## Chapter IV

### Some special classes of groups and the wreath product

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4.1. Groups satisfying finiteness conditions and the wreath product

If a group has a finite generating system, it is \textit{finitely generated}. If, further, it has a finite number of defining relations, it is \textit{finitely presented}. Baumslag (1961) has shown that if \( A \) and \( B \) are finitely presented groups, and \( B \) is finite, then the standard wreath product of \( A \) and \( B \) is a finitely presented group. We prove:

\textbf{4.1.1. Theorem.} If the set \( Y \) (and hence also \( B \)) is finite, then finite generation or finite presentation of the group \( A \) also implies that of the (general) wreath product \( AWrB (\approx AwrB) \).

\textbf{Proof.} Suppose \( Y \) (and hence also \( B \)) is finite and \( A \) is \textit{finitely generated}. Let \( K \) be a finite generating system of \( A \). Since an arbitrary element of \( A^Y \) is a finite product \( a_kx \), where \( a \in A \) and \( x \in Y \), the coordinate functions \( k_x \), where \( k \in K \), form a finite generating system of \( A^Y \). Thus \( A^Y \) is \textit{finitely generated}. And since an extension of a finitely generated group by a finitely generated group is also finitely generated (P. Hall, 1954, Lemma 1), it follows that \( AWrB \) is \textit{finitely generated}. 
Next suppose that $Y$ (and hence also $B$) is finite and $A$ is finitely presented. Again let $K$ be a finite generating system of $A$, with a corresponding finite generating system $L$ of $A^Y$. Since a finite number of defining relations in $A$ expressed in terms of the elements of $K$ can yield only a finite number of defining relations in $A^Y$ expressed in terms of the elements of $L$, we see that $A^Y$ is also finitely presented. Now an extension of a finitely presented group by a finitely presented group is again finitely presented (P. Hall, 1954, Lemma 1). Hence $A\wr B$ is finitely presented. //

A (finitely generated) group is a Noetherian group if every ascending chain of distinct subgroups of it is finite.

4.1.2. Theorem. If $A$ and $B$ are Noetherian, then $A\wr B$ (and hence $A\wr B$) is also Noetherian.

Proof. Let $A$ and $B$ be Noetherian.

If possible, let $A^Y$ be not Noetherian. Then there exists an infinite ascending chain of distinct subgroups of $A^Y$,

$$K_1 < K_2 < K_3 < \ldots$$
Choose any fixed element \( z \) in \( Y \). It then follows from Lemma 2.1.11 that there exist an infinite set of subgroups of \( A \),

\[ U_1, U_2, U_3, \ldots \]

where \( U_i \) is the subgroup of \( A \) formed by the images of \( z \) under the elements of the subgroup \( K_i \) of \( A^Y \). If \( z^f \in U_i \), \( f \in K_{i+1} \) (since \( K_i \) is contained in \( K_{i+1} \)), and so \( z^f \in U_{i+1} \). Thus,

\[ U_1 < U_2 < U_3 < \ldots \]

This contradicts the hypothesis that \( A \) is Noetherian. It follows that \( A^Y \) is Noetherian.

Since an extension of a Noetherian group by a Noetherian group is again Noetherian (P. Hall, 1954, Lemma 1), \( A \wr B \) is also Noetherian. //

A group is periodic if every element of it is of finite order. Some special types of periodic groups are locally finite groups, groups of finite exponent and p-groups.

A group is locally finite if every finitely generated subgroup of it is finite. A group is of finite exponent if
there is a positive integer \( n \) such that the equation \( x^n = 1 \) is a law in the group, and the smallest such integer \( n \) is the exponent of the group. A group is a p-group if every element of it has order a power of a prime \( p \).

4.1.3. Theorem. If \( A \) and \( B \) are periodic, then the restricted wreath product \( A \wr_r B \) is periodic.

If, in particular, \( A \) is of finite exponent, then the wreath product \( A \wr_r B \) is also periodic.

