Chapter I

The wreath product of groups

1.1. The holomorph of a group ... 1
1.2. The cartesian power and the direct power of a group ... ... 5
1.3. The wreath product and the restricted wreath product of groups ... ... 6
1.4. Standard wreath products ... 8
CHAPTER I

THE WREATH PRODUCT OF GROUPS

1.1. The holomorph of a group

A natural introduction to the concept of the wreath product of groups is through that of the holomorph of a group, discussed in this section.

Let $G$ be any group and $S_G$ the group of all permutations on $G$. Every $f \in G$ gives us a right multiplication $f \in S_G$, defined by the rule

$$x^{(f)} = xf, \quad x \in G.$$  \hspace{1cm} (1.1.1)

If $f$ and $g$ are any two right multiplications in $G$, so also is the product $f(g)^{-1}$. Hence the right multiplications in $G$ form a subgroup $R_G$ of $S_G$, the right regular representation of $G$. By Cayley's theorem there is an isomorphism

$$G \rightarrow R_G, \quad f \rightarrow f.$$ \hspace{1cm} (1.1.2)

Let $B$ be a group of automorphisms of $G$. Then $B$ is a subgroup of $S_G$, and we get a subgroup $H$ of $S_G$. 
generated by \( R_G \) and \( B \):

\[
H = \langle R_G , B \rangle . \tag{1.1.3}
\]

We call this group \( H \) the (generalized) holomorph of \( G \) with \( B \) (see Schenkman, 1965, p.90). We get

1.1.4. Theorem. The holomorph \( H \) of \( G \) with \( B \) is a split extension of \( G \) by \( B \):

\[
H = \mathcal{G} B = G B = B G .
\]

An arbitrary element of \( H \) is uniquely expressible as a pair \( b f \), where \( b \in B \) and \( f \in G \).

Multiplication in \( H \) is given by the rule

\[
bf \cdot fg = b e f c g \quad (b, e \in B, f, g \in G);
\]

and \( e^{-1} f c = f^e \) for any \( e \in B \) and any \( f \in G \).

Proof. We first show that \( H \) is a split extension (see Schenkman, 1965, p.91) of \( R_G \) by \( B \), i.e., \( R_G \) is normal in \( H \), \( H = R_G B (= B R_G) \) and the intersection of \( R_G \) and \( B \) is trivial.

Let \( e \in B \) and \( f \in R_G \). Then for every \( x \in G \) we get
SEC. 1.1. THE HOLOMORPH

\[ x(\overline{f}c) = (x(\overline{f}^c))^c = (xf)^c = \]
\[ = x^c f^c = (x^c)^c = x(x^c) = x(c f^c) .\]

Hence, \( \overline{f}c = c f^c \). (1.1.5)

It follows that \( R_G \) is a normal subgroup of \( H \), and
then the definition of \( H \) gives \( H = R_G B = B R_G \).

Let \( \overline{f} \) be any element in the intersection of \( R_G \)
and \( B \). Then for any \( x, y \in G \) we get

\[ x f y f = x(\overline{f}) y(\overline{f}) = (xy)(\overline{f}) = xyf .\]

Hence \( xf = x \), or \( f = 1 \). It follows that
\( \overline{f} = 1 \), and so the intersection of \( R_G \) and \( B \) is trivial.

Thus \( H \) is a split extension of \( R_G \) by \( B \).
Since $H = B \times R_G$, an arbitrary element of $H$ is expressible as a product $b \bar{f}$, where $b \in B$ and $\bar{f} \in R_G$. Suppose the element $b \bar{f}$ is also expressible as a product $d \bar{h}$, where $d \in B$ and $\bar{h} \in R_G$. We get

$$d^{-1} b = \bar{h}(\bar{f})^{-1},$$

where $d^{-1} b \in B$ and $\bar{h}(\bar{f})^{-1} \in R_G$. Hence $d^{-1} b$ and $\bar{h}(\bar{f})^{-1}$ belong to the intersection of $B$ and $R_G$, and are therefore trivial. Thus $d = b$ and $\bar{h} = \bar{f}$. It follows that the expression $b \bar{f}$ ($b \in B$, $\bar{f} \in R_G$) is unique for a given element of $H$.

If $b \bar{f}$, $c \bar{g} \in H$, then 1.1.5 gives

$$b \bar{f} c \bar{g} = b c \bar{f} \bar{g}.$$

We also get from 1.1.5 (or putting $b = c^{-1}$ in the last result) that

$$c^{-1} \bar{f} c = \bar{c} \bar{f}$$

for any $c \in B$ and any $\bar{f} \in R_G$.

