Chapter V

**Some embeddings**

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1</td>
<td>Embedding of extensions in a wreath product</td>
<td>95</td>
</tr>
<tr>
<td>5.2</td>
<td>Embedding of amalgams in a wreath product</td>
<td>100</td>
</tr>
<tr>
<td>5.3</td>
<td>Embedding of a permutational product in a wreath product</td>
<td>109</td>
</tr>
<tr>
<td>5.4</td>
<td>Equivalent embeddings</td>
<td>112</td>
</tr>
</tbody>
</table>
CHAPTER V

SOME EMBEDDINGS

5.1. Embedding of extensions in a wreath product

The following theorem of Krasner and Kaloujnine (1951) which has been given the name 'The Universal Embedding Theorem for Groups' (Dixon, 1967) — plays an important role in the embedding of groups and amalgams. In view of its importance different proofs of the theorem have been given by different authors (e.g.: B. H. Neumann, Hanna Neumann, and Peter M. Neumann, 1962; Higman, 1964; Reilly, 1969). We give a proof analogous to that of Reilly.

5.1.1. The Universal Embedding Theorem for Groups. If \( G \) is any extension of a group \( H \), then \( G \) can be embedded in the standard wreath product \( H \wr (G/H) \).

Proof. We have a natural epimorphism

\[
\alpha : G \to G/H, \quad g \mapsto gH.
\]

Take any transversal \( T \) of \( H \) in \( G \). Corresponding to this transversal \( T \) we get a 'countermap' of \( \alpha \),

\[
\alpha^* : G/H \to T, \quad tH \mapsto t \quad (t \in T).
\]
Let \( kH \) be any element of \( G/H \). Then there exists \( s \in T \) such that \( kH = sh \). Hence, \((kH)^{a^*a} = s^a = sh \). Thus for any \( kH \in G/H \) we get

\[(kH)^{a^*a} = kH. \quad (5.1.2)\]

For each \( a \in G \) define a map

\[ f_a : G/H \to G, \quad kH \to (k a^{-1} H)^{a^*} a (kH)^{-a^*}. \]

We note that

\[(k a^{-1} H)^{a^*} a (kH)^{-a^*} = (k a^{-1} H) a^a (kH)^{-a^*} \]

\[= (k a^{-1} H) aH (k^{-1} H) \quad \text{(using 5.1.2)} \]

\[= H, \quad \text{the unit element of } G/H. \]

It follows that the image \((k a^{-1} H)^{a^*} a(kH)^{-a^*}\) is contained in the kernel \( H \) of \( a \). Hence \( f_a \) is a map \( G/H \to H \), i.e.,

\[ f_a \in H(G/H). \]

Then for each \( a \in G \),

\[(aH)f_a \in H \wr (G/H). \]

Thus there exists a map

\[ \beta : G \to H \wr (G/H), \quad a \to (aH)f_a. \quad (5.1.3) \]
We shall see that $\beta$ is a monomorphism.

Let $a, b \in G$. Then

$$(a \cdot b)^\beta = a \cdot H \cdot b \cdot H \cdot f_{ab},$$

and

$$a^\beta \cdot b^\beta = a \cdot H \cdot b \cdot H \cdot f_{aH} \cdot f_{bH}.$$

Now for any $gH \in G/H$,

$$f_{bH} \cdot f_{aH} = (g \cdot b^{-1} \cdot H) \cdot f_{aH} \cdot (gH) \cdot f_{bH},$$

$$= (g \cdot b^{-1} \cdot a^{-1} \cdot H) \cdot a^{-1} \cdot (g \cdot b^{-1} \cdot H) \cdot a \cdot (g \cdot b^* \cdot H) \cdot a^{-1} \cdot (g \cdot b^{-1} \cdot H) \cdot a,$$

$$= (g \cdot b^{-1} \cdot a^{-1} \cdot H) \cdot a \cdot b \cdot (gH) \cdot a^{-1} \cdot (g \cdot b^{-1} \cdot H) \cdot a,$$

$$= (gH) \cdot f_{ab},$$

and so $f_{bH} \cdot f_{aH} = f_{ab}$.

Thus $(a \cdot b)^\beta = a^\beta \cdot b^\beta$, showing that $\beta$ is a homomorphism.

