GROUPS AND THE WREATH PRODUCT

A SURVEY

The concept of the wreath product of groups seems to have first emerged in a short note by Cayley (1878). He constructed the wreath product of a finite group by a cycle of order two. (An account of the early investigations has been given in B. H. Neumann, 1963 along with a bibliography of wreath products.)

The latent possibilities of the wreath product concept came to be known first through a number of papers by Kaloujnine and Krasner published during the years 1946-51.

An interesting result of Kaloujnine (1948) is that a Sylow p-subgroup of the symmetric group of degree $p^n$ is the n-th wreath power of the cyclic group of order p. Sylow subgroups of system normalizers of finite solvable groups and Sylow subgroups of locally finite groups also have been investigated respectively by Rose (1967) and Hartley (1971) with the help of the wreath product.

Krasner and Kaloujnine (1951) have shown that all extensions of a group A by a group B are embedded in a wreath product of A and B. This result has proved to be
of considerable utility and has earned for itself the name 'The Universal Embedding Theorem for Groups' (Dixon, 1967).

P. Hall (1954) first introduced the name 'wreath group'. Earlier, Polya (1937) had called it 'Kranzgruppe'.

**Pioneering work**

Four of the pioneers to utilize the wreath product for breaking new grounds were P. Hall, B. H. Neumann, Hanna Neumann and Gilbert Baumslag.

Hall (1954) has studied finitely generated solvable groups satisfying the maximal condition for normal subgroups with the help of the wreath product. He has shown that for solvable groups, the maximal condition for normal subgroups comes in generality strictly between the maximal condition for subgroups and the condition of being finitely generated.

It is known that finitely generated metabelian groups have nilpotent Frattini subgroup. Using the wreath product Hall (1961) has demonstrated the existence of 2-generator polyabelian groups with non-nilpotent Frattini subgroups.
Hall (1959) has established a sufficient condition for verbal completeness based on the wreath product, and has deduced the existence of an infinite number of non-isomorphic, countably infinite, locally finite, verbally complete $p$-groups with non-trivial centre.

With the aid of the wreath product Hall (1962, 1963) has constructed a countably infinite, characteristically simple, locally finite $p$-group with trivial Baer radical, and has investigated non-strictly simple groups.

Burns (1968) has supplemented the results of Hall, 1959, using the same tool. He has shown the existence of an equal number of similar $p$-groups with trivial centre.

Graham Higman, B. H. Neumann and Hanna Neumann (1949) had shown with the help of the generalized free product that an arbitrary countable group $G$ can be embedded in a 2-generator group $H$. B. H. Neumann and Hanna Neumann (1959) have given an alternative proof of this important result with the help of the wreath product. The powerful new method provides additional information: It shows that if the countable group $G$ belongs to a variety $U$, the 2-generator embedding group $H$ belongs to the product variety $U \vartriangleleft A$, where $A$ is the variety of all abelian groups. Consequently certain properties of $G$ can be made to persist in $H$. For example, if $G$ is a finite $p$-group,
H can be chosen as a finite p-group. The new method also leads to the interesting construction of a solvable non-Hopf with three generators.

The above embedding procedure of Neumann and Neumann has been extended in different directions: it has been applied to semi-groups and ordered groups (B. H. Neumann, 1960\textsuperscript{c} and 1960\textsuperscript{d}), transfinite generalizations of solvable groups (Kovacs and Neumann 1965) and non-Hopf groups (Dey, 1969). B. H. Neumann (1960\textsuperscript{c}) has proved that every countable semigroup $G$ can be embedded in a 3-generator semigroup $H$, and certain properties of $G$ can be made to persist in $H$. He (1960\textsuperscript{d}) has shown that analogues of Neumann and Neumann, 1959 hold for fully ordered groups. The main result of Kovacs and Neumann, 1965 states that every countable $\text{SI}^\ast$-group (or hyperabelian group) of length 1 can be embedded in a 2-generator $\text{SI}^\ast$-group of length $1 + 2$. An easy and interesting consequence of this result is that there exist $\text{SI}^\ast$-groups that are not locally solvable.