Because of the isomorphism 1.1.2 between $R_G$ and $G$, we identify $R_G$ with $G$ and $\bar{f}$ with $f$ for every $\bar{f} \in R_G$. This completes the proof of the theorem. //
1.2. The Cartesian power and the direct power of a group

Let $A$ be an arbitrary group and $Y$ an arbitrary nonempty set.

The set of all functions $f : Y \to A$ from $Y$ to $A$ is the *cartesian power of $A$ by $Y$* and is denoted by $A^Y$. It is immediately seen that $A^Y$ is a group under the multiplication rule

$$y(fg) = yf yg, \quad f, g \in A^Y, \quad y \in Y. \quad (1.2.1)$$

The unit element of the group $A^Y$ is the function

$$1 : Y \to A, \quad y \to 1. \quad (1.2.2)$$

The inverse $f^{-1}$ of an element $f$ of $A^Y$ is given by

$$y(f^{-1}) = (yf)^{-1}, \quad y \in Y. \quad (1.2.3)$$

For every $f \in A^Y$ we get a subset of $Y$ formed by the elements of $Y$ which have nontrivial images under $f$. It is the *support* of $f$.

If two elements $f$ and $g$ of $A^Y$ have finite support then clearly $fg^{-1}$ also has finite support. Hence
the elements of $A^Y$ which have finite support form a subgroup of $A^Y$. It is the **direct power** of $A$ by $Y$ and shall be denoted by $A{(Y)}$.

Let $B$ be a group of permutations on the set $Y$. For any $f \in A^Y$ ($A{(Y)}$) and any $b \in B$ define $f^b$ as an element of $A^Y$ ($A{(Y)}$) by the rule

$$y(f^b) = (y(b^{-1}))^f, \quad y \in Y. \quad (1.2.4)$$

We then get a well-known result:

1.2.5. **Lemma.** The permutation group $B$ is a group of automorphisms of $A^Y$ ($A{(Y)}$). (See, e.g., B. H. Neumann, 1960*, p.99.)

1.5. **The wreath product and the restricted wreath product of groups**

Let $A$ be an arbitrary group and $B$ a group of permutations on a nonempty set $Y$. ($A, B$ and $Y$ will always have these meanings in the following pages unless otherwise stated.)

By Lemma 1.2.5 $B$ is a group of automorphisms of $A^Y$. Hence the holomorph of $A^Y$ with $B$ exists. It is the
(cartesian, unrestricted or complete) wreath product (or interlacing) of A by B. We shall denote it by \( \text{A} \wr \text{B} \).

Theorem 1.1.4 now gives us

1.3.1. Theorem. The wreath product \( \text{A} \wr \text{B} \) is a split extension of \( \text{A}^\text{Y} \) by B:

\[
\text{A} \wr \text{B} = [\text{A}^\text{Y}] \text{B} = \text{A}^\text{Y} \text{B} = \text{B} \text{A}^\text{Y}
\]

An arbitrary element of \( \text{A} \wr \text{B} \) is uniquely expressible as a pair \( bf \), where \( b \in \text{B} \) and \( f \in \text{A}^\text{Y} \).

Multiplication in \( \text{A} \wr \text{B} \) is given by the rule

\[
bf cg = b^c f^cg \quad (b, c \in \text{B}, f, g \in \text{A}^\text{Y}),
\]

and

\[
c^{-1} fc = f^c \quad \text{for any } c \in \text{B}
\]

and any \( f \in \text{A}^\text{Y} \).

We immediately get 1.3.2. Corollary. An arbitrary element of \( \text{A} \wr \text{B} \) is uniquely expressible as a pair \( f b \), where \( f \in \text{A}^\text{Y} \) and \( b \in \text{B} \).

For putting \( c^{-1} = b \) in \( c^{-1} fc = f^c \) we get \( bf = f^{(b^{-1})} b \).
A^Y is the **base group** and B the **top group** of the wreath product.

By Lemma 1.2.5 B is also a group of automorphisms of the direct power A^Y. Hence the holomorph of A^Y with B exists, and is a subgroup of A\(\text{Wr}_B\). It is the **restricted (or discrete) wreath product (or interlacing)** of A by B. We shall denote it by A\(\text{wr}_B\).

There are obvious analogues of Theorem 1.3.1 and Corollary 1.3.2 for the restricted wreath product A\(\text{wr}_B\).

A^Y is the **base group** and B again the **top group** of the restricted wreath product.

Clearly the wreath product A\(\text{Wr}_B\) coincides with the restricted wreath product A\(\text{wr}_B\) if and only if Y is finite.