The kernel of $\beta$ is the set of elements $a \in G$ such that $a \cdot H \cdot f_a = 1$, i.e., such that

(i) $aH = H$ and

(ii) $k^{-1} \cdot a \cdot (kH)^{-1} \cdot a^{-1} = 1$ for any $kH \in G/H$. Clearly the kernel is trivial and so $\beta$ is a monomorphism.
We deduce

5.1.4. Corollary. If the group $H$ in Theorem 5.1.1 is a subgroup of a group $F$, then $G$ can be embedded in the standard wreath product $F \text{ Wr}(G/H)$.

For let $i$ be the inclusion map $H \to F$. Then $i$ is a monomorphism, and can therefore be extended to a monomorphism

$$i^* : H \text{ Wr } (G/H) \to F \text{ Wr } (G/H)$$

by Theorem 3.2.1. The product map $i^*$ embeds $G$ in $F \text{ Wr}(G/H)$.

The following result will be needed in subsequent discussions.

5.1.5. Lemma. If the subgroup $H$ in Theorem 5.1.1 has a transversal $T$ such that $T$ centralizes $H$ in $G$ (e.g., if $H$ is central in $G$), then the function $f_a$, $a \in H$, is the diagonal function $a^0$ in the cartesian power $H(G/H)$.

Proof. Suppose $H$ has a transversal $T$ in $G$ such that $T$ centralizes $H$ in $G$.

Let $kH \in G/H$. Then there exists $t \in T$ such that $kH = tH$. We get
SEC. 5.1. EMBEDDING OF EXTENSIONS

\[(kH)^{fa} = (th)fa\]

\[= (ta^{-1}H)^* a (th)^{-a*}\]

\[= (th)^* a (th)^{-a*} \quad \text{(since } a^{-1} \in H)\]

\[= ta t^{-1} \quad \text{(by definition of } a^*)\]

\[= a \quad \text{(since } a \in H \text{ and } T \text{ centralizes } H \text{ in } G)\]

Hence \( f_a = a^0 \) in \( H^{(G/H)} \). //

5.1.6. Corollary. If the subgroup \( H \) has a transversal \( T \) which centralizes \( H \) in \( G \) (e.g., if \( H \) is central in \( G \)), then the embedding

\[\beta : G \rightarrow H \wr (G/H)\]

is an extension of the diagonal embedding (see Corollary 2.1.7)

\[a^0 : H \rightarrow H \wr (G/H)\].

Proof. If \( a \in H \) we get

\[a^\beta = (aH)^{fa} = H^{fa}\]

\[= fa \quad \text{(since } H \text{ is the unitelement of } G/H)\]

\[= a^0 \quad \text{(by Lemma 5.1.5)}\]

\[= a(a^0)\].
Hence $\beta$ is an extension of $\alpha^0$ as stated. //

**Note.** P. Hall (1959, Lemma 10) has proved Corollary 5.1.6 for the special case when $G$ is finite and $H$ is central in $G$.

### 5.2. Embedding of amalgams in a wreath product

If $A$ and $B$ are two groups such that their intersection is a subgroup of both $A$ and $B$, then the union of $A$ and $B$ is an amalgam of $A$ and $B$ (Baer, 1949).

If there exists an injective map $\theta$ from the amalgam to a group $P$ such that the restrictions of $\theta$ to $A$ and $B$ are homomorphisms (monomorphisms) then the amalgam is embedded in the group $P$ (B. H. Neumann, 1963). We note

#### 5.2.1. Lemma. A necessary and sufficient condition such that an amalgam of two groups $A$ and $B$ with an intersection $H$ may be embedded in a group $P$ is that there exist monomorphisms $\alpha : A \to P$ and $\beta : B \to P$ such that $\alpha$ and $\beta$ agree on $H$ and the intersection of $A^\alpha$ and $B^\beta$ is $H^\alpha (= H^\beta)$.

**Proof.** We shall first show that the condition is necessary. Suppose that the amalgam of $A$ and $B$ is embedded in the group $P$ under an injective map $\theta$. Let $\alpha$ and $\beta$ be the restrictions
of \( \theta \) to \( A \) and \( B \) respectively. Then \( \alpha \) and \( \beta \) are monomorphisms \( A \to P \) and \( B \to P \), and clearly they agree on \( H \).