The proof is immediate from the following three lemmas.

4.1.4. Lemma. If \( A \) is periodic, so also is \( A^{(Y)} \).

Proof. Suppose \( A \) is periodic.

An arbitrary element of \( A^{(Y)} \) is a product of a finite number of coordinate functions \( a_x \), where \( a \in A \) and \( x \in Y \). Let \( m \) be the lowest common multiple of the orders of the elements \( a \). Since \( A \) is periodic, \( m \) is a finite positive integer. We get

\[
(x a_x)^m = x (a_x)^m = x (a^m)_x = 1,
\]
and so \( A^{(Y)} \) is periodic. //
4.1.5. **Lemma.** If $A$ is of finite exponent, so also is $A^Y$.

**Proof.** Suppose $A$ is of finite exponent $n$. If $f \in A^Y$, then for every $y \in Y$ we get

$$y(f^n) = (y^f)^n = 1.$$ 

Hence $f^n = 1$, which shows that $A^Y$ is of finite exponent. //

4.1.6. **Lemma.** An extension of a periodic group by a periodic group is also periodic.

**Proof.** Let $G$ be an extension of a periodic group $H$ by a periodic group $K$.

If possible, let $G$ have an element $g$ of infinite order. Suppose that under the given isomorphism $\alpha : G/H \to K$ the coset $gH$ has the image $k$. Since $K$ is periodic $k$ has a finite order $m$, say. Then under $\alpha$ we have the assignment $g^mH = 1$, where $1$ is the identity of $K$. Since $\alpha$ is an isomorphism we then get $g^mH = H$, the identity of $G/H$.

Here $g^m$ cannot have a finite order (otherwise the same would be true of $g$), and $H$ is periodic. Hence $g^m \not\in H$. 
and so \( g^n H \neq H \).

The contradiction establishes the lemma. //

4.1.7. Theorem. If \( A \) is finite and \( B \) locally finite, then both \( A \wr B \) and \( A \wr_r B \) are locally finite.

Proof. Suppose \( A \) is finite. Then \( A^Y \) is locally finite (B. H. Neumann, 1960\(^b\), Lemma 5.4).

An extension of a locally finite group by a locally finite group is again locally finite (Kurosh, 1960, Vol.2, p.153 and B. H. Neumann, 1960\(^a\), p.207). Hence it follows that if here \( B \) is locally finite, then \( A \wr B \) is locally finite.

Since a subgroup of a locally finite group clearly possesses local finiteness, the proof for the restricted wreath product \( A \wr_r B \) is now immediate. //

4.1.8. Theorem. If \( A \) and \( B \) are of finite exponent, then so also are \( A \wr B \) and \( A \wr_r B \).

Proof. Let \( A \) and \( B \) be of finite exponent. Then by Lemma 4.1.5 \( A^Y \) is of finite exponent.
An extension of a group of finite exponent by a group of finite exponent is again of finite exponent (B. H. Neumann, 1960, p.479). Hence the wreath product $A \wr B$ is of finite exponent under the condition stated.

And since a subgroup of a group of finite exponent is clearly of finite exponent, the restricted wreath product $A \wr_r B$ is also of finite exponent in the case under consideration.

4.1.9. Theorem. If $A$ and $B$ are $p$-groups, the restricted wreath product $A \wr_r B$ is a $p$-group.

If further $A$ is of finite exponent then $A \wr_r B$ is also a $p$-group.

Proof. Suppose $A$ is a $p$-group. Then the order of each element of $A$ is a power of $p$. Since an arbitrary element of $A^Y$ is a product of a finite number of coordinate functions $a_x^y (a \in A, x \in Y)$, we can calculate the lowest common multiple of the orders of the $a$'s. If this multiple is say $p^l$, then $(x_{a_x})^{p^l} = 1$. This shows that $A^Y$ is a $p$-group.

Suppose further that $A$ is of finite exponent $p^n$. If $f \in A^Y$, then for every $y \in Y$ we get
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Thus $A^Y$ is also a $p$-group in this case.

