CHAPTER I

INTRODUCTION AND GENERAL THEORY
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1.1. Introduction

Magnetohydrodynamics (or hydromagnetics) is the combination of hydrodynamics and electromagnetic theory. It is concerned, in the broad sense, with the motion of an electrically conducting fluid in the presence of a magnetic field; the fluid may be either a liquid or an ionized gas. The motion of the conducting fluid across the magnetic field generates electric currents which modify the magnetic field, and at the same time the electric currents react with the magnetic field to produce a body force which in turn modifies the motion. Thus, magnetohydrodynamics (often referred to as MHD) is the branch of science which deals with the study of interaction between the fluid motion and electromagnetic phenomena. The study was primarily inspired by geophysical and astrophysical problems and by problems associated with the fusion reactor.

Recently MHD has been the subject of intensive study and the importance of the study has been extended to many other kinds of associated problems - even in missile rockets. Technological problems like controlled thermonuclear fusion, thrust production for propulsive devices, pumping of liquid metals, high-temperature resistant coating, re-entry problem of ballistic
missiles, liquid metal lubrication, power conversion (i.e. extraction of electrical energy directly from a hot plasma*) etc. etc. need the study of the flow of an electrically conducting fluid.

As also stated earlier, the electrically conducting fluid may be either a liquid or an ionised gas. If the latter can be regarded as a continuous fluid and also if the properties of the component individual particles are accounted only through their effect on the viscosity and the thermal and electrical conductivities, then both a liquid and an ionised gas can be treated in a common theory. Here only a liquid will be dealt with. The term 'fluid' too will, however, be used in the thesis.

1.2. General equations

The general equations of Magnetohydrodynamics will be first discussed. These are the ordinary hydrodynamic and electromagnetic equations, modified to take into account of the interaction between the fluid motion and the magnetic field.

Electromagnetic part of the equations:

The classical Maxwell's equations with some characteristic approximations provide the electromagnetic part of the equations. In a large class of electromagnetic problems involving conductors but not

* We may use the term 'plasma' for ionised gas, since plasma is also an ionized gas.
concerned with rapid oscillations, low-frequency approximations are successfully used to treat them. Likewise, in the consideration of MHD problems not concerned with rapid oscillations the same type of approximations can be used, and these lead (as Klassen showed) to the neglect of the displacement current in Maxwell's equations. We shall not be concerned here with rapid oscillations, Maxwell's displacement current will, therefore, be ignored. As a consequence of this neglect, the accumulation of electric charge is also neglected in the equation of continuity of charge. This leads to the result that electric currents flow in closed circuits. Thus, if \( \mathbf{J} \) is the current density,

\[
\text{div} \mathbf{J} = 0, \quad (1.2.1)
\]
everywhere in the fluid. [In the equation of continuity of electric charge the term representing the rate of change of charge is, following low frequency approximation, of order \( v^2/c^2 \), where \( v \) is the material velocity and \( c \) is the velocity of light; its neglect is thus quite appropriate in most ordinary problems.]

With the displacement currents ignored, the electromagnetic equations are

\[
\begin{align*}
\text{Curl} \, \mathbf{H} & = 4 \pi \mathbf{J}, \\
\text{Curl} \, \mathbf{E} & = -\frac{\partial \mathbf{B}}{\partial t}, \\
\text{div} \, \mathbf{B} & = 0.
\end{align*}
\]  

(1.2.2)  

(1.2.3)  

(1.2.4)

The electromagnetic variables are all measured in electromagnetic units (EMU); \( \mathbf{E} \) and \( \mathbf{H} \) are the intensities of the electric and the magnetic fields, \( \mathbf{J} \) is the current density and \( \mu \) is the magnetic
permeability. In MHD the magnetic permeability is unity, however \( \mu \) will be retained in the equations to identify the units. The equation (1.2.4) is only an initial condition since (1.2.3) implies that
\[
\frac{\partial}{\partial t} \text{div} \vec{H} = 0.
\]
However, in steady problems, it is necessary as a governing equation.

In a stationary conductor Ohm's law states that
\[
\vec{J} = \sigma \vec{E},
\]
where \( \sigma \) is the electrical conductivity. In MHD, Ohm's law is modified to
\[
\vec{J} = \sigma \left[ \vec{E} + \mu \vec{v} \times \vec{H} \right], \tag{1.2.5}
\]
where \( \vec{v} \) is the material velocity. Through the occurrence of the velocity \( \vec{v} \) in the expression for \( \vec{J} \), the equations incorporate the effect of motion on the electromagnetic field.

The pondermotive force experienced by the matter (per unit volume) is
\[
\vec{F} = \mu \vec{J} \times \vec{H}, \tag{1.2.6}
\]
this goes under the name Lorentz force, and this is the force responsible for the modification of the (otherwise purely hydrodynamical) motion, and this is to be taken into account in the ordinary hydrodynamical equation of motion.