The notion of a 'variety of groups' (called a 'group manifold' in Smelkin, 1963) emerged in 1935. Hanna Neumann (1956) obtained the following properties of the algebraic structure of the set (it is a complete lattice) of varieties:

1. It is multiplicatively an ordered semigroup with unit element and zero element, (i) with cancellation of equal
non-zero right hand factors from inclusions and equations, and
(ii) with multiplication on the right distributing over the
lattice operations and over commutation.

II. Every non-zero and non-unit variety is expressible as a product of indecomposable varieties.

B. H. Neumann, Hanna Neumann and Peter M. Neumann (1962) have, with the help of the wreath product, established the left
analogue of I(i) and the uniqueness of II, and disproved the
left-analogue of I(ii).

Other investigations on varieties of groups with the
wreath product as a tool have been Gilbert Baumslag, 1963, Gilbert
Baumslag, B. H. Neumann, Hanna Neumann and Peter M. Neumann, 1964,
Brisley and Macdonald, 1969 and Benjamin Baumslag and Gilbert
Baumslag, 1971. A generalization of the wreath product, the
verbal wreath product, has been used for studying varieties of
groups by Smelkin (1963, 1964 and 1965), Burns (1967) and

It is known that in any group the product of two normal
solvable subgroups is solvable (Schenkman, 1965, p.217), the
product of two normal nilpotent subgroups is nilpotent (Scott,
1964, p.166), the product of two normal locally nilpotent sub-
groups is locally nilpotent (Schenkman, 1965, p.205) and the
product of two normal ZA-subgroups is a ZA-subgroup (Hall, 1961). Baumslag, Kovacs and Neumann (1965) have demonstrated with the help of the wreath product that the normal product theorem does not hold for locally solvable, locally SI*, SI, locally SN*, locally residually nilpotent and residually nilpotent subgroups.

Some allied concepts

Wreath products with central amalgamations have been introduced by B. H. Neumann (1957) under the name 'crown products', and have been used by him to study ascending derived series. A further generalization by Neumann (1963) is the 'twisted wreath product', and has been applied by him to demonstrate the existence of a splitting extension \( P^* \) of a non-trivial abelian group \( F^* \) by a non-trivial group \( B \) such that the commutator subgroup of \( P^* \) and \( F^* \) coincides with \( F^* \). A constructive procedure used by Gilbert Baumslag (1958) combines features of the crown product and the twisted wreath product.

Let \( G \) be a group, \( A \) and \( B \) subgroups of it, and \( X \) and \( Y \) normal subgroups of \( A \) and \( B \), respectively. If \( A \) and \( B \) have trivial intersection in \( G \), but the normal closure of \( A \) in \( G \) meets \( B \) exactly in \( Y \), and the normal closure of \( B \) in \( G \) meets \( A \) exactly in \( X \), then \( G \) is called 'a linked embedding of \( A \) and \( B \) with kernels \( X \) and \( Y \)'.
further, $G$ is generated by $A$ and $B$, the linked embedding is a 'linked product'. Hanna Neumann and James Wiegold (1960) have used the wreath product to obtain linked embeddings as well as some cases of linked products.

Using free products with an amalgamated subgroup, B. H. Neumann had extended in 1943 the classical theorem, 'Every abelian group can be embedded in a divisible abelian group' to arbitrary groups. With the aid of the wreath product Gilbert Baumslag (1959) has proved Neumann's result in a completely different way, and has further shown that (1) Every periodic group can be embedded in a periodic divisible group and (2) Every $p$-group can be embedded in a divisible $p$-group.

The wreath product construction adopted by Baumslag (1959) to solve the equation $x^ng = 1$, where $g$ is an element of a group $G$, has been used by Levin (1962) to solve the more general equation $x_{a_0} x_{a_1} \cdots x_{a_{n-1}} = 1$, with the $a_i$ in $G$.

Baumslag (1959, 1960a, 1960b, 1961), Gruenberg (1957), McCarthy (1968, 1970) and Berggren (1969) have investigated the conditions under which wreath products inherit certain properties from their ingredients. Baumslag (1959) has shown that the wreath product of a group $A$ by a group $B$ is nilpotent if and only if $B$ is a finite $p$-group and $A$ is a nilpotent
p-group of finite exponent. Buckley (1970) has given a new proof of this theorem with the help of polynomial functions. Baumslag's proof makes use of the following result which gives an easy method of constructing infinite p-groups that are not nilpotent: The restricted standard wreath product of a non-trivial group by an infinite group has trivial centre.