Since an extension of a $p$-group by a $p$-group is again a $p$-group (Scott, 1964, 6.1.2) the theorem follows. //

Theorem 4.1.9 immediately yields

4.1.10. Corollary. If $A$ and $B$ are $p$-groups and $B$ is regular and infinite, then the restricted wreath product $A \wr_r B$ is a non-nilpotent $p$-group.

If further $A$ is of finite exponent, then $A \wr_r B$ is also a non-nilpotent $p$-group.

Proof. If $B$ is regular and infinite, $A \wr_r B$ has trivial centre (Corollary 2.4.14) and is therefore non-nilpotent. Then Theorem 4.1.9 completes the proof of the first part of the corollary.

Since a subgroup of a nilpotent group is nilpotent, $A \wr_r B$ is non-nilpotent when $A \wr_r B$ is non-nilpotent. The second part of the corollary then follows. //

(The non-nilpotency of the restricted standard wreath product $A \wr_r B$ (a $p$-group), when $A$ and $B$ are $p$-groups and $B$ is infinite, was proved by Baumslag (1959).)
4.2. Solvable groups and the wreath product

We shall examine the effect of the solvability of the groups $A$ and $B$ on the wreath product $A \wr B$. We note.

4.2.1. Lemma. If the groups $A$ and $B$ are solvable of lengths $1$ and $m$ respectively, then $A \wr B$ and $A^r B$ are both solvable groups of length $< 1+m$.

Proof. Let $A$ and $B$ be solvable of lengths $1$ and $m$ respectively. Then $A^Y$ and $A^{(Y)}$ are solvable of length $< 1$ (Schenkman, 1965, VII.1.a, (vii) and (viii)).

Again an extension of a solvable group of length $1$ by a solvable group of length $m$ is a solvable group of length $< 1+m$ (Schenkman, 1965, VII.1.d). Hence the lemma.

We get the following two more precise results (which have been proved by McCarthy (1969) for the particular case of the standard wreath product).

4.2.2. Theorem. If $A$ is a solvable group of length $1$ and the group $B$ is abelian, then the wreath product $A \wr B$ is a solvable group of length $1+1$.

Proof. Let $A$ be solvable of length $1$ and $B$ abelian. Then by Lemma 4.2.1 $A \wr B$ is solvable of length $< 1+1$. We have
then only to show that the solvability length of $A\varPi B$ is $\geq 1-1$.

Take a fixed $b \in B$, $b \neq 1$, and a fixed $x \in Y$, and let $x^b = z$. Then, for every $a \in A$, we get in the derived group $(A\varPi B)'$ a commutator

$$c_a = a^{-1}_x b^{-1}_x a_x b = a^{-1}_x (x^b)$$

$$= a^{-1}_x a_z.$$

Let $G$ be the subgroup of the derived group $(A\varPi B)'$ generated by the set of commutators $c_a$, $a \in A$. Then an arbitrary element of $G$ is a finite product $c_a \cdots c_k$, where $a, \ldots, k \in A$. We get

$$z^{c_a \cdots c_k} = z^{c_a} \cdots z^{c_k} = a \cdots k.$$

Thus we get a map

$$\alpha : G \to A, \quad c_a \cdots c_k \mapsto a \cdots k.$$ 

It is at once seen that $\alpha$ is onto and homomorphic, and so $A$ is an epimorphic image of $G$. 


Since $G$ is a subgroup of the solvable group $AWrB$ and $A$ is solvable of length 1, it then follows that $G$ is solvable of length $\geq 1$. Hence the derived group $(AWrB)'$, which contains $G$, is also solvable of length $\geq 1$.

This shows that $AWrB$ has solvability length $\geq l+1$, and is thus solvable of length $1+1$. //

We get the following general result (which is also known for the permutational wreath product (Burns, 1968)).