Hydrodynamical part of the equations:

If \( \rho \) is the fluid density, the hydrodynamic equation of continuity is
\[
\frac{\partial \mathbf{f}}{\partial t} + \text{div} (\mathbf{f} \mathbf{v}) = 0. \quad (1.2.7)
\]
An alternative form of equation (1.2.7) is
\[
\frac{\partial \mathbf{f}}{\partial t} + \mathbf{f} \cdot \text{div} \mathbf{v} = 0, \quad (1.2.8)
\]
where \( \frac{\partial}{\partial t} \) is the time operator
\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi} + \mathbf{v} \cdot \text{grad}. \quad (1.2.9)
\]
For an incompressible homogeneous fluid, the equation of continuity reduces to
\[
\text{div} \mathbf{v} = 0. \quad (1.2.10)
\]
For an incompressible fluid of variable density, the above equation of continuity is to be supplemented by
\[
\frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla) \rho = 0, \quad (1.2.11)
\]
which means that the density of an element remains unchanged as it moves about. This is to be regarded as a fundamental equation when the density is not uniform.

In the equation of motion there appears a body force given by equation (1.2.6) of electromagnetic origin. If the only other body force is gravity with (vector) acceleration \( \mathbf{g} \), the equation of motion is then
\[
f \frac{d\mathbf{v}}{dt} = \text{-} \text{grad} \ p + \mathbf{f} \mathbf{g} + \mathbf{F} + \mu \mathbf{J} \times \mathbf{H}, \quad (1.2.12)
\]
where \( p \) is the pressure and \( \mathbf{F} \) is the viscous force per unit volume. For a Newtonian incompressible fluid \( \mathbf{F} \) assumes simply
\[
\mathbf{f} = \nu \nabla^2 \mathbf{v}, \quad (1.2.13)
\]
where \( \nu \) is the kinematic viscosity, \( \nabla^2 \) is the Laplacian operator.
and in cartesian co-ordinates \((x,y,z)\) it is given by

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.
\]

For problems involving heat distribution, the above equations are to be supplemented by the heat equation.

If \(U\) is the heat energy per unit mass, the heat equation is

\[
f \frac{\partial U}{\partial t} = \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right), \tag{1.2.14}
\]

where \(T\) is the temperature and \(k\) is the co-efficient of heat conduction. In the equation \(1.2.14\), the second, third and fourth terms on the right-hand side give the heating effect per unit volume due to heat conduction, viscosity and the flow of electric currents respectively. Of these, the first is normally the most important.

Equations \((1.2.1)\) to \((1.2.5)\), \((1.2.7)\), \((1.2.12)\) and \((1.2.14)\) constitute the basic equations of magnetohydrodynamics.

1.3. Discussion of the general equations

(a) If \((\text{electrical conductivity})\) is uniform in space, by equations \((1.2.5)\), \((1.2.2)\), \((1.2.3)\) and \((1.2.4)\), we obtain

\[
\frac{\partial \mathbf{H}}{\partial t} = \text{Curl} (\mathbf{\nabla} \times \mathbf{H}) + \gamma \mathbf{\nabla} \cdot \mathbf{H}, \tag{1.3.1}
\]

where

\[
\gamma = (4\pi \mu_\sigma)^{-1}. \tag{1.3.2}
\]

Equation \((1.3.1)\) gives the change in the magnetic field. The first term on the right-hand side is the contribution due to the motion of
the fluid, while the second term represents the diffusion of magnetic field due to finiteness of electrical conductivity.

If the material is at rest the equation (1.3.1) becomes

$$\frac{\partial H}{\partial t} = \gamma \nabla \times H. \quad (1.3.3)$$

This has the form of a diffusion equation; the quantity $\gamma$ can be called the magnetic diffusivity. This equation indicates that the field 'leaks' through the material from point to point. A decay of the field results and time of decay can be seen to be of the order $L^2 \gamma^{-1} = 4\pi \mu / L^2$, where $L$ is the length comparable with the dimension of the region in which current flows.

Again, if the material is in motion, but has negligible electric resistance, the equation (1.3.1) becomes

$$\frac{\partial H}{\partial t} = \text{Curl} (\nabla \times H). \quad (1.3.4)$$

This equation for $\vec{H}$ is analogous to the equation for vorticity in the ideal fluid theory. Thus equation (1.3.4) ensures that the magnetic lines of force move with the material as if they are glued to it. Following Alfvén one may say that the lines of force are 'frozen' into the material. In the frozen-in-field case the currents are simply to be regarded as given by the equation (1.2.2).

When neither term of the right-hand side of the equation (1.3.1) is negligible, the change in the field $\vec{H}$ is affected partly by transportation and partly by diffusion. We thus say that when the material is in motion and is of finite conductivity, the lines of force tend to be carried about with the material and at the same
time they leak through it. Using dimensional consideration, the transport effect dominates the leak if \( LV > \gamma \) where \( L \) and \( V \) are the characteristic length and velocity associated with the field. By analogy with the Reynolds number

\[
R = \frac{VL}{\nu},
\]

(1.3.5)
of ordinary hydrodynamics, we can define the 'magnetic Reynolds number' \( R_m \) by

\[
R_m = \frac{VL}{\gamma}.
\]

(1.3.6)

Thus the condition for transport to dominate the leak is that \( R_m \) shall be large compared with unity.

This condition is only rarely satisfied in the laboratory (for mercury \( \gamma \) is about \( 8 \times 10^3 \text{ cm}^2/\text{sec} \)) and hence the lines of force slip readily through the material. But in cosmic masses the condition is satisfied because of enormous size of \( L \) and hence the leak of the lines of force is slow and they can be regarded as nearly frozen into the material.