Liebeck (1962) has provided a lower bound for the class of a nilpotent wreath product. He has given a simple construction for a set of non-nilpotent metabelian groups which satisfy a finite Engel condition and has shown that there are nilpotent groups of arbitrarily large nilpotency class for which the nilpotency class is equal to the Engel class.

Inheritance investigations

Scrutton (1966), Meldrum (1967, 1970) and Shield (1971) have supplemented the above investigations of Baumslag and Liebeck. Connections between roots in groups and the wreath product have been established by Baumslag (1960a and 1960b). He (1961) has shown that the wreath product of a non-trivial finitely presented group A by a finitely presented group B is a finitely presented group if and only if B is finite. Baumslag (1966) has obtained a representation of the wreath product of two torsion-free abelian groups in a power series ring.
Let $P^*$ be a property of groups which is inherited by subgroups and finite direct products. Further suppose that any extension of a group $A$ by a group $B$, where $A$ is residually $P^*$ and $B$ has property $P^*$, is residually $P^*$. Then $P^*$ is called a 'star property'. (It has been called 'root property' in Gruenberg, 1957.) For example, finiteness, being of $p$-power order and solvability are star properties; nilpotency is not (Hanna Neumann, 1967, p.73). Gruenberg (1957) has shown that if $A$ and $B$ are residually $P^*$, where $P^*$ is a star property, then the restricted standard wreath product of $A$ and $B$ is residually $P^*$ if and only if either $B$ has property $P^*$ or $A$ is abelian.

An infinite group is called 'just-infinite' if all the proper quotient groups of it are finite (i.e., if every non-trivial normal subgroup of it has finite index.). Thus, the infinite cyclic group and the infinite dihedral group are just-infinite groups. McCarthy (1968) has shown that the standard wreath product of two non-trivial groups $A$ and $B$ is just-infinite if and only if $B$ is finite and $A$ is non-abelian just-infinite, and has deduced that there exist solvable just-infinite groups of every (positive) derived length and solvable just-infinite groups of every positive rank having derived length at most 3. He (1970) has also shown that the unrestricted wreath product of two infinite cyclic groups is residually just-infinite.
Berggren (1969) has studied the class of all finite groups every element of which is conjugate to its inverse (for example, every symmetric group of finite degree is such a group). He has shown that the wreath product of any group of this class with the cyclic group of order two again belongs to this class. It follows that every Sylow 2-subgroup of a symmetric group is a member of the particular class.

The theory of the wreath product has been applied to studies of the following objects in finite solvable groups: System normalizers and nilpotent self-normalizing subgroups in certain classes of finite solvable groups (Carter, 1962); formations of finite solvable groups with pronormal system normalizers (Rose, 1967); covering subgroups of a finite solvable group corresponding to a normal system and a saturated formation of the group (Prentice, 1969). Carter has given an easy method for construction of A-groups (that is, groups in which every Sylow subgroup is abelian) based on the wreath product.

**Embedding of amalgams**

If the intersection of two given groups is a subgroup of each of the two groups, then their union is called the 'amalgam' of the two groups. According to a classical theorem of Otto Schreier (1927), every amalgam of two groups can be embedded in the generalized free product of the amalgam.
B. H. Neumann (1960\textsuperscript{a}, 1960\textsuperscript{b}) gave a simpler proof of Schreier's theorem with the help of the permutational product of groups and provided additional informations through the new method. Continuing Neumann's investigations, Wiegold (1962) has given a method for embedding an amalgam of two groups $A$ and $B$ for which their intersection $H$ is normal in $A$ and satisfying a certain condition in the wreath product of $B$ by $A/H$.

Higman (1964) has obtained sufficient conditions for the embeddability of an amalgam in a wreath product. He has deduced a criterion for the embeddability of an amalgam of two finite $p$-groups in a finite $p$-group, which holds equally well for all finitely generated nilpotent groups.