4.2.3. Theorem. If $A$ is a solvable group of length 1 and the group $B$ is solvable of length $m$, then the wreath product $AWrB$ is a solvable group of length $1+m$.

Proof. Suppose $A$ and $B$ are solvable of lengths 1 and $m$ respectively. Then by Lemma 4.2.1 $AWrB$ is solvable of length $\leq 1+m$. We have then to show that the solvability length is $\geq 1+m$.

In the group $B$ and its successive derived groups $B(1)$, $B(2)$, ..., $B^{(m-1)}$ take fixed elements $b_1 \in B$,

$b_2 \in B(1)$, ..., $b_m \in B^{(m-1)}$ such that $b_1 \notin B^{(1)}$,

$b_2 \notin B^{(2)}$, ..., $b_{m-1} \notin B^{(m-1)}$ and $b_m \neq 1$. Take a fixed
x \in Y and let \( x^{(b_1)} = z \). We get for any \( a \in A \),

\[
ax^{-1}b_1^{-1}axb_1 = ax^{-1}(ax)^{b_1} = ax^{-1}ax^{-1}b_1.
\]

For simplicity put

\[
\begin{align*}
(a_1, b_1) & = ax^{-1}a_1, \\
(a_2, b_2) & = ax^{-1}a_2, \quad \ldots, \\
(a_{x^1b_1b_2}, b_1b_2) & = ax^{-1}a_1, \quad \ldots, \quad .
\end{align*}
\]

Then the commutator of \( ax \) and \( b_1 \) is \( ax^{-1}a_1 \),

the commutator of \( ax, b_1 \) and \( b_2 \) is

\[
ax^{-1}a_1 ax^{-1}a_1 b_2 = ax^{-1}(ax)^{-b_2}(a_1)^{b_2}
\]

\[
= ax^{-1}a_1^{-1}a_2^{-1}a_12 \quad \text{(by Lemma 2.1.12)},
\]

the commutator of \( ax, b_1, b_2 \) and \( b_3 \) is

\[
ax^{-1}a_1 ax^{-1}a_1 a_2 ax^{-1}a_1 a_2 a_12 b_3
\]

\[
= ax^{-1}a_1 ax^{-1}a_1 a_2 a_12 b_3 (a_3)^{-b_3(a_1, a_2)^{-b_3(a_2, a_12)^{-b_3(a_12, b_3)}
\]

\[
= ax^{-1}a_1 a_2 a_3 a_12 a_13 a_23 a_123,
\]
Similarly, the commutator of $a_x^{-1}$ and $b_1$ is $a_x a_1^{-1}$, the commutator of $a_x^{-1}$, $b_1$ and $b_2$ is $a_x^{-1} a_1 a_2 a_1^{-1} a_2$, the commutator of $a_x^{-1}$, $b_1$, $b_2$, and $b_3$ is $a_x a_1^{-1} a_2^{-1} a_3^{-1} a_1 a_2 a_3^{-1} a_1 a_2 a_3^{-1} a_1 a_2 a_3^{-1}$.

Let $c_a$ be the commutator of $a_x$, $b_1$, $b_2$, ..., $b_m$ if $m$ is odd and the commutator of $a_x^{-1}$, $b_1$, $b_2$, ..., $b_m$ if $m$ is even. Then $c_a$ is an element of the derived group $(A \wr B)^{(m)}$.

Let $G$ be the subgroup of $(A \wr B)^{(m)}$ generated by the set of commutators $c_a$, $a \in A$. For any $a$, ..., $k \in K$ we get

$$(c_a \ldots c_k) = a \ldots k,$$

whether $m$ is odd or even. Thus we get an epimorphism

$$a : G \to A, c_a \ldots c_k \to a \ldots k.$$

Since $G$ is a subgroup of the solvable group $A \wr B$ and $A$ is solvable of length 1, it follows that $G$ is solvable of length $\geq 1$. Hence the derived group $(A \wr B)^{(m)}$ containing $G$ is also solvable of length $\geq 1$. 
We infer that $AWrB$ has solvability length $> 1+m$.
Thus $AWrB$ is a solvable group of length $1+m$.