The image \( H^\alpha \) is contained in both \( A^\alpha \) and \( B^\beta \), and hence in their intersection \( I \). To prove the opposite relation consider an arbitrary element \( z \in I \). Since \( \theta \) is a monomorphism, \( z \) has a unique preimage \( x \) in the amalgam. Since \( z \) is contained in both \( A^\alpha \) and \( B^\beta \), \( x \) is contained in both \( A \) and \( B \) — and hence in \( H \). It follows that \( z \in H^\alpha \), and so \( I \) is contained in \( H^\alpha \). Thus \( H^\alpha = I \), the intersection of \( A^\alpha \) and \( B^\beta \).

To show that the condition is also sufficient suppose that there exist monomorphisms as stated. Then we get a map \( \theta \) from the amalgam to the group \( P \) such that the restrictions of \( \theta \) to \( A \) and \( B \) are homomorphisms, viz., \( \alpha \) and \( \beta \) respectively. Since \( \alpha \) and \( \beta \) agree on \( H \), the map is well-defined.

Let two elements \( x \) and \( y \) of the amalgam have the same image under \( \theta \). If \( x, y \in A \), then \( x^\theta = y^\theta = y^\alpha \), and so \( x = y \). If \( x, y \in B \), then \( x^\beta = y^\theta = y^\beta \), and so \( x = y \). If \( x \in A \) and \( y \in B \), we get \( x^\alpha = x^\theta = y^\theta = y^\beta \). Since the intersection of \( A^\alpha \) and \( B^\beta \) is \( H^\alpha (= H^\beta) \) and since \( \alpha \) and \( \beta \) agree on \( H \), it then follows that there exists an element \( z \in H \) such that \( x^\alpha = z^\alpha \) and \( y^\beta = z^\beta \) — and so \( x = z = y \). Thus the map \( \theta \) is injective. //
Wiegold (1962, Theorem 1.4) has established the following result:

Let $Q$ be an amalgam of two groups $A$ and $B$ such that their intersection $H$ is a normal subgroup of $B$ and there exists a transversal of $H$ in $B$ consisting of elements $t_i$, $i \in B/H$. To every $i \in B/H$ let there exist an endomorphism $\alpha_i$ of $A$ such that

$$h \alpha_i = t_i h t_i^{-1}$$

for every $h$ in $H$, and let the intersection of the kernels of all the endomorphisms $\alpha_i$ be trivial.

Then the amalgam $Q$ can be embedded in the standard wreath product $A \wr (B/H)$.

We prove

5.2.2. Theorem. Let $Q$ be an amalgam of two groups $A$ and $B$ such that their intersection $H$ is a normal subgroup of $B$. Let there exist a transversal $T$ of $H$ in $B$ such that $T$ centralizes $H$ in $B$.

Then the amalgam $Q$ can be embedded in the standard wreath product $A \wr (B/H)$.
Proof. We have a natural monomorphism (see Corollary 2.1.7)

\[ \alpha^0 : A \to \text{AWr}(B/H), \quad a \mapsto \alpha^0, \]

where \( \alpha^0 \) is the diagonal function

\[ \alpha^0 : B/H \to A, \quad bH \mapsto a. \]

As in Theorem 5.1.1 and Corollary 5.1.4 we get a map

\[ \alpha^* : B/H \to T, \quad tH \mapsto t \quad (t \in T) \]

corresponding to the transversal \( T \) and a map

\[ f_b : B/H \to A, \quad tH \mapsto (t b^{-1} b^0) \rightarrow \alpha^* \]

for every \( b \in B \), leading to a monomorphism

\[ \beta : B \to \text{AWr}(B/H), \quad b \mapsto (bH)f_b. \]

Let \( b \in H \). Then \( b (\alpha^0) = b^0 \). And

\[ b^\beta = (bH)f_b = Hf_b \quad (\text{since } b \in H) \]

\[ = f_b \quad (\text{since } H \text{ is the unitelement of } B/H) \]

\[ = b^0 \quad (\text{by Lemma 5.1.5}). \]