The structure of the wreath product has been specially studied in a few investigations. Neoughton (1962) has described completely the way in which the automorphism group of the unrestricted standard wreath product is built up from certain distinguished subgroups whose choice arises naturally from the way in which the wreath product is formed from its component groups. Peter M. Neumann (1964) has provided a considerable amount of information on the structure of standard wreath products. He has characterized the derived group in a restricted standard wreath product and has done the same as far as possible for unrestricted standard wreath products. He has also investigated direct product decompositions and isomorphisms of
standard wreath products. The last investigation has yielded two unique factorization theorems. Seksenbaev (1966) and Kappe and Parker (1970) have determined the set $S_G$ of self-centralizing elements and the set $R_G$ of elements with trivial centralizer in a wreath product $G$, and have given some necessary and some sufficient conditions for $G = \langle S_G \rangle$ and $G = \langle R_G \rangle$. Rose (1968 and 1969) has studied the wreath product of a cyclic group of prime order by a finite group.

Recent investigations

B. H. Neumann (1960) first applied the theory of the wreath product to partially ordered groups. He showed that the complete wreath product cannot be linearly ordered, but it can be turned into a partially ordered group in such a way that a 'large' subgroup of it becomes linearly ordered. Reilly (1969) has used this result to obtain an embedding of $o$-groups in $o$-simple groups, and has also shown that if an $l$-group $G$ is a lexicographic extension of an $l$-group $A$ by an $o$-group $B$ then $G$ is $l$-isomorphic to an $l$-subgroup of the unrestricted standard wreath product of $A$ and $B$.

It is known that the order-preserving permutations on a linearly ordered set form an $l$-group. Lloyd (1964) has used the wreath product of two such $l$-groups for establishing the
embeddability of a certain 1-group in a divisible 1-group as an 1-subgroup.

Holland (1969) has contributed a generalized notion of wreath product to the theory of permutation groups. The notion of the generalized wreath product has subsequently been applied to ordered permutation groups by Holland and McCleary (1969). McCleary (1970) has used the theory of sets with valuation to further the theory of wreath products.

Some other recent investigations with the help of the wreath product have been:

Complementation in groups (Christensen, 1964 and Dixon, 1967);

Groups in which the commutator subgroup coincides with the set of commutators (Rhemtulla, 1969);

Generalization of Hamiltonian and nilpotent groups (Dixon, Poland and Rhemtulla, 1969);

The power-commutator structure of finite p-groups (Arganbright, 1969);

Conjugacy of normalizers corresponding to a saturated formation (Hawkes, 1970);
Odd \( p \) groups as fixed point free automorphism groups (Berger, 1971);
The subnormal structure of some classes of soluble groups (McDougall, 1972).

**The present work**

A general study of the (cartesian) wreath product and its special types is the object of the present work. The following is a brief analysis.

We have introduced the wreath product as a holomorph of a group and have obtained a number of results concerning some important subgroups of the wreath product, for example, the derived group, the centre, and centralizers and normalizers of particular subgroups. (Chapters I and II)

Some properties of homomorphisms and automorphisms of wreath products have been established. Existence of certain "E-subgroups" in wreath products and certain (covariant) functors from the category of groups and homomorphisms to itself has then been deduced. Existence of an isomorphism between a wreath product and a monomial group has been demonstrated. (Chapter III)

A number of results have been obtained regarding wreath products of groups satisfying finiteness conditions (such as
finite generation, finite presentation, the maximal condition for subgroups, periodicity, local finiteness, finiteness of exponent), solvable groups, hypercentral groups, nilpotent groups and divisible groups. (Chapter IV)

We have given a proof of the famous theorem of Krasner and Kaloujnine on embedding of group extensions. It has yielded a theorem on the embeddability of certain group amalgams in standard wreath products. Results on embeddability of special amalgams in special groups and of permutational products in wreath products have been deduced from this theorem. Existence of embeddings equivalent to diagonal embeddings and coordinate embeddings has been established. (Chapter V)

The structures of the wreath products of partially ordered groups and wreath products of lattice-ordered groups and order-preserving homomorphisms have been studied in some detail. We have shown that a wreath product of certain partially-ordered groups is a topological group. Our results in this area include an extension of the Embedding Theorem of Krasner and Kaloujnine to partially-ordered groups. (Chapter VI)