(b) Retaining viscous and magnetic forces, the equation of motion (1.2.12) for an incompressible homogeneous fluid is

\[
f \left\{ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right\} = \nabla p + \nu \nabla^2 \vec{v} + \mu \nabla \times \vec{H},
\]

(1.3.7)
or,

\[
\frac{\partial \vec{v}}{\partial t} + \nabla \times \vec{v} = -\nabla \gamma + \nu \nabla^2 \vec{v} + \frac{\mu \epsilon}{\nu} \left( \vec{E} + \mu \nabla \times \vec{H} \right) \times \vec{H},
\]

(1.3.8)

where

\[
\vec{\omega} = \frac{\partial \vec{v}}{\partial t} + \frac{1}{2} \nabla^2 \vec{v}.
\]

(1.3.9)
In writing the magnetic term in (1.3.8), use is made of the equation (1.2.5).

The mechanical force $\mathbf{J} \times \mu \mathbf{H}$ of electromagnetic origin is perpendicular to the magnetic field. It thus has no direct influence on the motion along the lines of force, but it opposes the motion across the lines of force. If $L$ and $H$ be the characteristic length and field strength, the ratio of the magnetic force to the viscous force is

$$M^2 = \frac{\mu L^2 H^2}{(\sigma \nu)}.$$ \hspace{1cm} (1.3.10)

Thus, the magnetic drag will be dominant over the viscous drag if $M$ is large as compared to unity. If $M \ll 1$, the viscous drag will be dominant, in which case the problem will approach the ordinary hydrodynamical problem. The number $M$, which equals $\mu HL(\sigma \nu)^{1/2}$, is called Hartmann number, since its importance was first discovered by J. Hartmann (1937).

In general, a magnetic field opposes motion perpendicular to it and always tends to oppose the onset of turbulence in a conducting fluid.

1.4. A simple illustration

As a simple illustration of magnetohydrodynamic equations laid down above, we can cite Hartmann flow. We consider the steady (parallel) flow of a viscous homogeneous incompressible conducting fluid between two stationary non-conducting parallel planes at $z=\pm L$.
in presence of a uniform magnetic field $\mathbf{H}_0$ along the $z$-direction (i.e., perpendicular to the plane walls). The field acquires a component parallel to the motion, and let $h_x$ be this component. The electric field $\mathbf{E}$ reduces to be static and uniform, and acts in the $y$-direction having thus zero component normal to the boundary walls. Thus in the steady state, we have from equations (1.3.7) and (1.2.5)

$$-\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial z} \frac{\partial \phi}{\partial z} = -\frac{\mu}{\sigma} \mathbf{H}_0 \cdot \mathbf{j}_y = -\frac{\mu}{\sigma} \mathbf{H}_0 (E - \frac{\mu}{\sigma} \mathbf{v} \cdot \mathbf{H}_0). \quad (1.4.1)$$

And equation (1.2.2) gives

$$4\pi \mathbf{j}_y = \frac{\partial h_x}{\partial z}. \quad (1.4.2)$$

Here all the variables $\phi$, $\mathbf{j}_y$, and $h_x$ are functions of $s$ only, and the pressure gradient

$$-\frac{\partial p}{\partial x} = P$$

is the same at all points of the field. Equation (1.4.1) can be written as

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{\mu}{\sigma} \frac{\partial \phi}{\partial z} \left[ \mathbf{j}_y - \frac{E}{\mu \mathbf{H}_0} - \frac{P}{\sigma \mu \mathbf{H}_0^2} \right]. \quad (1.4.3)$$

The solution subject to the condition $\phi = 0$ at $s = \pm L$ is

$$\phi = \left( \frac{E}{\mu \mathbf{H}_0} + \frac{P}{\sigma \mu \mathbf{H}_0^2} \right) \left( 1 - \frac{\cosh \frac{Ms}{L}}{\cosh M} \right), \quad (1.4.4)$$

where $M$ is the Hartmann number defined by

$$M = \frac{\mu \mathbf{H}_0 L}{\frac{\mu}{\sigma} \mathbf{H}_0} \left( \frac{E}{\mu} \right)^{1/2}. \quad (1.4.5)$$
The expression for \( v \) (equation (1.4.4)) shows that a flow is possible without the pressure gradient \( P \). This may happen if the uniform field \( B \) is imposed in proper direction; for example, by passing the electric current in the fluid parallel to \( y \)-axis. This is, in fact, the basis of the magnetic pump.

If no imposed current flows in the fluid, we must have

\[
\frac{L}{-L} J dz = 0.
\]

This determines the electric field and leads to

\[
(P + \mu EH_0) \tanh M = PM.
\]

Then (1.4.4) becomes

\[
v = \frac{PM}{\sigma \mu E H_0^2} \left( \frac{\cosh M - \cosh \frac{Ms}{L}}{\sinh M} \right).
\]

This shows that for small \( M \) i.e., when viscosity dominates over induction drag the velocity profile is effectively parabolic. If \( M \) is large viscosity is unimportant except in a thin boundary layer near the walls and away from the walls \( v \) is nearly a constant. These explain the importance of the Hartmann number.

The field component \( h_x \) is found to be

\[
h_x = H_0 R_e \left( \frac{\sinh \frac{Mr}{L} - r/L \sinh M}{M \cosh M - \sinh M} \right),
\]

where \( R_e = 4\pi \mu LV \) is the magnetic Reynolds number. The fraction on the right is always finite and tends to zero as \( M \to \infty \). Thus \( h_x \) is small compared with \( H_0 \), and the lines of force are not greatly distorted.
by flow, if $R_m$ is small.

