives a particular stream line in the flow \( \psi_0 \).

Then

\[ \cosh \left( \xi - \xi_0 \right) \sin \eta = \sqrt{K^2 + \sin^2 \eta}, \]

and so

\[ c \cosh \xi \sin \eta = a \sqrt{K^2 + \sin^2 \eta} + bK, \]

and

\[ c \sinh \xi \sin \eta = b \sqrt{K^2 + \sin^2 \eta} + aK. \]

Also on the stream line \( K \)

\[ q_0 \frac{\delta}{\delta s} = \frac{1}{2} \left( q_0 \sin \eta \frac{\delta}{\delta \eta} \right)^2 = - \frac{U(a+b) \cosh \left( \xi - \xi_0 \right) \sin \eta}{c^2 \left( \cosh^2 \xi - \cos^2 \eta \right)} \frac{\delta}{\delta \eta}, \]

in which \( \xi \) is to be expressed in terms of \( K \) and \( \eta \).
Hence, on the stream line $K$ the equation (2.3.9) simply becomes

$$\frac{\partial \omega}{\partial \gamma} = \frac{A' \sin^4 \gamma \cos^2 \gamma}{\sqrt{k^2 + \sin^2 \gamma} \left[ a \sqrt{k^2 + \sin^2 \gamma} + b \gamma^2 - c \sin \gamma \cos \gamma \right]} \tag{2.3.11}$$

where

$$A' = \sqrt{\frac{\pi}{2}} (a+b)b^2 = 2A\sqrt{\Lambda}b. \tag{2.3.12}$$

Equation (2.3.11) gives the change of $\omega_1$ with respect to $\gamma$ on the stream line $K = \text{constant}$; this is to be integrated keeping $K$ constant.

When the cylinder has a small eccentricity $e=\frac{c}{a}$, so that

$$b = a(1-e^2/2) \quad \text{neglecting higher powers of } e,$$

the right-hand side of (2.3.11) can be expanded in powers of $e$ as

$$\frac{A'}{a^2} = \left[ \frac{\sin^4 \gamma \cos^2 \gamma}{\sqrt{k^2 + \sin^2 \gamma} (\sqrt{k^2 + \sin^2 \gamma} + \gamma^2)} \right] \left[ 1 + e^2 \left\{ \frac{k}{(\sqrt{k^2 + \sin^2 \gamma} + \gamma^2)} + \frac{\sin^2 \gamma \cos^2 \gamma}{(\sqrt{k^2 + \sin^2 \gamma} + \gamma^2)^2} \right\} \right. + \left. \ldots \ldots \right]$$

$$= \frac{A'}{a} \left[ \frac{\sin^4 \gamma \cos^2 \gamma}{\sqrt{k^2 + \sin^2 \gamma} (\sqrt{k^2 + \sin^2 \gamma} - \gamma^2)} \right]$$

$$+ e^2 \left\{ \frac{\gamma \cos^2 \gamma}{(\sqrt{k^2 + \sin^2 \gamma} - \gamma^2)^3} + \frac{\cos^4 \gamma}{\sin \gamma \sqrt{k^2 + \sin^2 \gamma}} (\sqrt{k^2 + \sin^2 \gamma} - \gamma^2)^4 \right\} \right.$$}

$$+ \ldots \ldots \right]. \tag{2.3.13}$$

(The constant $A'$ being independent of co-ordinates is left intact; otherwise, it is also to be expanded in powers of $e$.)

\[43\]
The first term within [ ] can be expressed as

\[-K \cos^2 \gamma + \frac{3}{2} \left\{ \sin \gamma \cos \gamma \sqrt{K^2 + \sin^2 \gamma} \right\} + \frac{1 - 4k^2}{3} \sqrt{K^2 + \sin^2 \gamma} + \frac{4}{3} \frac{\varepsilon_k^2 (1 + k^2)}{\sqrt{K^2 + \sin^2 \gamma}}\]

while the second term as

\[-4K^3 \cot^2 \gamma + K (6k^2 - 3) \cos^2 \gamma + 4K \cos \gamma \sin^2 \gamma - 4K^2 \frac{\varepsilon_k}{\sin \gamma} (\cot \gamma \sqrt{K^2 + \sin^2 \gamma})\]

\[+ \frac{2}{5} (1 - 6k^2) \frac{\varepsilon_k}{\sin \gamma} (\sin \gamma \cos \gamma \sqrt{K^2 + \sin^2 \gamma}) - \frac{1}{6} \frac{\varepsilon_k}{\sin \gamma} (\sin^3 \gamma \cos \gamma \sqrt{K^2 + \sin^2 \gamma})\]

\[+ \frac{1}{5} (16k^4 - 54k^2 + 1) \sqrt{K^2 + \sin^2 \gamma} + \frac{2}{5} K^2 (-ak^4 + 3k^2 + 11) \frac{1}{\sqrt{K^2 + \sin^2 \gamma}}\]

Then substituting in (2.3.11) and then integrating from \( \gamma \) to \( \pi \) (for \( K > 0, 0 \leq \gamma \leq \pi \)) with the condition that \( \omega_1 = 0 \) at \( \gamma = \pi \) (i.e. at \( x = -\infty \)), we get

\[\omega_1 (K, \gamma) = \frac{\alpha}{\varepsilon_k^2} \left[ K (\alpha - \frac{3}{4} \sin 2\gamma) + \frac{1}{6} \sin 2\gamma \sqrt{K^2 + \sin^2 \gamma} - \frac{1 - 4k^2}{3} I_1 \right.\]

\[- \frac{4k^2}{3} I_2 + \varepsilon_k \left\{ K (\alpha - \frac{3}{4} \sin 2\gamma - \frac{1}{8} \sin 4\gamma) \right\} \]

\[+ 6k^3 (\alpha - \frac{1}{4} \sin 2\gamma) + 4K^2 \cot \gamma (K - \sqrt{K^2 + \sin^2 \gamma}) \]

\[- \frac{1}{10} \sin 2\gamma \sqrt{K^2 + \sin^2 \gamma} (\sin^2 \gamma + 12k^2 - 2) - \frac{16k^4 - 54k^2 + 1}{5} I_1 \]

\[- \frac{2}{5} K^2 (-ak^4 + 3k^2 + 11) I_2 \} + \ldots \]