Thus \( b (\alpha^0) = b^\beta \) for every \( b \in H \), i.e., \( \alpha^0 \) and \( \beta \) agree on \( H \).
Finally we have to show that the intersection $I$ of $A(a^0)$ and $B^\beta$ is just $H(a^0)$. Let $u$ be an arbitrary element of this intersection $I$. Then there exist $a \in A$ and $b \in B$ such that $u = a^0$ and $u = (bH)f_b$. We get
\[ bH = a^0(f_b)^{-1} \in A(B/H). \]
Hence $bH$ is contained in the intersection of $B/H$ and $A(B/H)$. Since this intersection is trivial, $bH = H$, or $b \in H$. It follows that $u = f_b = b^0 \in H(a^0)$, and so the intersection $I$ is contained in $H(a^0)$. The opposite inclusion relation being obvious this completes the proof. //

The following well known result is now immediate.

5.2.3. Corollary. If $Q$ is an amalgam of two groups $A$ and $B$ such that their intersection $H$ is central in $B$, then $Q$ can be embedded in the standard wreath product $AWr(B/H)$.

B. H. Neumann (1960°, p.204) has asked: 'If $A$ and $B$ are periodic, can the amalgam be embedded in a periodic group?' He adds, 'Nothing is known in the case when $H$ is finite or when $H$ is central only in $A$ or $B$.' We get the following result in this regard:

5.2.4. Corollary. Let $Q$ be an amalgam of a group $A$ of finite exponent and a periodic group $B$ such that the intersection $H$ of $A$ and $B$ is a normal subgroup of $B$. Let
there exist a transversal of $H$ in $B$ which centralizes $H$ in $B$. Then the amalgam $Q$ can be embedded in a periodic group.

Proof. We know from theorem 5.2.2 that the amalgam $Q$ can be embedded in the standard wreath product $A \wr (B/H)$.

Since $B$ is periodic, so also is the factor group $B/H$. For let $bH$ ($b \in B$) be an arbitrary element of $B/H$. Here $b$ has a finite order, say $m$. Then we get $(bH)^m = b^mH = H$, and so $bH$ has a finite order.

Since $A$ is of finite exponent, it follows from Theorem 4.1.3 that the standard wreath product $A \wr (B/H)$ is a periodic group.

B. H. Neumann (1960, Corollary 6.5) has proved that an amalgam of two groups $A$ and $B$ of finite exponent is embeddable in a group of finite exponent if the intersection $H$ is central in one of the constituents $A$ and $B$. We show:

5.2.5. Corollary. Let $Q$ be an amalgam of two groups $A$ and $B$ of finite exponent such that their intersection $H$ is a normal subgroup of $B$. Let there exist a transversal of $H$ in $B$ which centralizes $H$ in $B$.

Then the amalgam $Q$ can be embedded in a group of finite exponent.
Proof. Since $B$ is of finite exponent, so also is the factor group $B/H$. For let $B$ have exponent $n$ and let $bH$, where $b \in B$, be an arbitrary element of $B/H$. We then get $b^n = 1$, and so

$$(bH)^n = b^n H = H,$$

showing that $B/H$ is of finite exponent.

Since $A$ and $B/H$ are of finite exponent, it follows from Theorem 4.1.8 that the standard wreath product $A \wr (B/H)$ is also of finite exponent. Theorem 5.2.2 now completes the proof. //

Higman (1964) has shown that an amalgam of two finite $p$-groups $A$ and $B$ with an intersection $H$ is embeddable in a finite $p$-group if and only if there exist chief series $(A_i^)$ and $(B_i^)$ of $A$ and $B$, respectively, such that the two chief series of $H$ obtained on taking intersections with $(A_i^)$ and $(B_i^)$ are equal. Our corresponding result is

5.2.6. Corollary. Let $Q$ be an amalgam of two $p$-groups $A$ and $B$ such that $A$ is of finite exponent, the intersection $H$ of $A$ and $B$ is normal in $B$ and there exists a transversal of $H$ in $B$ which centralizes $H$ in $B$. Then $Q$ can be embedded in a $p$-group $R$. 
If, in particular, \(A\) and \(B\) are finite \(p\)-groups, the \(p\)-group \(R\) can also be chosen finite.