There is a good agreement between theory and experiment for high values of fields and low speed flow, but at high flow speeds and magnetic fields the pressure gradient becomes too high which appears to be caused by the breakdown of laminar flow.

It is worth mentioning that investigation of the type of the problem discussed above is of importance from the point of view of electromagnetic measuring devices such as flowmeters. With the help of such flowmeters the rates of flow of fluids like liquid metals, blood and sea water etc. can be determined by measuring the potential difference induced in these fluids by motions through transverse magnetic fields.

1.5. Magnetic energy

A magnetic field possesses energy $\frac{\mu H^2}{2\pi}$ per unit volume. Thus the total magnetic energy is

$$W_H = \frac{1}{4\pi} \int H^2 d\tau ,$$

(1.5.1)

the integration being taken over the volume occupied by the field. The rate of increase of the magnetic energy is

$$\frac{dW_H}{dt} = \frac{1}{4\pi} \int \mu \left[ \mathbf{H} \cdot \nabla \times \mathbf{H} + \gamma \mathbf{H} \cdot \nabla^2 \mathbf{H} \right] d\tau$$

(on the use of the equation (1.3.1))

$$= \frac{1}{4\pi} \int \left[ \mathbf{H} \cdot \nabla \mathbf{H} \right] d\tau - \nu \int \left[ \mathbf{H} \cdot \nabla \mathbf{J} \right] d\tau ,$$
which can further be reduced (with the help of equations (1.2.2) and (1.2.5)) to

\[
\frac{dW_H}{dt} = - \int \nabla \cdot (J \times \mu H) \, dt - \int \nabla \cdot \mathbf{D} \, dt - \frac{1}{4\pi} \int (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{n} \, ds
\]

(1.5.2)

where \( \mathbf{n} \) is a unit vector normal to the surface bounding the field.

The third term on the right-hand side represents the Poynting flux of energy flowing over the surface bounding the field. If the field extends to infinity, then this surface integral vanishes since in MHD both \(|\mathbf{E}|\) and \(|\mathbf{H}|\) are at most of order \([\text{distance}]^{-2}\). Thus in this case

\[
\frac{dW_H}{dt} = - \int (J^2 / c) \, dt - \int \nabla \cdot (J \times \mu H) \, dt.
\]

(1.5.3)

The first term on the right-hand side represents the conversion of magnetic energy into joule heat, at the rate of \( J^2 / c \) per unit volume. The second term represents the work done by the material against the magnetic force \( J \times \mu H \) during the motion. Since magnetic force can be shown to be equivalent to a hydrostatic pressure \( \mu H^2 / 8\pi \), together with a tension \( \mu H^2 / 4\pi \) along the lines of force, and if the motion is such as to produce no changes in density, the hydrostatic pressure does no work; and so the changes in magnetic energy result from work done against the tension \( \mu H^2 / 4\pi \) along the lines of force, i.e., any extension of the lines of force increases the magnetic energy. Thus the second term in (1.5.3) can be seen in incompressible fluids as the energy gain resulting from the stretching of the magnetic lines of force by the fluid elements dragging them around with their motions.
An electromagnetic field possesses magnetic energy as well as electrical energy. But, in the magnetohydrodynamic approximation, the electrical part of energy is small as compared with the magnetic part; hence in MHD considerations, only the magnetic energy can be regarded as representing the total electromagnetic energy of the field.

1.6. Thermal convection in Magnetohydrodynamics (laminar)

Heat transfer problems for electrically conducting fluids are becoming increasingly important in view of their applications in many fields of recent origin. An externally applied magnetic field tends to reduce the aerodynamic drag and the rate of heat transfer at the walls.

Problems on thermal convection may be classified into forced and free convections. A convection process which takes place due to motion caused by an external agency such as pressure gradient or a pump or blower is termed as forced convection. If the motion is caused by the action of gravity on the fluid, which arises as a result of density variation due to temperature differences, it is called free convection or natural convection. Such variations of density give rise to a buoyancy force which causes relative motion.

We shall now consider the equations governing the flow when a temperature variation is imposed in the fluid. The effects of the temperature variation will be taken into account only in so far as it directly acts to cause (or modify) the motion. Thus, the
physical properties of the fluid such as viscosity, electrical conductivity, thermal conductivity etc; (which in general vary with temperature though slightly) will be regarded as constants. This kind of assumption is usually made in problems relating to thermal convection when the temperature differences involved are small.

We shall confine here to free convection; considering first a homogeneous and incompressible fluid, the motion in free convection arises because of the density variation due to temperature differences. The variation of the density $\varrho$ with temperature $T$ may be taken in the form

$$\varrho = \varrho_0 \left\{1 - \alpha(T-T_0)\right\}, \quad (1.6.1)$$

where $\alpha$ denotes the coefficient of thermal expansion and $\varrho_0$ the density corresponding to a certain mean temperature $T_0$. (The above equation in fact expresses the equation of state for the problem under consideration.)