A finite solvable group all whose Sylow subgroups are abelian is an $A$-group (P. Hall, 1940). Carter (1962, p.555) has given a method for constructing such groups with the help of the wreath product concept. The result contained in the following lemma is a little more general.

4.2.4. Lemma. Let $p_1, \ldots, p_{r+1}$ be distinct primes.
If $G_{r+1}$ is a finite abelian $p_{r+1}$-group and $G_i$ are finite abelian $p_i$-groups of permutations on sets $X_i$, $i = 1, \ldots, r$, respectively, then the repeated wreath product

$$G = G_{r+1} \text{Wr}(G_r \text{Wr}(\ldots(G_2 \text{Wr} G_1)\ldots))$$

is an $A$-group.

Proof. Let the repeated wreath product

$$P_k = G_k \text{Wr}(\ldots(G_2 \text{Wr} G_1)\ldots)$$

be the permutation group on the product set $Z_k$, $k = 2, \ldots, r$.
Then,

$$|P_2| = |G_2|^{\mid X_1\mid} |G_1|, \quad |P_3| = |G_3|^{\mid Z_2\mid} |P_2|,$$

\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots
We knew that the base group is a normal subgroup of the wreath product. Hence,
\[ |P_r| = |G_r|^{|Z_{r-1}|} |P_{r-1}|, \quad |G| = |G_{r+1}|^{|Z_r|} |P_r|. \]

Further, the primes \( p_{r+1}, p_r, p_3, p_2 \) do not divide respectively \( |P_r|, |P_{r-1}|, \ldots, |P_2| \) and \( |G_1| \).

It follows that the subgroups 4.2.5 along with the subgroup \( G_1 \) form a complete set of Sylow subgroups of \( G \). And clearly all these subgroups are abelian. //

4.3. Hypercentral groups, nilpotent groups and the wreath product

If \( G \) is any (nontrivial) group we can always obtain a series of characteristic subgroups of \( G \),
\[ 1 = Z_0 \trianglelefteq Z_1 \trianglelefteq Z_2 \trianglelefteq \ldots \trianglelefteq Z_i \trianglelefteq \ldots, \]
such that each factor group \( Z_{i+1} / Z_i \) is the centre of the factor group \( G/Z_i \).
It is the upper central series of $G$.

If there exists an ordinal $\alpha$ such that $G = Z_\alpha$, then $G$ is a hypercentral group (or a $ZA$-group). If $\alpha$ is the first ordinal such that $G = Z_\alpha$, then $\alpha$ is the length of $G$.

It is obvious from the condition 4.3.1 that a hypercentral group has a nontrivial centre, and a group $G$ is hypercentral if and only if every nontrivial homomorphic image of $G$ has a nontrivial centre.

A hypercentral group of finite length is a nilpotent group.

Clearly a subgroup of a hypercentral group is hypercentral and a subgroup of a nilpotent group is nilpotent.

We shall obtain necessary conditions such that a wreath product may be hypercentral or nilpotent.

4.3.2. Theorem. If the restricted wreath product $A \wr_r B$ is hypercentral and the permutation group $B$ is regular, then $A$ is a hypercentral $p$-group and $B$ is a finite $p$-group for the same prime $p$. 
If, in particular, $A \cap B$ is nilpotent, then $A$ is a nilpotent $p$-group of finite exponent.

For the proof we shall need the following result (an analogue of Lemma 3.1 of Liebeck, 1962).

4.3.3. Lemma. Let $a \in A$, $b \in B$, $x \in Y$, and for $m = 1, 2, \ldots$ denote $a$ simply by $a_m$.

Then the commutator $c_m$ of $a_x$, $b$, $b$, $\ldots$ to $m$'s is the product

$$a_x^{(-1)}a_i \ldots a_x^{(-1)}a_1^{m+1}(m) \ldots \ldots \ldots a_x^{(-1)}a_m^{m}$$

where $\binom{m}{i}$ are binomial coefficients.