\((K \geq 0, 0 \leq \gamma \leq \pi) (2.3.14)\)
where

\[ I_1 = \int_\eta^\infty \sqrt{K^2 + \sin^2 \eta} \, d\eta = a^{-1} \left[ E(a, \frac{\eta}{2}) + E(a, \phi) \right] \]

and

\[ I_2 = \int_\eta^\infty \frac{d\eta}{\sqrt{K^2 + \sin^2 \eta}} = a \left[ F(a, \frac{\eta}{2}) + F(a, \phi) \right]. \]

\(P, F\) being the (Jacobi's) elliptic integrals of the first and second kind respectively and

\[ a^2 = \frac{1}{1+K^2}, \quad \phi = \frac{\pi}{2} - \eta. \]

\[ \left[ K(a, -\phi) = -E(a, \phi); \quad P(a, -\phi) = -P(a, \phi) \right]. \]

For negative values of \(K\), it is convenient to take \(\eta\) as negative \((-\pi < \eta < 0)\) so that one has to integrate (2.3.11) from \(-\pi\) to \(\eta\). It is thus found that

\[ \omega_1(-\pi, -\eta) = \omega_1(K, \eta) \]

(2.3.16)

(0 \leq \eta \leq \pi).

This shows as expected that \(\omega_1\) is continuous on \(x\)-axis (\(|x| > a\)).

To determine \(\nabla^2 \psi_1\), we first take Curl of both sides of the second equation of (2.3.2b); then on simplification with the values of \(\chi_0\) and \(\chi_1\), we get

\[ q_0 \frac{d}{dz} \nabla^2 \psi_1 = \frac{V^2 a+b}{c^3 (\text{Cosh}_\eta - \text{Cos}_\eta)^3} \left[ \sin^3 \eta - \text{Cosh}_\eta \cdot \sin 3 \eta + \sin 3 \eta \right]. \]
On the stream line \( k \)-constant this reduces to

\[
\frac{\partial}{\partial \eta} (\nabla^2 \phi) = -\frac{A'' \sin \gamma (\sqrt{k^2 + \sin^2 \gamma} + bk)}{\sqrt{k^2 + \sin^2 \gamma} \left[ (a \sqrt{k^2 + \sin^2 \gamma} + bk)^2 - c^2 \sin^2 \gamma \cos^2 \gamma \right]^2}
\]

\[
= -A'' \left[ \frac{-3 \cos \gamma}{\sqrt{k^2 + \sin^2 \gamma}} \left( \sqrt{k^2 + \sin^2 \gamma} - k \right) \right] + \frac{\sin^2 \gamma}{\sqrt{k^2 + \sin^2 \gamma}} \left( \sqrt{k^2 + \sin^2 \gamma} - k \right)^2
\]

\[
+ e^2 \left\{ \frac{-3}{2} \frac{k \cot^2 \gamma}{\sqrt{k^2 + \sin^2 \gamma}} \left( \sqrt{k^2 + \sin^2 \gamma} - k \right)^2 \right\} + \frac{1}{2} \frac{k (\sqrt{k^2 + \sin^2 \gamma} - k)^2}{\sqrt{k^2 + \sin^2 \gamma}}
\]

\[
+ \frac{-3 \cot \gamma + 3 \cos \gamma}{\sqrt{k^2 + \sin^2 \gamma}} \left( \sqrt{k^2 + \sin^2 \gamma} - k \right)^3 + \ldots \ldots \ldots \]

(when expanded in powers of \( e \))

\[
\text{(2.3.17)}
\]

where

\[
A'' = V_{ab}^2
\]

(which is kept as it is, without expanding in powers of \( e \)).

This gives the change of \( \nabla^2 \psi_1 \) (= \( \nabla_{\eta}^2 \)) with respect to \( \eta \) on the stream line \( k \)-constant. Integrating as in the case of \( \nabla_1 \) from \( \eta \) to \( x \) (for \( k > 0 \), \( 0 < \eta \leq x \)) and also noting that at \( \eta = x \)

(i.e., \( x = -\infty \)) \( \nabla_{\eta}^1 = 0 \), we get for any point \( (\eta) \) on the curve

\( k = \text{constant}(>0) \).
\( \Psi_1(k, \eta) = \frac{A''}{a} \left[ - (\eta - \sin 2\eta) - 4kI_k + K(3 + 4k^2)I_2 \right. \\
\left. + e^2 \left[ - \frac{1}{2}(\eta - \frac{3}{2}\sin 2\eta - \frac{1}{2}\sin 4\eta) + 24k^2(\eta - \frac{1}{3}\sin 2\eta) \\
+ k \sin 2\eta \sqrt{k^2 + \sin^2 \eta - 9k \cot \eta (k - \sqrt{k^2 + \sin^2 \eta})} \\
+ 8k(2k^2 - 5)I_k + k(-16k^4 - 8k^2 + \frac{15}{2})I_2 \right] + ... ... ... \right] \),

\((0 < k, \ 0 < \eta \leq \pi) \quad (2.3.18)\)

where \( I_1 \) and \( I_2 \) are given by \((2.3.15)\).

Considering the simultaneous negative values of \( k \) and \( \eta \), one can also show that

\( \Psi_1(-k, -\eta) = - \Psi_1(k, \eta) \quad (2.3.19) \)

\((0 < \eta < \pi, \ 0 < k)\).

So unlike \( \omega_1 \), \( \Psi_1 \) is discontinuous on the (+ve) x-axis in the fluid but it is continuous on the (-ve) x-axis like \( \omega_1 \).