**Proof.** Since \(B\) is a \(p\)-group, so also is the factor group \(B/H\). For let \(bH\), where \(b \in B\), be an arbitrary element of \(B/H\).

Since \(B\) is a \(p\)-group, there is a positive integer \(n\) such that the order of \(b\) is \(p^n\). We then get

\[(bH)^{p^n} = b^{p^n}H = H,\]

whence \(B/H\) is a \(p\)-group.

Then by Theorem 4.1.9 the standard wreath product \(A \text{Wr}(B/H)\) is also a \(p\)-group. We use Theorem 5.2.2 and put \(R = A \text{Wr}(B/H)\) and the proof is complete. //

B. H. Neumann (1960, p. 300 and Theorem 4) has shown that an arbitrary amalgam of two solvable groups \(A\) and \(B\) may not be embeddable in a solvable group, but if \(A\) and \(B\) are solvable groups of lengths \(l\) and \(m\), respectively, and the intersection of \(A\) and \(B\) is central in either \(A\) or \(B\) then the amalgam can be embedded in a solvable group of length \(\leq l + m - 1\). We prove

**5.2.7. Corollary.** Let \(Q\) be an amalgam of two solvable groups \(A\) and \(B\) of lengths \(l\) and \(m\), respectively, such
that their intersection $H$ is normal in $B$ and there exists a transversal of $H$ in $B$ which centralizes $H$ in $B$. Then $Q$ can be embedded in a solvable group $R$ of length $\leq 1 + m$.

If, in particular, $A$ and $B$ are finite solvable groups, the solvable group $R$ can be chosen finite.

**Proof.** Since $B$ is a solvable group of length $m$ we know that $B/H$ is a solvable group of length $\leq m$ (Schenkman, 1965, VII.1.a(x)). Then by Lemma 4.2.1 the standard wreath product $\text{Awr}(B/H)$ is a solvable group of length $\leq 1 + m$.

To complete the proof we use Theorem 5.2.2 and put $R = \text{Awr}(B/H)$.

It is known that an arbitrary amalgam of two nilpotent groups may not be embeddable in a nilpotent group (see, e.g., Wiegold, 1959, p.152). We get

**5.2.8. Corollary.** Let $Q$ be an amalgam of a nilpotent $p$-group $A$ of class $c$ and finite exponent $p^s$ and a finite $p$-group $B$ of class $k$ and order $\delta$ such that their intersection $H$ is normal in $B$ and there exists a transversal of $H$ in $B$ which centralizes $H$ in $B$. Then $Q$ can be embedded in a nilpotent $p$-group $R$ of class

$$\leq c s (\delta^* - 1) + \max(c, k), \text{ where } \delta^* = \frac{\delta}{|H|}.$$
If \( A \) is finite, \( R \) can be chosen finite.

**Proof.** Since \( B \) is a finite \( p \)-group, we see as in the proof of Corollary 5.2.6 that \( B/H \) is a finite \( p \)-group. The class of \( B/H \) is \( \leq k \) (Schenkman, 1965, VI.1.e), and its order is \( \delta^* \). Then by Theorem 4.3.7 the standard wreath product \( A \text{Wr}(B/H) \) is a nilpotent group of class \( \leq c_s (\delta^*-1) + \max(c,k) \).

Finally we use Theorem 5.2.2 and put \( R = A \text{Wr}(B/H) \).

Analogously we get on using Theorem 4.3.7 and Theorem 5.2.2:

**5.2.9. Corollary.** Let \( Q \) be an amalgam of a hypercentral \( p \)-group \( A \) of length \( \omega \) and a finite \( p \)-group \( B \) of class \( k \) such that their intersection is \( H \) is normal in \( B \) and there exists a transversal of \( H \) in \( B \) which centralizes \( H \) in \( B \). Then \( Q \) can be embedded in a hypercentral \( p \)-group of length \( \leq \omega \chi + k \), where \( \omega \) is the first infinite ordinal.

**5.3. Embedding of a permutational product in a wreath product**

Suppose we have an amalgam of two groups \( A \) and \( B \) with an intersection \( H \). Let \( S \) and \( T \) be left transversals of \( H \) in \( A \) and \( B \) respectively.
Let the triple \((s, t, h)\) be an arbitrary element of the cartesian product \(S \times T \times H\). Then \(s \in S\), \(t \in T\) and \(h \in H\). We note that for any element \(a \in A\) the product \(s \cdot h \cdot a\) is an element of \(A\). Hence there exist unique elements \(s' \in S\) and \(h' \in H\) such that \(s \cdot h \cdot a = s' \cdot h'\).