Proof. Making use of the calculations for the different commutators in the proof of Theorem 4.2.3 and putting $b_1 = b_2 = \ldots = b$ we see that the commutator of $a_x$ and $b$ is $a_x^{-1}a_1$, the commutator of $a_x$, $b$ and $b$ is $a_x a_1^{-2}a_2$. Thus the lemma is true for $m = 1$ and $m = 2$.

Now suppose that the lemma is true for a particular positive integer $m$. Then we get
Thus the lemma is true for \( m+1 \), and so it follows that it is true for all positive integral values of \( m \).

Proof of Theorem 4.3.2. Suppose \( \text{AwrB} \) is hypercentral and \( B \) is regular.

It immediately follows that \( A \) and \( B \) are hypercentral.

Since \( \text{AwrB} \) is hypercentral, it has a nontrivial centre. It follows from Corollary 2.4.14 (\( B \) being regular) that \( B \) is finite.

We know that in a hypercentral group elements of relatively prime orders commute (Schenkman, VI.3.j). But commutativity does not hold, in general, in the wreath product (Corollary 2.4.2). It follows that \( A \) and \( B \) are p-group for the
Suppose \( \text{AwrB} \) is nilpotent. Then, as above, \( A \) is a nilpotent \( p \)-group. If possible, let \( A \) be not of finite exponent.

Since \( B \) is finite, every element of it has finite order. Let \( b \in B \) be of order \( t \) and \( p \) be any prime greater than \( t \) and \( n \), where \( n \) is the class of nilpotency of \( \text{AwrB} \). Since \( A \) is not of finite exponent it possesses an element \( a \) of order \( > 2^p \).

By Lemma 4.3.3 the commutator \( c_p \) of \( a_x, b, b, \ldots \) to \( p \) \( b \)'s is

\[
(a_x)^{(-1)^p} \ldots (a_i)^{(-1)^{p+i}} (a_p)^{p}.
\]

Hence \( c_p \in A(Y) \), and so \( c_p \) is contained in \( \text{AwrB} \).

Here \( \text{AwrB} \) is nilpotent of class \( n \), and \( p > n \).

It follows that \( c_p = 1 \) (Kurosh, 1960, Vol.II, p.214).

Since \( p > t \), there exists a positive integer \( s \) such that \( s t < p < (s+1) t \). We get

\[
\left( x_{(b^t)} \right)^{a_t} = a_x.
\]
and similarly,

\[ a_{2t} = a_{3t} = \cdots = a_{st} = a_x. \]

Hence the exponent of \( a_x \) in \( e \) is

\[ e = (-1)^P + (-1)^{P+t} \binom{P}{t} + (-1)^{P+2t} \binom{P}{2t} \]

\[ + \cdots + (-1)^{P+st} \binom{P}{st} \]

\( \neq 0 \) (for \( e = (-1)^P \) is clearly a multiple of \( p \)).

Also,

\[ |e| < |(-1)^P (1+1)^P| = 2^P. \]

Since \( a \) is of order \( > 2^P \) we then see that \( a^e \neq 1 \). Hence, \( (a_x)^e \neq 1 \); and so the commutator \( c_p \neq 1 \), a contradiction.

It follows that \( A \) is of finite exponent under the conditions stated. //

The following two results are now immediate.

4.3.4. Corollary. If the wreath product \( A \wr B \) is hypercentral and the permutation group \( B \) is regular, then \( A \) is a hypercentral \( p \)-group and \( B \) is a finite \( p \)-group for the same prime \( p \).
If, in particular, $A \wr B$ is nilpotent, then $A$ is a nilpotent $p$-group of finite exponent.