The value of \( \Psi_1(-\nabla^2 \psi_1) \) at the point \((k, \eta)\) can also be determined from \( \omega_1 \) by differentiation. Taking the vector product of the second equation of \((2.3.2b)\) with \( \nabla \eta \) we have

\( \nabla^2 \psi_1(\nabla \psi_0 \times \nabla \eta_0) = \nabla \omega_1 \times \nabla \eta_0 \)

\( = (\frac{\partial \omega_1}{\partial k} \nabla k + \frac{\partial \omega_1}{\partial \eta} \nabla \eta) \times \nabla \eta_0 \).
On using the values of $\gamma_0$, $\chi_0$ and expressing $\xi$ in terms of $K$ and $\eta$, the above equation gives (after some simplification)

$$
\xi_1 = \xi(t) = - \frac{V_A b}{2k} \frac{\partial \omega_1}{\partial k} + \frac{V_A}{2} \frac{a\sqrt{k^2 + \sin^2 \eta} + bk}{\sin \eta} \frac{\partial \omega_1}{\partial \eta}
$$

$$
= - \frac{V_A b}{2k} \frac{\partial \omega_1}{\partial k} + \frac{V_A^*}{2} \frac{a\sqrt{k^2 + \sin^2 \eta} + bk}{\sin \eta} \times
$$

$$
\frac{\sin^3 \eta \cos \eta \left[ a\sqrt{k^2 + \sin^2 \eta} + bk \right]}{\sqrt{k^2 + \sin^2 \eta} \left[ a\sqrt{k^2 + \sin^2 \eta} + bk \right] - c\sin^2 \eta \cos \eta}
$$

(by (2.3.11))

$$
= - \frac{V_A b}{2k} \frac{\partial \omega_1}{\partial k} + \frac{V_A^*}{2} \times
$$

$$
\frac{\sin^3 \eta \cos \eta \left[ a\sqrt{k^2 + \sin^2 \eta} + bk \right]}{\sqrt{k^2 + \sin^2 \eta} \left[ a\sqrt{k^2 + \sin^2 \eta} + bk \right] - c\sin^2 \eta \cos \eta}
$$

$$
= - \frac{V_A b}{2k} \frac{\partial \omega_1}{\partial k} + \frac{V_A^*}{2} \left[ \frac{\sin \eta}{\sqrt{k^2 + \sin^2 \eta}} \right] \left( \sqrt{k^2 + \sin^2 \eta} - k \right)
$$

$$
+ \epsilon^2 \left\{ \frac{k}{2} \cot \eta \left( \sqrt{k^2 + \sin^2 \eta} - k \right)^2 \cos \eta \cot \eta \left( \sqrt{k^2 + \sin^2 \eta} - k \right)^3 \right. 
$$

$$
+ \ldots \ldots \right\} \quad (2.3.20)
$$

But from (2.3.14)
\[ \frac{d \omega_1}{dk} = \frac{A'}{a^2} \left[ (x - \gamma - \frac{1}{2} \sin 2\eta) + 4kI_1 - K(3 + 4k^2)I_2 - \frac{k}{2} \frac{\sin 2\eta}{\sqrt{K^2 + \sin^2 \eta}} \right. \\
+ e^2 \left\{ (x - \gamma - \frac{3}{4} \sin 2\eta - \frac{1}{4} \sin 4\eta) - 24k^2(x - \gamma - \frac{3}{4} \sin 2\eta) \right. \\
+ 4k \cot \eta (3k - \frac{3k^2 + 28 \sin^2 \eta}{\sqrt{K^2 + \sin^2 \eta}}) - \frac{k}{16} \frac{\sin 2\eta}{\sqrt{K^2 + \sin^2 \eta}} (36k^2 + 25 \sin^2 \eta - 2) \\
- 2k(6k^2 - 15)I_1 - K(-16k^4 - 6k^2 + 9)I_2 \right\} + \ldots \ldots \ldots \ldots \] 
for \[ \frac{dI_1}{dk} = kI_2 \] 
and \[ \frac{dI_2}{dk} = - \frac{1}{k(k^2 + 1)} I_1 + \frac{1}{2k(k^2 + 1)} \frac{\sin 2\eta}{\sqrt{K^2 + \sin^2 \eta}}. \]

Substituting above value of \( \frac{d \omega_1}{dk} \) in (2.3.20) and on simplification we get the value of \( \omega_1 \) as in (2.3.18).

Equations (2.3.14) and (2.3.16) give now respectively the values of \( \omega_1 \) and \( \varphi_1 \) for any point \( (\eta) \) on the original line \( K \)-constant. Then, replacing \( K \) by \( \sinh(\xi - \xi_0) \sin \eta \) we will get the values of \( \omega_1 \) and \( \varphi_1 \) for the point \( (\xi, \eta) \) in the fluid.

But the elliptic integrals appearing in the results of \( \omega_1 \) and \( \varphi_1 \) introduce difficulties when one tries to express
the results in terms of $\xi$ and $\eta$, and thus the results for them are left as functions of $K$ and $\gamma$.

Because of the difficulties just stated above, we do not proceed further to obtain $\varphi_1$ from $\nabla^2 \varphi_1$. But if we regard $\nabla^2 \varphi_1$ as known for any point $(\xi, \eta)$ then we can write for $\varphi_1$, by analogy with electrostatic potential theory* as

$$
\varphi_1(\xi, \eta) = -\frac{1}{2\pi} \iint \log\left(\frac{r_P}{r_Q}\right) (\nabla^2 \varphi_1)(\xi', \eta') \, r' \, d\xi' \, d\eta', \quad (2.3.21)
$$

* We may regard the vorticity $\nabla^2 \varphi_1$ as equivalent to a distribution of electrostatic line charges parallel to the axis of the cylinder; then the determination of $\varphi_1$ is the same as that of two-dimensional potential which will vanish on the surface of the cylinder.
integrate is to be taken throughout the fluid.

In the expression for $\omega_1$ and $\zeta_1$ — equations (2.3.14) and (2.3.18), the functions which will appear as coefficients of $e^4, e^6, \ldots$ are very complicated; they are expected to remain finite for all values of $n$. Hence, when $e$ is small, the terms beyond the second will give negligible contributions.

Because of the difficulties encountered above, we do not further attempt to obtain $\gamma_2, \varphi_2, \omega_2$ and similar functions for higher order approximations. However, with the first approximation a fairly good idea of the flow pattern and the lines of force can be derived. These will be discussed in section 8.5., and in the next section we consider the drag force on the cylinder.