Thus for every \(a \in A\) we get a map

\[
\tilde{a} : S \times T \times H \rightarrow S \times T \times H,
\]

\[
(s, t, h) \rightarrow (s', t, h').
\]

The following results (i), (ii), (iii) are new easily obtained (see B. H. Neumann, 1960, pp.181-186):

(i) The map \(\tilde{a} : (s, t, h) \rightarrow (s', t, h')\) is a permutation on the set \(S \times T \times H\).

(ii) For every \(b \in B\) there is a permutation

\[
\tilde{b} : (s, t, h) \rightarrow (s, t'', h''), t'' \in T, h'' \in H\]

on the set \(S \times T \times H\) such that \(t \cdot h \cdot b = t'' \cdot h''\).

(iii) The subsets \(\tilde{A}\) and \(\tilde{B}\) of the symmetric group on the set \(S \times T \times H\), formed by the permutations \(\tilde{a}\) and \(\tilde{b}\) \((a \in A, b \in B)\), respectively, are subgroups of the symmetric group, and there are isomorphisms.
\[ \alpha' : \bar{A} \to A, \bar{a} \to a \]
and
\[ \beta' : \bar{B} \to B, \bar{b} \to b \]
such that \( \alpha' \) and \( \beta' \) agree on the intersection \( \bar{H} \) of \( \bar{A} \) and \( \bar{B} \) and \( H = ( \bar{H} )^{\alpha'} = ( \bar{H} )^{\beta'} \).

Let \( P \) be the subgroup of the symmetric group on \( S \times T \times H \) generated by \( \bar{A} \) and \( \bar{B} \). This subgroup \( P \) is the permutational product of the given amalgam of \( A \) and \( B \) corresponding to the transversals \( S \) and \( T \) (B. H. Neumann, 1960\textsuperscript{a}).

We get the following embedding theorem.

5.3.1. Theorem. Let \( P \) be the permutational product of an amalgam of two groups \( A \) and \( B \) whose intersection \( H \) is normal in \( B \), and let there exist a transversal of \( H \) in \( B \) centralizing \( H \) in \( B \).

Then \( P \) can be embedded in the standard wreath product \( A \text{Wr}(B/H) \).

Proof. We know from the foregoing that there exist isomorphisms
\[ \alpha' : \bar{A} \to A \text{ and } \beta' : \bar{B} \to B \]
such that \( \alpha' \) and \( \beta' \) agree on the intersection \( \bar{H} \) of \( \bar{A} \) and \( \bar{B} \) and \( H = ( \bar{H} )^{\alpha'} \).
And we know from Theorem 5.2.2 that there exist monomorphisms

\[ \alpha^0 : A \rightarrow \text{AWr}(B/H), \quad \beta : B \rightarrow \text{AWr}(B/H) \]

such that \( \alpha^0 \) and \( \beta \) agree on the intersection \( H \) of \( A \) and \( B \) and the intersection of \( A(\alpha^0) \) and \( B^\beta \) is \( H(\alpha^0) \).

Put \( \alpha^* = \alpha^0\alpha^0 \) and \( \beta^* = \beta^\beta \).

Then \( \alpha^* \) and \( \beta^* \) are monomorphisms

\[ \tilde{A} \rightarrow \text{AWr}(B/H) \quad \text{and} \quad \tilde{B} \rightarrow \text{AWr}(B/H), \]

respectively. Clearly \( \alpha^* \) and \( \beta^* \) agree on the intersection \( \tilde{H} \) of \( \tilde{A} \) and \( \tilde{B} \), and the intersection of \( (\tilde{A})^{\alpha^*} \) and \( (\tilde{B})^{\beta^*} \) is \( (\tilde{H})^{\alpha^*} \).