For if $A \wr B$ is hypercentral or nilpotent, so also is $A \wr B$. //

4.3.5. Corollary. If either of $A \wr B$ or $A \wr B$ is hypercentral, the two coincide.

For if $A \wr B$ or $A \wr B$ is hypercentral, $B$ is finite—whence $Y$ is finite, and so $A \wr B = A \wr B$. //

To obtain sufficient conditions such that a wreath product may be hypercentral or nilpotent we shall make use of the following lemma due to Wiegold (1962, Lemma 2.1):

4.3.6. Lemma. Let $G$ be a group generated by a normal subgroup $N$ and a subgroup $B$ such that $G/N$ is a finite $p$-group of class $k$ and order $\delta$, and the commutator subgroup of $N$ and $B$ is a $p$-group.

(i) If now $N$ is a hypercentral group of length $\omega$, then $G$ is a hypercentral group of length $\leq \omega x + k$, where $\omega$ is the first infinite ordinal.

(ii) And if $N$ and $B$ are nilpotent of classes $d$ and $e$ respectively and the commutator subgroup of $N$ and $B$ is of
exponent $p^s$, then $G$ is nilpotent of class $\leq d s(\delta-1) + \max (d, e)$.

We now get

4.3.7. Theorem. Let $A$ be a hypercentral $p$-group of length $x$ and the group $B$ a finite $p$-group of class $k$ (and hence $Y$ finite). Then the wreath product $A \Wr B (= A \wr B)$ is a hypercentral $p$-group of length $\leq bx + k$.

If, in particular, $A$ is a nilpotent $p$-group of class $c$ and finite exponent $p^r$ and $B$ is of order $\delta$, then $A \Wr B (= A \wr B)$ is a nilpotent $p$-group of class $\leq cs(\delta-1) + \max (c, k)$.

Proof. Let $B$ be a finite $p$-group of class $k$ and order $\delta$.

We know that $A^Y$ is a normal subgroup of $A \Wr B$, and $A \Wr B$ is generated by $A^Y$ and $B$. The factor group $(A \Wr B)/A^Y$ is isomorphic to $B$, and is therefore a finite $p$-group of class $k$ and order $\delta$.

Again, an arbitrary element of the commutator subgroup of $B$ and $A^Y$ is a product of commutators $b^{-1} f^{-1} bf$ where $b \in B$ and $f \in A^Y$. Since the commutator $b^{-1} f^{-1} bf$ is the product $f^{-b} f$, it is contained in the base group $A^Y$. Hence we know that the commutator subgroup of $B$ and $A^Y$ is actually a subgroup of $A^Y$. 
Now let $A$ be a hypercentral $p$-group of length $x$. It is easily seen that $A^X$ is a hypercentral group of length $\leq x$. Also by Theorem 4.1.9 $A^X$ is a $p$-group and so the commutator subgroup of $B$ and $A^X$ is a $p$-group. Then the first part of our theorem follows from Lemma 4.3.6(i) on taking $N = A^X$ and $G = A \wr B$.

Next suppose $A$ is a nilpotent $p$-group of class $c$ and exponent $p^s$. Then $A^X$ is nilpotent $p$-group of class $d$, $\leq c$ (Schenkman, 1965, VI.1.5(v) and Theorem 4.1.9 and of exponent $\leq p^s$ (Lemma 4.1.5). It follows that the commutator subgroup of $B$ and $A^X$ is a $p$-group of exponent $\leq p^s$. Suppose it is of exponent $p^r$, $r \leq s$. Then by the second part of Lemma 4.3.6, $A \wr B$ is nilpotent of class

$$\leq d \cdot r \cdot (\delta - 1) + \max (d, k)$$

$$\leq c \cdot s \cdot (\delta - 1) + \max (c, k).$$

4.4. Divisible groups and the wreath product

For his proof of Neumann's theorem, 'Every group can be embedded in a divisible group', Baumslag (1959) used the following result (see Lemma 4.1): If $A$ is an arbitrary group and $B$ a cyclic group of order $n$, then $A$ can be embedded in the
standard wreath product $A \wr B$ such that every element of $A$ has $n$th root in $A \wr B$.