2.4. The drag force on the cylinder.

The force experienced by the cylinder, will be only in the direction of the $x$-axis, as everything is symmetrical about it. So the drag force per unit length of the cylinder is given by

$$D = -\int p \cos \varphi \, ds,$$

where $p$ is the fluid pressure and $\varphi$ the angle made with the $x$-axis by the outward normal at $ds$ on the ellipse and the integration is taken once round the ellipse. [The e.m.force $4\pi$ (Curl $\mathbf{H} \times \mathbf{H}$) is a volume body force; one would get a surface e.m.force on the cylinder only if the field $\mathbf{H}$ were discontinuous at the cylinder.]
By the property of ellipse

\[
\frac{\sin \phi}{a \sin \eta} = \frac{\cos \phi}{b \cos \eta} = \frac{1}{\sqrt{a^2 \sin^2 \eta + b^2 \cos^2 \eta}},
\]

and

\[ds = \sqrt{a^2 \sin^2 \eta + b^2 \cos^2 \eta} \, d\eta.\]

Hence

\[D = -b \int_{\phi=\phi_0}^{\phi} p \cos \eta \, d\eta. \tag{2.4.1}\]

Now, from the definition of \( \omega \),

\[\frac{\partial}{\partial \eta} = \omega = 1 + \frac{1}{2} (u^2 + v^2),\]

where \( u \) and \( v \) are the \( x \) and \( y \) components of velocity,

\[
\omega = C_0 - \frac{1}{2} (u_0^2 + v_0^2) + \frac{1}{m} (\omega_1 - u_0 u_1 - v_0 v_1) + \ldots \ldots \ldots ,
\]

where \( u_0, v_0 \) correspond to \( \psi_0 \) and \( u_1, v_1 \) to \( \psi_1 \), etc., The non-magnetic term will contribute nothing to the integral in (2.4.1); therefore, to the first approximation of \( \frac{1}{m} \)

\[D = -\frac{d\phi}{\eta} \int_{\phi=\phi_0}^{\phi} (\omega_1 - u_0 u_1 - v_0 v_1) \cos \eta \, d\eta. \tag{2.4.2}\]

On the cylinder \( (\phi = \phi_0) \)
\[ u_o = \frac{bU(a+b)}{b^2 + c^2 \sin^2 \eta} \sin^2 \eta \]
\[ v_o = -\frac{bU(a+b)}{b^2 + c^2 \sin^2 \eta} \sin \eta \cos \eta \]

and
\[ u_1 \cos \phi + v_1 \sin \phi = 0 \]

(as demanded by the condition of zero normal component of velocity).

\[ \therefore \text{ Equation (2.4.2) becomes} \]

\[ D = -\frac{bU}{\eta} \int_{\eta_0}^{\eta} \int_{\zeta_0}^{\zeta} u_1 \cos \eta \, d\eta \]
\[ + \frac{A''}{\eta} \int_{\eta_0}^{\eta} \int_{\zeta_0}^{\zeta} \frac{u_1 (a^2 \sin^2 \eta + b^2 \cos^2 \eta)}{b^2 + c^2 \sin^2 \eta} \cos \eta \, d\eta , \]

(2.4.3)

where
\[ A'' = \frac{bU(a+b)}{a} \]  
(2.4.4)

Again, the \( x \)-component of the second equation of (2.3.2b) gives

\[ v_o \nabla^2 \psi_1 = \frac{\partial \psi_1}{\partial x} \]
\[ \therefore \psi_0 = -v_1 y, \frac{\partial \psi_0}{\partial x} = 0 . \]

Integrating this over the whole region of the fluid, one finds
\[ \iint v_0 \nabla^2 \psi_1 \, dx \, dy = \iint \frac{\partial \bar{\omega}_1}{\partial x} \, dx \, dy \]

\[ = \int \left( \bar{\omega}_1 \right)_{x=\pm \infty} \, dy - b \int_{x=\pm \infty} \bar{\omega}_1 \cos \eta \, d\eta , \]

by using Green's theorem and using the fact that \( \bar{\omega}_1 = 0 \) at \( x = \pm \infty \).

Also since \( \nabla^2 v_0 = 0 \),

\[ \iint v_0 \nabla^2 \psi_1 \, dx \, dy = \iint \left( v_0 \nabla^2 \psi_1 - \psi_1 \nabla^2 v_0 \right) \, dx \, dy \]

\[ = - \int_{x=\pm \infty} \frac{1}{b^2} \left( v_0 \frac{\partial \psi_1}{\partial x} - \psi_1 \frac{\partial v_0}{\partial x} \right) \sqrt{a^2 \sin^2 \eta + b^2 \cos^2 \eta} \, d\eta , \]

using Green's theorem and omitting the integral on a large circle. (\( \psi_1 \) cannot contain any term of the order \( r^3 \) as will be verified later). Since \( \psi_1 = 0 \) on the cylinder, the integral on the right-hand side equals

\[ - \int_{x=\pm \infty} A^m \frac{u_1 (a^2 \sin^2 \eta + b^2 \cos^2 \eta)}{b^2 + c^2 \sin^2 \eta} \cos \eta \, d\eta . \]

\[ \therefore - \int_{x=\pm \infty} A^m \frac{u_1 (a^2 \sin^2 \eta + b^2 \cos^2 \eta)}{b^2 + c^2 \sin^2 \eta} \cos \eta \, d\eta \]

\[ = \int \left( \bar{\omega}_1 \right)_{x=\pm \infty} \, dy - b \int_{x=\pm \infty} \bar{\omega}_1 \cos \eta \, d\eta . \]
Hence from (2.4.3)

\[ D = - \frac{c}{\eta} \int \omega_1(x+) \, dy \]

\[ = - \frac{2c}{\eta} \int_0^\infty (\omega_1)(x+) \, dy. \]  

(2.4.5)

To evaluate this integral it is to be noted that for \( K > 0, \ 0 < \eta < \infty \)

\[ \omega_1(K, \eta) = - \int_{\eta}^\infty \frac{\partial \omega_1(K, \eta)}{\partial \eta} \, d\eta. \]