With the neglect of the dissipative heat effects due to viscosity and the flow of electric currents in the fluid, and denoting the thermal diffusivity by $\chi$ ($\chi = \frac{k}{\varrho_0 c_p}$, $c_p$ being the specific heat at constant pressure) the equation for temperature distribution (cf. equation (1.2.14) is

$$\frac{\partial T}{\partial t} + (\nabla \cdot \varrho) T = \chi \nabla^2 T. \quad (1.6.2)$$

The fundamental assumption is now made, following Boussinesq and Rayleigh, that the fractional part $\alpha(T-T_0)$ in the variation of density (1.6.1) is small and that, in view of this,
the density variation can be disregarded except for its effect in producing buoyancy forces. Therefore, the variation of the density \( f \) is taken into account only in the term \( \mathbf{g} \) of the equation of motion (1.2.12), and elsewhere \( f \) is regarded as a constant equal to \( f_0 \).

Thus the equation of continuity of motion is still used in the form (1.2.10), while the equation of motion becomes modified to

\[
\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial \mathbf{v}}{\partial t} + (\nabla \cdot \mathbf{v}) = -\nabla \phi - \mathbf{a}(T-T_0) + \nu \nabla^2 \mathbf{v} + \frac{\mathbf{J} \times \mathbf{H}}{\rho_0},
\]

where

\[
\phi = \frac{B}{f_0} + \phi,
\]

\( \phi \) being the gravitational potential.

The field equations are not affected by the assumptions made above and thus remain unaltered.

The non-magnetic problem of a horizontal layer of viscous fluid heated from below was first considered by Lord Rayleigh (1916), and later more extensively by Jeffreys (1926, '28). Experimental results were first obtained by Bénard (1900, '01). A linear method based on the method of marginal stability' was employed, and the essential results obtained were that if a characteristic number called the Rayleigh number \( (R_a = \frac{g \Delta T |g|^4}{\kappa \nu^3}) \) defined in (1.7.11) exceeds a certain critical value, thermal instability sets in and the resulting flow pattern has a stationary cellular structure. The exact value of the critical Rayleigh number depends on boundary
conditions such as whether the horizontal plane surfaces bounding the fluid are free or rigid. In the case of a horizontal temperature gradient imposed in a vertical layer of fluid, there appears to be no such specific significance of $R_a$ except that it characterises the (immediately ensuing) convective motion; for in that case thermal effects prevent there being any equilibrium state.

1.7. Thermal convection in MHD (contd.); equations and parameters involved

Convective flow in MHD is a generalisation of that in ordinary hydrodynamics. The magnetic problem of a horizontal layer of a uniform, incompressible, viscous and electrically conducting fluid heated from below was first considered independently by Thompson (1961) and Chandrasekhar (1952, 1954). Cowling (1957) and others also discussed this problem. Experimental work was first done by Nakagawa (1955, 1957) and Lehnert and Little (1957). The authors found that the magnetic field inhibits the onset of thermal instability by convection. A short analysis of the problem is being presented here.

The basic equations governing the convection (obtained earlier) are

\[
\begin{align*}
\text{div} \vec{H} & = 0, \\
\frac{\partial \vec{H}}{\partial t} - \text{Curl}(\vec{v} \times \vec{H}) & = \gamma \nabla^2 \vec{H},
\end{align*}
\]

(1.7.1)
\[
\text{div } \vec{v} = 0, \\
\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\nabla \bar{\omega} - \alpha (T - T_0) \vec{g} + \nu \nabla^2 \vec{v} + \frac{\sigma}{4\pi \rho_0} (\text{Curl } \vec{H}) \times \vec{H}, \\
\frac{\partial T}{\partial t} + (\vec{v} \cdot \nabla) T = \kappa \nabla^2 T. 
\] (1.7.3)

[ In writing the second equation of (1.7.2), use is made of the equation (1.2.2). It may be noted that the equations hold whether there is maintained in the fluid a vertical or horizontal temperature gradient.]

The above equations involve three diffusivities, viz., \(\eta\) (magnetic diffusivity), \(\nu\) (viscous diffusivity) and \(\kappa\) (thermal diffusivity). In order to reduce the complexity of the problem rendered by these diffusivities, it is assumed that

\[
\eta > \kappa > \nu, 
\] (1.7.4)

and that in fact \(\kappa/\eta\) is fairly small. These inequalities are attained for conducting fluids under normal terrestrial conditions. (As for example, for mercury at room temperature \(\eta/\kappa \sim 1.6 \times 10^6\), \(\nu/\kappa \sim 1/40\), and for liquid sodium at temperature 200°C, \(\eta/\kappa \sim 1.6 \times 10^3\) \(\nu/\kappa \sim 1/135\).)

The assumption that \(\kappa/\eta\) is fairly small implies that the perturbation in the field is small compared with the undisturbed imposed field. For, writing

\[
\vec{H} = \vec{H}_0 + \vec{h} 
\] (1.7.5)

where \(\vec{H}_0\) is the (uniform) undisturbed imposed field and \(\vec{h}\) the
perturbation in the field caused by the motions and the induced currents in the fluid, and then substituting in the field equations (1.7.1), one has

$$\text{div } \vec{h} = 0,$$

$$\frac{\partial \vec{h}}{\partial t} - \nabla \times (\vec{h} \times \vec{E}_0) = \gamma \nabla^2 \vec{h}. \quad (1.7.6)$$

Again, from the heat equation (1.7.3), for convection far away from the walls of the containing vessel, a dimensional comparison of the convective term and the conduction term gives $|\vec{v}| \sim \gamma/L$, where $L$ is some representative length (comparable with the dimensions of the vessel). Then, a dimensional comparison of the diffusion term and the transport term in the second equation of (1.7.6) shows that $|\vec{h}|$ is comparable with $|(x/\gamma)\vec{E}_0|$. Hence, the perturbation field is small compared with the undisturbed field.