1.4. Standard wreath products of groups

If B is an arbitrary group, then its right regular representation R_B is a group of permutations on the set B. The wreath product A\(\text{Wr}_B\) and the restricted wreath product A\(\text{wr}_B\) are called **standard wreath products** of A and B, and
are also denoted by $A_{WRB}$ and $A_{wrb}$ respectively. Because of 1.1.2 we identify $bGg$ with $bCB$.

We note that in the standard wreath products $A_{WRB}$ and $A_{wrb}$ we have $y^b = y b$ for all $y, b \in B$. Hence in these wreath products we define $f^b$ ($f \in A^B$ or $A^{(B)}$, $b \in B$) by the rule

$$y(f^b) = (yb^{-1})^f, \quad y \in B. \quad (1.4.1)$$

We get another type of 'standard wreath products' - viz., the permutational wreath products. First we consider some preliminary results.

1.4.2. Lemma. If $A$ is also a permutation group on a set $X$, the wreath product $A_{WRB}$ and the restricted wreath product $A_{wrb}$ are permutation groups on the product set $X \times Y$ (B. H. Neumann, 1960*, p.102).

For an arbitrary element $f b$ ($f \in A^Y$ and $b \in B$) of $A_{WRB}$ is a permutation on the set $X \times Y$ under the rule

$$(x, y)^{fb} = (x(y^f), y^b), (x, y) \in X \times Y, \quad (1.4.3)$$

and a similar result holds for $A_{wrb}$.

It is known that wreath multiplication of permutation groups is associative (P. Hall, 1959, p.313 and B.H. Neumann,
1960e, p.104). We immediately deduce

1.4.4. Lemma. The unique factorization theorem does not hold for the wreath product of permutation groups: We may have

\[ AWrB = A^* Wr B^* \quad \text{or} \quad AwrB = A^* \wr B^* \]
even though \( A \) is not isomorphic with \( A^* \) and \( B \) is not isomorphic with \( B^* \).

(P. M. Neumann (1964) has shown that the unique factorization theorem holds for the standard wreath products)

A permutation group \( G \) on a set \( S \) is **transitive** if for any \( s, t \in S \) there is \( g \in G \) such that \( t = s^g \).

If, further, given any \( g \in G \), \( g \neq 1 \), and any \( s \in S \), \( s^g \neq s \), then \( G \) is **regular**. Clearly a subgroup of a regular group is regular. We note:

1.4.5. Lemma. If \( A \) and \( B \) are transitive, so also are the permutation groups \( AWrB \) and \( AwrB \).

**Proof.** Suppose \( A \) and \( B \) are transitive. Let \((x, y), (u, v) \in X \times Y\); then \( x, u \in X \) and \( y, v \in Y \).

Since \( A \) and \( B \) are transitive, there are \( a \in A \)
and \( b \in B \) such that \( u = x^a \) and \( v = y^b \). Take \( f \in A^Y \)
such that \( y^f = a \). Then \((u, v) = (x^f, y^f) = (x, y)^{fb}\).
It follows that $\mathbb{A} \wr \mathbb{B}$ is transitive.

The proof for $\mathbb{A} \wr \mathbb{B}$ is analogous. //

Suppose $\mathbb{A}$ and $\mathbb{B}$ are arbitrary groups. Then the right regular representations $R_\mathbb{A}$ and $R_\mathbb{B}$ are permutation groups on the sets $\mathbb{A}$ and $\mathbb{B}$, respectively. The wreath product $R_\mathbb{A} \wr R_\mathbb{B}$ and the restricted wreath product $R_\mathbb{A} \wr R_\mathbb{B}$ are also called standard wreath products of $\mathbb{A}$ and $\mathbb{B}$ and denoted by $\mathbb{A} \wr \mathbb{B}$ and $\mathbb{A} \wr \mathbb{B}$ respectively. They are also known as permutational wreath products of $\mathbb{A}$ and $\mathbb{B}$. We note:

1.4.6. Lemma. Permutational wreath products are transitive permutation groups, but are not regular.

Proof. The first part follows immediately from Lemma 1.4.5. To see the second part, let $(x, y) \in \mathbb{X} \times \mathbb{Y}$ and choose $f \in A(Y)$, $f \neq 1$, such that $y^f = 1$. Then $f \in A \wr B$, $f \neq 1$, and $(x, y)^f = (x(y^f), y) = (x, y)$. Hence $A \wr B$ is not regular, and so the overgroup $A \wr B$ is also not regular. //

We immediately deduce (in continuation of Lemma 1.4.5)

1.4.7. Corollary. The wreath product or the restricted wreath product of two regular permutation groups may not be regular.