Then making \( x = - \) (as \( x \to -\infty, \ K = y' = \frac{y}{a+b}, \ \eta = 0 \))

\[ (\omega_1)(x+) = - \int_0^\infty (\frac{\partial \omega_1}{\partial \eta})_{y'} \, d\eta. \]

\[ \therefore \text{ From (2.4.5)} \]

\[ D = \frac{2c}{\eta} \int_0^\infty \int_0^\infty (\frac{\partial \omega_1}{\partial \eta})_{y'} \, d\eta, \]
or,

\[
D = \frac{2e}{\eta m} \int_0 \int_a \frac{v^2 b^2}{a^2} \left[ \frac{\cos^2 \eta}{\sqrt{y^2 + (a+b)^2 \sin^2 \eta}} \right] \left( \frac{y^2 + (a+b)^2 \sin^2 \eta}{y^2 + (a+b)^2 \sin^2 \eta} \right) \, dy
\]

\[
+ \frac{e^2}{(a+b)^2} \left\{ \frac{y \cot^2 \eta}{\sqrt{y^2 + (a+b)^2 \sin^2 \eta}} \right\} \left( \frac{y^2 + (a+b)^2 \sin^2 \eta}{y^2 + (a+b)^2 \sin^2 \eta} \right)^3
\]

\[
+ \frac{\cos^4 \eta}{\sin^2 \eta \sqrt{y^2 + (a+b)^2 \sin^2 \eta}} \left( \frac{y^2 + (a+b)^2 \sin^2 \eta}{y^2 + (a+b)^2 \sin^2 \eta} \right)^4 \right] \, dy
\]

(by (2.3.13))

\[
= \frac{\int v^2 A \omega^2}{\eta m U} \int_0 \left[ \sin^2 \eta \cos^2 \eta + e^2 \left\{ \frac{1}{4} \sin^2 \eta \cos^2 \eta + \frac{1}{2} \sin^2 \eta \cos^4 \eta \right\} + \ldots \right] \, d\eta
\]

\[
= \frac{v^2 A \omega^2}{\eta m U} \int \left( 1 + \frac{1}{2} e^2 + \ldots \right)
\]

(2.4.6)

Since

\[
\eta_m = (4n \mu / a - 1) \quad \text{and} \quad v^2_A = \frac{\mu}{4n \rho} \frac{h^2}{a^2}
\]

\[
D = \frac{n u \mu^2 b^2 (a+b)^2}{8 \alpha^2} \left( 1 + \frac{1}{2} e^2 + \ldots \right)
\]

(2.4.7)

This shows that to the order of approximations used, the drag force experienced by the cylinder is independent of the fluid density.
If we put \( e = 0 \) i.e., \( a = b \) in the above result, then we shall get the case for a circular cylinder; and in that case the drag force is

\[
D_C = \frac{\pi}{2} a^2 \mu \alpha H_0^2 U,
\]

which confirms with the result obtained by Singh (unpublished).

2.5. General discussions

We shall discuss here the general features of the pressure distribution, flow pattern and lines of force as represented by the solutions (2.3.14), (2.3.18) and (2.3.8).

For \( \omega_1 \):

As shown by (2.3.14), for given values of \( a \) and \( e \), \( \omega_1 \) is proportional to \( V^2 \); also for any given \( K \) (i.e., on the stream line \( \psi_0 \)) \( \omega_1 \) increases with \( |\gamma| \), though it is stationary at \( |\gamma| = \frac{\pi}{2} \).

Since \( \omega_1 = 0 \) when \( |\gamma| = \pi \) for all \( K \), it follows that \( \omega_1 \) can nowhere be positive and that at \( \gamma = 0 \) (i.e., \( x = +\infty \)) \( \omega_1 \) has a negative maximum value, which depends solely on \( K \) i.e., \( y \) (since as \( x \to \infty \), \( K - y' = y/(a+b) \)). In other words,

as \( x \to -\infty \), \( \omega_1 \to 0 \)

and

as \( x \to +\infty \), \( \omega_1 \to f(y) \);
the function \( f(y) \) (from (2.3.14)) is

\[
f(y) = \frac{a'}{\alpha^2} \left[ ny' - \frac{2}{3} \{ (5\alpha^2 - 4)E + 4(1 - \alpha^2)y \} + \right.
\]
\[
\left. + e^{2} \left( ny' - \frac{2ny'(1 - \alpha^2)}{\alpha^2} \right) - \frac{2}{5\alpha^3} \left[ (71\alpha^2 - 86 + 14)E + 2(27 - 19\alpha^2 - 8\frac{2}{\alpha^2})y \right] \right] + \ldots \]

(2.5.1)

where

\[
\alpha = (1 + y^2)^{-\frac{1}{2}}, \quad F = F(a, \frac{\pi}{2}) \quad \text{and} \quad E = E(a, \frac{\pi}{2}) .
\]

From this we can show that (see appendix at the end of this chapter),

\[
f(y) = - \frac{2}{3} \frac{a'}{\alpha^2} \left( 1 + \frac{3}{\alpha^2} e^2 + \ldots \right) \quad \text{for} \quad y = 0 \quad (\therefore E(1, \frac{\pi}{2}) = 1)
\]

\[
= \frac{a'}{\alpha^2} \left[ ny' - \frac{2}{3} + e^2 \left( - 7ny' - \frac{5}{3} \right) + \ldots \right] \quad \text{for} \quad |y| < 1
\]

\[
= \frac{25a'}{64a^2} y^{-3} \quad \text{as} \quad |y| \to \pm \infty
\]

The function \( a^2 f(y)/a' \) (approximating up to \( a^2 \)) has, for \( \epsilon = 0.1 \), the curve as shown in figure 2 (page 743).

Also values of \( \frac{a^2 \omega_1}{a'} \) for \( K = 1 \) and some values of \( \epsilon \) as given by (2.3.14) are tabulated in Table 1 (page 71). In figure 3 (page 744) are plotted distribution of \( \omega_1 \) for \( K = 1 \) and \( \epsilon = 0.1 \). The stationary character of \( \omega_1 \) at \( \eta = \frac{\pi}{2} \) can be seen in the curve also.
The significance of the fact that $\omega_1 < 0$ is as follows:

$\omega$ is $p_\rho + \frac{1}{2} v^2$ which is a constant in irrotational motion. Thus $\omega_1 < 0$ coupled with $\omega_1 = 0$ at $x = -\infty$ implies that either one needs a pressure gradient along a stream line to overcome the magnetic drag or the motion is slowed down by the drag or both. Thus to the first approximation there is a fall below the original pressure, the fall being greater on the positive side of $x = 0$. At large distances from the cylinder on the negative side of $x$-axis or for $|y|$ large the pressure tends to its original value i.e., remains unaffected by the field.