Thus the second equation of (1.7.6) approximates to

$$\frac{\partial \vec{h}}{\partial t} - \nabla \times (\vec{v} \times \vec{E}_0) = \gamma \nabla^2 \vec{h}, \quad (1.7.7)$$

and also the magnetic term in the equation of motion of (1.7.2) approximates to $(-\gamma/4\pi \vec{E}_0) \cdot (\nabla \times \vec{H}_0)$, so that the equation of motion becomes

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\nabla \bar{p} - \alpha (T-T_0) \vec{g} + \nu \nabla^2 \vec{v} + \frac{\mu}{4\pi \vec{E}_0} (\nabla \times \vec{H}_0). \quad (1.7.8)$$

A dimensional comparison of the magnetic term and the viscosity term shows that the magnetic drag will be dominant.
over the viscous drag or not, according as

\[ M > 1, \]

where

\[ M = \frac{\rho u H_0 L}{\frac{\varepsilon}{L^2 \nu}} \] \( (1.7.9) \)

this number \( M \) being the usual Hartmann number.

Again, (since from the heat equation \( |\nabla| \sim \kappa/L \) the ratio of the viscosity term to the inertia term is

\[ \frac{\nu}{\kappa} = \varepsilon'; \] \( (1.7.10) \)

this number \( \varepsilon' \) is usually called the Prandtl number and it depends only on the material properties of the fluid. The Prandtl number measures the relative importance of the viscous forces and that of the inertia forces. In fluids like water the Prandtl number \( \varepsilon' \) is greater than unity; but, for most conducting liquids under terrestrial conditions; it is less than unity. (As for example, for mercury at room temperature \( \nu/\kappa \sim 1/40 \) and for liquid sodium at 200°C, \( \nu/\kappa \sim 1/135 \).)

Also, if \( \Delta T \) be a temperature gradient comparable with the actual undisturbed gradient, then the absolute value of the ratio of the buoyancy term to the viscosity term in the equation (1.7.8) is

\[ \frac{\kappa \nu L^4 |\Delta T|}{\nu^2} = R_a. \] \( (1.7.11) \)

This number depends on the imposed temperature gradient as well as on the material properties of the fluid, and indicates the
nature of the convection that will arise in the fluid. The number $R_a$ is called the Rayleigh number.

The significance of the relative magnitudes of the inertia, buoyancy and viscosity terms remains the same as in non-magnetic problem. Thus, the numbers $\sigma$ and $R_a$ play similar roles in the magnetic problem as in non-magnetic problem.

As in the non-magnetic problem, a linear theory can be followed for the convection in the neighbourhood of marginal stability, as done by Thompson (1961), Chandrasekhar (1952, '54) and others. In the convective motion in the neighbourhood of marginal stability, the variations of different variables from their static values are regarded as small so that their squares and products can be neglected. This is, indeed, the basis of the linear theory in the classical Rayleigh–Jeffreys problem. The condition for marginal stability is obtained by setting $\frac{\partial \psi}{\partial t} = 0$, as shown by Jeffreys (1926, '28), and Pellew and Southwell (1940).

Following the linear theory, the value of the critical Rayleigh number at which the fluid first becomes unstable has been found to depend not only on the boundary conditions as whether the bounding horizontal surfaces of the fluid are free or rigid, but also on the parameter

$$Q = \frac{\mu^2 \cos^2 \theta}{1.7.12},$$

where $\theta$ is the angle made by the imposed field ($H_o$) with the vertical. Thus, it appears that a horizontal field exerts no influence whatsoever on the stability. Chandrasekhar (1954) argued
that if the field $H_0$ is not in the vertical direction, convection when it first sets in appears in the form of 'longitudinal rolls' extended in directions parallel to vertical planes through the lines of force, and 'transverse rolls' (which give the convection, when the field is vertical, a cellular pattern) are suppressed. It is for the longitudinal rolls with axes horizontal and parallel to the field that a horizontal field exerts no influence on the stability.

With the field vertical and very strong or with small viscosity, the condition of limiting stability is found to be

$$R_a (\text{Rayleigh number}) = \chi^2 \eta^2$$

(Chandrasekhar(1952)), and as mentioned already, a magnetic field exerts a stabilising influence on convection. It was also shown by Chandrasekhar(1952) that if $\gamma > \chi$ (which is assumed in the present consideration) instability arises as cellular convection (i.e., as a stationary pattern of motion) and if $\chi > \gamma$ instability arises as cellular convection or as motion in oscillations with increasing amplitude (following Addington one may say 'overstability') depending on the magnitude of the velocity of the magnetohydrodynamic wave.

When the imposed temperature gradient is in the horizontal direction, thermal effects, as in the non-magnetic case, prevent there being any equilibrium state, and consequently a linear theory discussed above is, in general, not appropriate except in certain limiting cases.
Thermal convection in Non-homogeneous (stratified) fluids

We have so far considered thermal convection in homogeneous fluids. We shall now consider here the case when the fluid is not homogeneous, and discuss briefly the modifications to be made in the equations (1.6.1), (1.2.10) and (1.6.3) when the non-homogeneity is supposed to be due to mass stratification. Equations governing the convective flow in the marginal state in such a non-homogeneous fluid case will however be discussed and derived in details in chapter IV.