We get the following generalization of Baumslag's lemma for the wreath product $A \wr B$.

4.4.1. Lemma. If $B$ is a finite cyclic regular permutation group of order $n$, every element of the diagonal $A^0$ has an $n$th root in the wreath product $A \wr B$.

Proof. Let $a^\circ$ be an arbitrary element of $A^0$ and $B$ be generated by $b$. We get for any given $x$ in $Y$,

$$(ba_x)^n = (a_x)^{b^{n-1}} \cdots (a_x)^b a_x$$

$$= a^{(x(b^{n-1}))} \cdots a^{(x^b)} a_x.$$

Since $B$ is regular,

$$x^{(b^{n-1})}, \ldots, x^b \neq x.$$

Hence,

$$(ba_x)^n = (a_x) x^x = a.$$

Next let $y \in Y$, $y \neq x$. Since $B$ is regular, one and only one of the elements $x^{(b^{n-1})}, \ldots, x^b$ is $y$. 
Hence,
\[(ba_x)^n = y^y = a.\]

Thus we get for every \(y \in Y\), \((ba_x)^n = a\).

It follows that \(a^y = (ba_x)^n\), as stated. //

4.4.2. Note. We see that the condition that \(B\) is regular is necessary:

Let \(Y\) be the set of numbers 1, 2, 3, 4, 5 and \(b\) the permutation \((1 2 3)(4 5)\). Then

\[
\begin{align*}
b^2 &= (1 3 2), \\
b^3 &= (4 5), \\
b^4 &= (1 2 3), \\
b^5 &= (1 3 2)(4 5),
\end{align*}
\]

and so the group \(B\) is not regular.

We see that

\[
(ba_1)^6 = (a_1)^b^5 (a_1)^b^4 (a_1)^b^3 (a_1)^b^2 (a_1)^b
\]

\[= a_3 a_2 a_1 a_3 a_2.\]

Then \(2(ba_1)^6 = a^2\), and so \((ba_1)^6 \neq a^6\). //
The following result shows that divisibility of the factors is inherited by the wreath product and the restricted wreath product.

**4.4.3. Lemma.** If \( A \) and \( B \) are divisible groups, so also are \( A \wr B \) and \( A \triangleright B \).

**Proof.** Let \( b f \in A \wr B \), \( b \in B \) and \( f \in A^Y \), and \( n \) a given integer.

Since \( B \) is divisible, there exists an element \( c \) in \( B \) such that \( c^n = b \). We shall now construct \( g \in A^Y \) such that \( (cg)^n = bf \).

For any \( g \in A^Y \) we get

\[
(cg)^n = c^n g(c^{n-1}) g(c^{n-2}) \ldots g^e g.
\]

Put \( c^{-1} = p \). Then for every \( y \in Y \),

\[
y(g(c^{n-1}) g(c^{n-2}) \ldots g^e g) = (y^{p^{n-1}}) g(y^{p^{n-2}}) g \ldots (y^p) g y g.
\]

We now construct our required \( g \) as follows: If \( y^e = y \), \( y^g \) is chosen such that

\[
(y^g)^n = y^f.
\]
(Since $A$ is divisible this is always possible.)

If $y^c \neq y$, we choose $y^g$, $(y^p)^g$, ..., $(y^{p^{n-2}})^g$ arbitrarily in $A$ and $(y^{p^{n-1}})^g$ such that the product of all these chosen elements is $y^f$.

Then for every $y \in Y$ and for the $g$ constructed above we get

$$y(g^{c^{n-1}} g^{c^{n-2}} \ldots g^c g) = y^f.$$  

Hence, $g^{c^{n-1}} g^{c^{n-2}} \ldots g^c g = f$, and so $(cg)^n = bf$, as stated.

It is obvious that if $f \in A^{(Y)}$, then the element $g$ as constructed above lies in $A^{(Y)}$. Hence the result also holds for $A_{wr}B$. //