For $\xi_1$ and $\psi_1$:

The vorticity $\xi_1 (= \nabla^2 \psi_1)$ (as given by (2.3.18)) does not depend on the speed $u$ of the original streaming motion at $x = -\infty$, but is simply proportional to $V_A^2$ (the square of the Alfvén velocity). As asserted before, it is continuous on the ($-ve$) $x$-axis but discontinuous on ($+ve$) $x$-axis, the amount of discontinuity being

$$-\frac{2\pi V_A^2}{a} (1 + \frac{1}{2} e^2 + ...);$$

also it has opposite signs on opposite sides of the $x$-axis. This discontinuity of $\xi$ implies that there is a vorticity defect in the downstream side. As Tamada (1962) has shown in the case of flow past a circular cylinder, there is a region near the axis of $x$, where the
velocity decreases and vorticity is pre-dominant and where the kinetic energy decreases. Hence one may infer the existence of a vorticity wake where vorticity is significant outside which it is insignificant. The profile of trial at far downstream may be obtained by putting $\gamma = 0$ (only in the upper plane) and $K = y'$ in (2.5.18). Thus we obtain

$$S_{1x} = \frac{A''}{a} \left[ \left\{ -m + \frac{E'}{a} \left[ (4-a^2)E' - 4E' \right] \right\} \right]$$

+ $e^2 \left\{ - \frac{a^2}{2} + 24xy' \left( \frac{E'}{a} \left[ 16(2-5a^2)E' - (a^4 - 45a^2 + 32)E' \right] \right) \right\} + \ldots \right\}$

(2.5.3)

where

$$a = (1+y'^2)^{-\frac{1}{2}}, \quad f = f(a, \frac{a}{B}) \quad \text{and} \quad K = K(a, \frac{a}{B}).$$

From this we can show that (see appendix at the end of this chapter)

$$S_{1x} = \begin{cases} \frac{A''}{a} \left[ -m - 6y' \log y' + 4(3 \log 2 - 2) y' \right] \\ + e^2 \left\{ - \frac{a^2}{2} - 15y' \log y' + 3(10 \log 2 - 16) y' \right\} + \ldots \right\}, \quad \text{for } y < 1 \\
- \frac{B}{24} A' \quad \text{for } y > 1. \end{cases}$$

(2.5.4)

The function $\frac{A''}{a} S_{1x}$ (approximating up to $e^2$) has, for $a = 0.1$, the curve as shown in figure 4 (page 74c).
For any $k(>0)$, $\gamma_1$ increases from $\eta = 0$ nearly upto

$\eta = \frac{\alpha}{3}$ and then decreases upto $\eta = \frac{2\alpha}{3}$ and then again increases finally vanishing at $\eta = \alpha$ (i.e., at $x = -\infty$). For $k < 0$, the behaviour is just reversed. Numerical values of $\gamma_1$ for $k=1$ and some values of $\alpha$ are tabulated in Table II (page 72). These values also confirm the above mentioned behaviour of $\gamma_1$.

Again, by taking the $x$-component of the second equation of (2.3.2b) and then eliminating $\frac{\partial \psi}{\partial x}$ with the help of equation (2.3.9) $\left( q_0 \frac{\partial \psi}{\partial x} = u_0 \frac{\partial \psi}{\partial x} + v_0 \frac{\partial \psi}{\partial y} \right)$, we get

$$\gamma_1 = -\frac{2U}{u_0} \left[ -\frac{(at+b)}{2c^2} + \frac{\sin 2\eta}{(\cosh \frac{\eta}{c} - \cos \eta)} + \frac{1}{V_A} \frac{\partial \psi}{\partial y} \right],$$

where (from (2.3.4))

$$u_0 = \frac{U(at+b)}{c} \frac{\cosh \frac{\eta}{c} \cosh \left( \frac{\eta}{c} - \frac{\eta_0}{c} \right) - \cosh \frac{\eta}{c} \cos \eta \cosh \frac{\eta}{c} - \cos \eta \cosh \frac{\eta}{c}}{\cosh^2 \frac{\eta}{c} - \cos^2 \eta}.$$  

This expression for $u_0$ can be simplified as

$$u_0 = \frac{U(at+b)}{c} \frac{\cosh \frac{\eta}{c} \cosh \left( \frac{2\eta}{c} - \frac{2\eta_0}{c} \right) - \cosh \frac{\eta}{c} \cos \frac{\eta}{c} \cosh \frac{\eta}{c} - \cos \frac{\eta}{c} \cosh \frac{\eta}{c}}{\cosh^2 \frac{\eta}{c} - \cos^2 \frac{\eta}{c}}.$$  

If $r$ be the distance from the origin, then

$$r^2 = c^2 \left( \cosh^2 \frac{\eta}{c} - \sin^2 \eta \right).$$
so that,
\[ a(\cosh^2 \gamma - \cos^2 \eta) - b \sinh \gamma \cosh \gamma = a \left( \frac{K}{\eta^2} - \cos 2\eta \right) - b \sqrt{\frac{K}{\eta^4} - \frac{K^2}{\eta^6} \cos 2\eta} - \frac{1}{4} \sin^2 2\eta. \]

Hence, by (2.5.5), it seems that curves \( \int \varphi_1 = \text{constant} \) have a tendency to form loops on both sides of \( y \)-axis.

For large values of \( K \) (outside the trail), the leading term in \( \varphi_1 \) (2.3.18) is of \( O(1/K^2) \) and hence \( \varphi_1 \to 0 \) as \( |K| \to \infty \) more slowly than \( \omega_1 \).

Using the above results, one can now have, to the first approximation, an idea of the nature of motion in the fluid.

Since
\[ \varphi_1 = -\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}, \]

the equation (2.5.5) shows that at large distance from the cylinder \( u_1 \) can be replaced by \( U^{-1} \omega_1 \) (\( v_1 \) is expected to tend to zero). Thus
\[ u = u_0 + \frac{1}{\eta} u_1 \to U + \frac{1}{\eta} U \omega_1 \to U + \frac{1}{\eta} U f(y) \quad \text{as } x \to \infty \]
\[ \to U \quad \text{as } x \to \infty \]

and
\[ v = v_0 + \frac{1}{\eta} v_1 \to 0 \quad \text{as } x \to \infty \]

where \( f(y) \) is as stated earlier, always negative and equal to
\[-\frac{2}{3} \frac{K'}{a^2} (1 + \frac{5}{6} \cdot e^2 + \ldots) \quad \text{for } y = 0\]

and

\[\rightarrow 0 \quad \text{as } y \rightarrow \infty\]

The figure 5 shows the velocity profile at large distance of x on the right side. Hence the flow at large distance of x on the positive side is slower than at large distance on the negative side and as a consequence the stream lines will be widened on the (+ve) side of x-axis (as shown in figure 6). It is because of this

\[\text{Fig. 5. The velocity profile at large distance of } x \text{ on the right.}\]

\[\text{Fig. 6. Sketch of the stream lines on both sides of } x\text{-axis.}\]
widening of the stream lines on the (+ve) side of x-axis that the cylinder experiences a drag force.