The effect of mass diffusion on the thermosolutal convection in a fluid layer has been studied by Stern (1960), where he considered the case of linear opposing gradients of two properties between horizontal boundaries at fixed concentrations. Since then many others, including Veronis (1965, '68), Sani (1966), Turner and Stommel (1964), Turner (1965, '68, '74) have developed the idea of double diffusion.

We can argue that in reality a fluid is originally non-homogeneous. The gravitational effects of even a very small amount of this original non-homogeneity may turn out to be quite significant. In thermal convection of originally non-homogeneous fluids we have to take account of both original and thermal stratifications.

Banerjee (1969, '71, '72) has given an analytical treatment of thermal convection of originally non-homogeneous fluids. He has shown that introduction of an original non-homogeneity in the
fluid alters the character of the marginal state of Rayleigh-Jeffreys problem and directly leads to overstable solutions. Experimental results of Shirtcliffe (1967, '69) also has shown that the first instability does occur as a growing oscillation (see also Caldwell (1974)).

We shall now show how the ordinary equations of thermal convection (obtained in section 1.6) are to be modified when the fluid considered is not homogeneous due to stratification.

Let us assume that the original non-homogeneity (stable) is of the exponential type, viz.,

\[ f = f_0 e^{-\gamma z}, \]  

(1.8.1)

where \( z \) is the vertical coordinate and \( \gamma \) the stratification constant which is assumed to be small.

The density variation due to temperature variation is given by equation (1.6.1); and the resultant density distribution which arises due to the interaction between the original and thermal stratifications may be written (Banerjee (1972)) as

\[ f = f_0 \left[ e^{-\gamma z} - \alpha(T - T_0) \right], \]  

(1.8.2)

since \( \alpha \) and \( \gamma \) are small.

The equation of continuity, viz.,

\[ \text{div} \, \vec{v} = 0 \]

is now to be supplemented by the incompressibility condition

\[ \frac{\partial f}{\partial t} + (\vec{v} \cdot \nabla) f = 0. \]  

(1.8.3)
Again, with Boussinesq approximation (i.e., the variation of density is taken into account only in the buoyancy term), the equation of motion is modified to

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\nabla\tau - \left\{ \gamma s + \kappa (T - T_0) \right\} \vec{g} + \nu \nabla^2 \vec{v} + \frac{\mu}{\sigma_0} \vec{j} \times \vec{H}$$

(1.8.4)

(cf. equation (1.6.3)).

The heat equation will be used still in the form of equation (1.6.2).

As regards field equations, they remain unaffected.

The non-magnetic problem of thermal convection in a stratified fluid has been tackled by Banerjee (1971, '72) by following a linear theory as done in the classical Rayleigh-Jeffreys problem.

We shall make an attempt in chapter IV to extend the problem when the fluid is electrically conducting and when there is imposed a uniform magnetic field.

1.9. Two-dimensional steady case of magnetohydrodynamic thermal convection (fluid homogeneous)

In this section and in the next, we shall deal with the two-dimensional steady case, the fluid being uniform and incompressible. In the two-dimensional steady case when the field \( (\vec{H}_0) \) and the motion \( \vec{v} \) are supposed to be everywhere parallel to a
vertical plane, say, the x-y plane with y-axis vertically upwards and \( \frac{\partial}{\partial t} = 0 \), the field equation (1.7.7) and the equation of motion (1.7.8) can be further reduced in addition to the fact that \( \frac{\partial}{\partial t} = 0 \).

In the steady state, equation (1.7.7) becomes

\[-\text{Curl} (\vec{v} \times \vec{H}_0) = \eta \nabla^2 \vec{E},\]

which reduces to

\[(4\pi f a) \text{Curl} \vec{n} = \frac{1}{\eta} (\vec{v} \times \vec{H}_0) + \frac{1}{\eta} \text{grad} \vec{n}, \tag{1.9.1}\]

where \( \vec{n} \) is some scalar function. In fact \( \frac{1}{4\pi \eta} \text{grad} \vec{n} \) stands for the electric intensity \( \vec{E} \) (which now satisfies \( \text{Curl} \vec{E} = 0 \)). Since \( \vec{v}, \vec{H}_0 \) and hence \( \vec{n} \) are all parallel to the x-y plane, the current density \( \vec{J} \) and \( (\vec{v} \times \vec{H}_0) \) are perpendicular to this plane i.e., parallel to \( z \)-axis, but independent of \( z \). Thus \( \text{grad} \vec{n} \) is a constant vector parallel to \( z \)-axis. But, in the absence of convection there is no electric current (\( \text{Curl} \vec{H} = 0 \)); so it is appropriate to assume that for the convective motion also the total electric current in the fluid across the x-y plane is zero. On this assumption the constant vector \( \text{grad} \vec{n} \) reduces to zero. Hence, equation (1.9.1) becomes

\[(4\pi f a) \text{Curl} \vec{n} = \frac{1}{\eta} (\vec{v} \times \vec{H}_0). \tag{1.9.2}\]

Substituting this value of \( \text{Curl} \vec{n} \), the equation of motion (1.7.8) becomes

\[ (\vec{v} \cdot \text{grad}) \vec{v} = -\text{grad} \vec{\omega} - a(T - T_0) \vec{g} + \nu \nabla^2 \vec{v} + \frac{\mu}{4\pi f o \eta} ((\vec{v} \times \vec{H}_0) \times \vec{H}_0). \tag{1.9.3}\]

In the steady state, the heat equation (1.7.3) becomes
\[(\vec{v}.\text{grad})T = \kappa \nabla^2 T. \]  
\hspace{1cm} (1.9.4)

The perturbation field \(\vec{h}\) and the velocity vector \(\vec{v}\) also satisfy the equations

\[\text{div} \vec{h} = 0 \]  
\hspace{1cm} (1.9.5)

and

\[\text{div} \vec{v} = 0. \]  
\hspace{1cm} (1.9.6)

The equations (1.9.2) to (1.9.6) constitute the fundamental equations of magnetohydrodynamic convection in the two-dimensional steady case.