It follows from Lemma 5.2.1 that the amalgam of \( \tilde{A} \) and \( \tilde{B} \) is embedded in \( \text{AWr}(B/H) \). Hence the permutational product \( P \) is also embedded in \( \text{AWr}(B/H) \). //

5.4. Equivalent embeddings

Let \( P, Q, R, S \) be groups such that there are embeddings (i.e., monomorphisms) \( \alpha : P \rightarrow Q \) and \( \beta : R \rightarrow S \). If now there exist isomorphisms \( \alpha : P \rightarrow R \) and \( \theta : Q \rightarrow S \).
such that $\alpha \theta = \pi \beta$, i.e., such that the diagram

\[
P \xrightarrow{\alpha} Q
\]

\[
\pi \downarrow \theta
\]

\[
R \xrightarrow{\beta} S
\]

commutes, then $\alpha$ and $\beta$ are equivalent embeddings or embeddings of the same type (P. Hall, 1959, p. 315). The concept has been used by Burns (1968) for constructing certain special groups.

The following result gives us some examples of equivalent embeddings.

5.4.1. Lemma. For every automorphism of the group $A$ there is (1) an embedding $\theta^0 : A \to A \wr B$ equivalent to the diagonal embedding $\alpha^0 : A \to A \wr B$ (see Corollary 2.1.7), and (2) an embedding $\theta_x : A \to A \wr B$ equivalent to the coordinate embedding $\alpha_x : A \to A \wr B$ (Corollary 2.1.7).

Proof. (1) Let $\alpha$ be an automorphism of $A$. It yields a map $\theta^0 : A \to A \wr B$, $a \to (a^\alpha)^0$, where $(a^\alpha)^0$ is the diagonal function $y \to a^\alpha$, $y \in Y$.

For any $a, u \in A$, we get
and so $\theta^0$ is a homomorphism. Again

\[
(a^\alpha)^0 = 1 = a^\alpha = 1 = a = 1,
\]

showing that $\theta^0$ is a monomorphism.

The automorphism $\alpha$ of $A$ also yields an automorphism $\alpha^*$ of the wreath product $A \wr B$ as in Theorem 3.3.1.

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha^0} & A \wr B \\
1 & \mapsto & \alpha^* \\
A & \xrightarrow{\theta^0} & A \wr B
\end{array}
\]

Let $a \in A$. Then

\[
(a^0)^* = a^0 \alpha \quad \text{(by 3.2.4)}
\]

\[
= (a^\alpha)^0,
\]

since an arbitrary element $y$ of $Y$ has the same image under both $a^0 \alpha$ and $(a^\alpha)^0$.

Then for every $a \in A$,

\[
a(\alpha^0 a^*) = (a^0)^* = (a^\alpha)^0
\]

\[
= a \theta^0 = a (1 \theta^0),
\]
where $1$ is the identity map $A \to A$. It follows that $\alpha^0 \alpha^* = 1 \theta^0$, i.e., the diagram given in the last page commutes.

(2) The automorphism $\alpha$ also yields a map

$\theta_x : A \to A \wr B$, $a \to (a^x)^x$, where $(a^x)^x$ is the coordinate function under which $x \in Y$ has the image $a^x$ and all other elements of $Y$ have trivial images.

It is immediately seen (as in the case of $\theta^0$) that the map $\theta_x$ is actually a monomorphism. And the automorphism $\alpha$ yields an automorphism $\alpha'$ of the restricted wreath product $A \wr B$ as in Theorem 3.3.1.

Let $a \in A$. Then we get

$$(a^x)^{\alpha'} = a^x \alpha \quad \text{(by 3.2.4)}$$

$$= (a^x)^x,$$

since an arbitrary element $y$ of $Y$ has the same image under both $a^x \alpha$ and $(a^x)^x$.

Again, by definition of $a^x$ and $\theta_x$, we have

$$(a^x)^{\alpha'} = a^x \alpha' \quad \text{and} \quad (a^x)^x = a^x \theta_x.$$
Hence it follows that, as required,

\[ \alpha_x \alpha' = 1 \theta_x, \]

where 1 is the identity map \( A \to A \), i.e., the diagram

\[ \begin{array}{ccc}
A & \overset{\alpha_x}{\longrightarrow} & A \\
\downarrow{\theta_x} & & \downarrow{\alpha'} \\
A & \longrightarrow & A \\
\end{array} \]

commutes. //