For the magnetic field:

We have

\[ H_z = H_{0f} + \frac{1}{\eta} H_{1f} \]

\[ \frac{H_{0c} \sinh \gamma \sin \eta}{h} + \frac{H_{0a}}{4h} \eta^2 \left[ 1 - \cosh 2 \zeta_0 (\cosh 2 \gamma - \sinh 2 \gamma) \right] \cos 2 \eta \]

and

\[ H_\eta = H_{0\eta} + \frac{1}{\eta} H_{1\eta} \]

\[ -\frac{H_{0c} \cosh \gamma \sin \eta}{h} - \frac{H_{0a}}{4h} \eta^2 \left[ \cosh 2 \zeta_0 (\cosh 2 \gamma - \sinh 2 \gamma) \right] \sin 2 \eta \]

and the lines of force are given by

\[ \eta \sinh \gamma \sin \eta + \frac{1}{\eta} \frac{b_0 u (a + b)}{8} \left[ 1 - \cosh 2 \zeta_0 (\cosh 2 \gamma - \sinh 2 \gamma) \right] \sin 2 \eta = \text{constant} \]

\( (\gamma > \zeta_0) \).

Thus the field, unlike the velocity of the fluid, remains uniform and parallel to the x-axis at large distances from the cylinder.
2.6. Validity of the expansions (2.3.1)

In this section we shall discuss briefly the convergence of the expansions (2.3.1).

From equations (2.3.5), (2.3.6), (2.3.7) we have, so far as orders of magnitude are concerned, that near the cylinder

\[ \psi_0 \sim Ua, \quad \chi_0 \sim V_A a, \quad \omega_0 \sim C_0. \]

From equations (2.3.2b), (or from the results derived from them)

\[ \psi_1 \sim V_A^2 a^2, \quad \chi_1 \sim UV_A a^2, \quad \omega_1 \sim V_A^2 Ua, \]

and from equations (2.3.2c)

\[ \psi_2 \sim \left( \frac{V_A}{U + U^2 V_A} \right) a^3, \quad \chi_2 \sim \left( V_A^3 + U^2 V_A \right) a^3, \quad \omega_2 \sim \left( V_A^4 + U^2 V_A \right) a^2. \]

Similarly

\[ \psi_3 \sim \left( \frac{V_A^2}{U} + V_A^4 + U^2 V_A \right) a^4, \quad \chi_3 \sim \left( \frac{V_A^5}{U} + U V_A^3 + U^3 V_A \right) a^4, \]

\[ \omega_3 \sim \left( \frac{V_A^6}{U} + U V_A^4 + U^3 V_A \right) a^3; \]

and so on.

Putting \( V_A/U = \beta \), then \( \beta \) is a pure number as \( V_A \) has the dimension of a velocity. (This \( \beta \) is in fact, the Alfvén number.)
If $\beta = O(1)$, then the expansions (2.3.1) can be expressed as

$$\psi = U_0 \left[ \sum n + m_1 \left( \frac{U_A}{\eta_m} \right) + m_2 \left( \frac{U_A}{\eta_m} \right)^2 + m_3 \left( \frac{U_A}{\eta_m} \right)^3 + \ldots \right],$$

$$\chi = V_A \left[ \sum n + n_1 \left( \frac{U_A}{\eta_m} \right) + n_2 \left( \frac{U_A}{\eta_m} \right)^2 + n_3 \left( \frac{U_A}{\eta_m} \right)^3 + \ldots \right],$$

$$\omega = \omega_0 + V_A^2 \left[ k_1 \left( \frac{U_A}{\eta_m} \right) + k_2 \left( \frac{U_A}{\eta_m} \right)^2 + k_3 \left( \frac{U_A}{\eta_m} \right)^3 + \ldots \right],$$

where $a, n, k$'s are dimensionless and expected to be of order unity. Then the series are convergent if $R_m$ (magnetic Reynolds number) $= \frac{U_A}{\eta_m} \ll 1$. If $\beta$ is large, the condition for the convergence is that $\beta^2 R_m$ should be small.

2.7. Conclusions

We have considered the problem of two-dimensional flow of an inviscid, incompressible and electrically conducting fluid past an elliptic cylinder of small eccentricity in presence of an aligned uniform magnetic field. In contrast to other works in similar problems the analysis has been carried out by retaining inertia terms in the equations and using a kind of perturbation
technique. The first order corrections to undisturbed flow and field are obtained and the drag force experienced by the cylinder is also calculated. The results agree with that of Tamada (1962) and Singh (unpublished) for the case of circular cylinder.

The drag force experienced by the cylinder is found to be independent of fluid density to the order of approximations considered. The physical reason for the drag force to be independent of the fluid density may be attributed to the fact that the drag force is only due to the lines of force, fluid being inviscid. The lines of force tend to slow down the fluid motion more on the downstream side of the cylinder than on the upstream side. So the stream lines will be widened on the downstream side, and because of this widening on the downstream side, the cylinder experiences a drag force.

It is also found that one needs a pressure gradient along a stream line to overcome the magnetic drag or the motion is slowed down by the drag or both. Thus there is a fall below the original pressure, the fall being greater on the positive side of x-axis.

And the field unlike the velocity of the fluid remains uniform and parallel to the x-axis at large distances from the cylinder.
2.3. Some remarks on MHD flow past solid bodies

Certain common features seem to be recognizable in the results of a number of studies of the problem of steady flow past solid bodies in the presence of a magnetic field, with different approximations. The features are, broadly speaking, those which appear to distinguish flows of low conductivity from those of high conductivity or more accurately, flows of low magnetic Reynolds number $R_m$ from those of high $R_m$. There are unmistakable analogies between the influences of the magnetic and the actual Reynolds numbers i.e., between electric resistance and viscosity in their effects upon flow patterns. Thus the features that distinguish between low-$R_m$ and high-$R_m$ flows are often analogous to those that distinguish between flows of small and large Reynolds numbers.