It can be noted that the perturbation field depends only on the ensuing motion (equation (1.9.2)) and has nothing to do now in the determination of the fluid motion (equation (1.9.3)).

1.10.

The equations (1.9.2) to (1.9.6) can be conveniently expressed in non-dimensional forms by writing

\[
\begin{align*}
(x,y) &= L(x',y'), \\
\vec{v} &= xL^{-1}\vec{v}', \\
\vec{h} &= H_0\vec{h}', \\
\omega &= x^2L^{-2}\omega', \\
T-T_0 &= |T_1-T_0|\theta,
\end{align*}
\]  
\hspace{1cm} (1.10.1)

where the dashes denote non-dimensional quantities, \(L\) a certain scale length, \(T_1\) a certain reference temperature and \(\theta\) the
non-dimensional measure of temperature. On substitution in the
equations (1.9.2) to (1.9.6) and then suppressing the dashes,
one has

\[ \text{Curl } \mathbf{h} = - \frac{2}{\gamma} \left( \mathbf{e} \times \nabla \right), \]  
(1.10.2)

\[ (\nabla \cdot \mathbf{v}) \mathbf{v} = -\nabla \mathbf{\omega} + G_x \mathbf{e} + \beta \left( \mathbf{e} \times (\mathbf{e} \times \nabla) \right), \]  
(1.10.3)

\[ (\nabla \cdot \mathbf{e}) \mathbf{e} = \psi^2 \mathbf{e}, \]  
(1.10.4)

\[ \text{div } \mathbf{h} = 0, \quad \text{div } \nabla = 0, \]  
(1.10.5)

where

\[ G_x = \frac{a \Delta T - T_o L^3}{\kappa^2}, \quad \text{and} \quad \beta = \frac{\mu H_0^2 (e \times \mathbf{e})}{4 a \kappa_0^3 \gamma \kappa^2}. \]  
(1.10.6)

and \( \mathbf{e} \) a unit vector in the upward direction, and \( \mathbf{e} \) a unit vector in the direction of \( \mathbf{H}_0 \). The number \( \beta \) is the usual Prandtl number; and the numbers \( G_x \) and \( \beta \) may be called the modified Grashof number and Hartmann number (modified by replacing the kinematic viscosity \( \nu \) by the thermal diffusivity \( \kappa \)) respectively.

For most conducting liquids under normal terrestrial conditions, the Prandtl number \( \beta \) is less than unity. Hence, if \( M_x \) be sufficiently large, then both the inertia and viscous terms are less important than the magnetic term in equation (1.10.3), leading to their neglect in some problems. The neglect of viscosity is justified in the main body of the fluid; but in the immediate neighbourhood of rigid boundaries the effect of viscosity is not negligible as can be seen from the occurrence of higher derivatives.
in the viscosity term coupled with the condition of no slip velocity on the rigid walls. The viscosity effect is in fact confined to an extremely thin surface layer and the forces within the layer have no appreciable effect on the main convection. Again, when a thermal boundary layer develops along the walls, because of \( \Gamma_x \) being large, this thermal layer will be more significant and thicker than the viscous layer. These had been well explained by Singh and Cowling (1963) in discussing the magnetic free-convection problem of boundary layer flow up a hot vertical plate. The same type of free-convection problem had been considered by Gupta (1961), Sparrow and Cess (1961), Riley (1964), D'as (1967) and Kuiken (1970). Singh and Cowling (1963) showed also that the condition for formation of a well-developed thermal boundary layer is

\[
B \gg 1, \quad (1.10.7)
\]

where the number \( B \) is defined by

\[
B = \frac{\Gamma_x}{M_x^2}, \quad (1.10.8)
\]

\( M_x \) being large enough so that the magnetic drag dominates over the viscous drag.

\[1.11. \quad \text{Outline of the thesis}\]

This thesis deals with a few problems of MHD flow past solid bodies and thermal convection in MHD. The next chapter (i.e., chapter II) will deal with the two-dimensional steady flow of an
inviscid, electrically conducting fluid past an elliptic cylinder in presence of a uniform magnetic field $\mathbf{H}_0$ in the direction of the undisturbed flow; chapter III will deal with the problem of two-dimensional flow of an incompressible, electrically conducting and viscous fluid past a circular cylinder placed midway between two parallel plates in presence of a uniform magnetic field parallel to the walls.

In chapter IV, we shall discuss the magnetohydrodynamic problem of thermal convection in a horizontal layer of fluid heated from below, the fluid being non-homogeneous.

Equations are marked at the right-hand side. A decimal system is used to indicate the chapter, section and number of the equation; for example, equation (1.2.3) indicates the third equation in the second section of chapter I. References are cited giving the names of the authors followed by years of publication.
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