Flow patterns when the magnetic Reynolds number is small are analogous to flows of small Reynolds number, such as those treated in the Stokes— and Oseen— type approximations. The analogy arises from the fact that in both the situations the diffusion process dominates the convection process. Thus the flow past a solid body results in a large vortical wake, for the disturbance due to the body diffuses rapidly into the fluid as it is carried downstream.

But in MHD there is a new remarkable phenomenon appearing, that is, the formation of two wakes instead of one; one upstream and the other downstream. The physical reason seems
to be that small disturbances propagate without diffusion along the lines of force for sufficiently strong field with a velocity relative to the fluid equal to the Alfvén velocity. Such disturbances reach infinity always in the downstream, and upstream if the flow speed is less than Alfvén velocity. Thus the lateral extent of these wakes is determined by the speed of diffusion in relation to the effective transport speed, which is the Alfvén-wave propagation relative to the moving medium.

In general, the low-$R_m$ flows are characterised by large 'wakes' of vorticity and electric current; diffusion being a dominant process. At moderate values of $R_m$, there must be vortical wakes extending outward from the solid bodies in the directions determined by the effective convection velocity, which is the resultant of the true convection and Alfvén propagation. As $R_m$ is increased, such wakes become narrow, for at sufficiently large $R_m$ the region of vortical current-carrying flow lose the character of wakes and become narrow diffusion zones lying along wave-like disturbances. These disturbances are standing Alfvén waves, their directions determined by the resultant of Alfvén propagation and convection by the moving fluid. Thus in the case of large-$R_m$ steady flow, the picture is dominated by standing Alfvén waves. The effect of electric resistance is only to diffuse these waves, and thus to cause them to alternate at a distance from the solid obstacle.

In the present problem we assume the magnetic diffusivity
\( \eta_m = \left(4\pi N/\sigma\right)^{-1} \) to be large or the magnetic Reynolds number to be small. Thus this is a low-\( R_m \) case and so all characteristics of the low-\( R_m \) flows can be observed in the consideration of the present problem.
Table I  
Values of $\tilde{\omega}_1 (= \frac{2\tilde{\omega}_1}{k})$ for $k=1.0$ and for different values of $\epsilon$.

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Table II

Values of $\gamma'_1(=\frac{a}{2^\gamma})$ for $K=1.0$ and for different values of $e$.

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APPENDIX

In deriving the approximate values of \( f(y) \) and
\[
\sum_{x = \infty}^x
\]
for small and large values of \( y \) (equations (2.6.2) and (2.6.4)) we made use of approximate values of \( F \) and \( E \).

For small values of \( y \), we may put
\[
\alpha^2 = \frac{1}{1+y'^2} = 1 - \varepsilon^2, \quad \varepsilon^2 \ll 1
\]

\[
\varepsilon = \int_0^{\frac{x}{2}} \left(1 - \varepsilon^2 \sin^2 \theta \right) - \frac{1}{2} \, d\theta
\]

\[
= \frac{x}{2} - \varepsilon^2
\]

\[
= \int_0^{\frac{x}{2}} (\cos^2 \theta + \varepsilon^2 \sin^2 \theta) - \frac{1}{2} \, d\theta + \int_{\frac{x}{2}}^{\frac{x}{2}} (\cos^2 \theta + \varepsilon^2 \sin^2 \theta) - \frac{1}{2} \, d\theta
\]

\[
= \frac{x}{2} - \varepsilon^2
\]

\[
= \int_0^{\phi'} \frac{d\theta}{\cos \theta} + \int_0^{\phi'} (\phi'^2 + \varepsilon^2) - \frac{1}{2} \, d\phi' \quad (\phi = \frac{x}{2} - \phi')
\]

\[
= \left[ \frac{1}{2} \log \frac{1+\sin \theta}{1-\sin \theta} \right]_0^{\frac{x}{2}} - \left\{ \log \left[ -\phi' + (\phi'^2 + \varepsilon^2) \right] \right\}_0^{\phi'}
\]

\[
= - \log \varepsilon + 2 \log 2.
\]

But \( \varepsilon = y' \).

Therefore we have
\[
y = - \log y' + 2 \log 2. \quad (1)
\]
And \( x = \int \left( 1 - \alpha^2 \sin^2 \theta \right)^{\frac{1}{2}} \, d\theta \)

\[
\frac{3}{2} \cos \theta \, d\theta
\]

\[= 1. \quad (\text{II})\]

Again, for large \( y \), \( \alpha \) becomes very small and we have the expansions

\[
y = \frac{3}{2} \left[ 1 + \left( \frac{1}{2} \right)^2 \alpha^2 + \left( \frac{1}{2} \cdot \frac{3}{2} \right)^2 \alpha^4 + \left( \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \right)^2 \alpha^6 + \left( \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} \right)^2 \alpha^8 + \ldots \right], \quad (\text{III})
\]

\[
x = \frac{3}{2} \left[ 1 - \left( \frac{1}{2} \right)^2 \alpha^2 - \frac{1}{3} \left( \frac{1}{2} \cdot \frac{3}{2} \right)^2 \alpha^4 - \frac{1}{6} \left( \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \right)^2 \alpha^6 - \frac{1}{7} \left( \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} \right)^2 \alpha^8 + \ldots \right], \quad (\text{IV})
\]

Also \( \alpha \) can be expanded as

\[
\alpha = y^{-1} \left( 1 - \frac{1}{2} y^{-2} + \frac{3}{8} y^{-4} - \ldots \right). \quad (\text{V})
\]

Using (I) - (V) in (2.5.1) and (2.5.3) one can get (2.5.2) and (2.5.4) respectively.
FIG. 2. \( f(y) \) IN THE TRAIL (FOR \( \varepsilon = 0.1 \))
Fig. 3. Distribution of $\tilde{\omega}_k$ for $k=10$ and $\tilde{\omega}_0$.
FIG. 4. PROFILE OF VORTICITY DEFECT IN THE TRAIL (FOR $e = 0.1$)
References

Hasimoto, H. (1966) : Read before the 13th annual meeting of Physical Society of Japan held at Kyoto University.