Chapter - 2

Measure of Associativity for an Inexact Groupoid of Degree one.

2.1. Introduction

The set of transformations from a singleton set into an ordered subset of unit interval is closed with respect to the operation, absolute difference and termed as inexact groupoid of degree one and is denoted by E but it is not associative. The main purpose of this chapter is to introduce the measure of associativity for E. This gives a measure by which E deprives of being a group. Material incorporated in this chapter is published in the Mathematical Gazette, volume 80, Number 488, July, 1996.

2.1.1 An overview on measure of commutativity -

We know that the measure of commutativity for a group is defined by Joseph as a ratio between the number of commutating ordered pairs and the square of the order of the group. This ratio is a positive fraction lying between 0 and 1. If this ratio is equal to one, then the group is commutative and otherwise it is non-commutative.

On the otherhand, there is another method introduced by Machale for the determination of Joseph measure of a finite group. He proved that the Joseph measure of a finite group depends upon the number of conjugacy classes of the finite group. Therefore to find the measure of commutativity for a non-commutative finite group, it will be sufficient to determine the number of conjugate classes of the group. For a finite group G, if K(G) be the number of conjugate classes of G, then Joseph measure of commutativity for G is the ratio between K(G) and order of G. If H be a subgroup of a finite group G, then J(G) ≤ J(H), where J(G) and J(H) denote the Joseph measure of commutativity for G and H respectively.
If $G$ be a non - commutative finite group and $p$ be the least prime number and $p$ divides the order of $G$, then $J(G) \leq \frac{(p^2 + p - 1)}{p^3}$. If $Z(G)$ be the centre of $G$, and the order of $G/Z(G)$ is equal to $p^2$, then $J(G) = \frac{(p^2 + p - 1)}{p^3}$.

One of the most interesting results of Joseph measure for a non - commutative finite group $G$ is $J(G) \leq 5/8$.

In conformity with the Joseph measure of commutativity for a finite group, we introduce the notion of measure of associativity for an inexact groupoid $E$ of degree one. The measure of associativity for $E$ depends upon the number of associative triples $(a, b, c)$ of the elements of $E$, where the triple $(a, b, c)$ satisfying $(a \Delta b) \Delta c = a \Delta (b \Delta c)$ is called associative. The measure of associativity for an inexact groupoid $E$ of degree one is the ratio of the number of associative triples in $E$ and the cube of the order of $E$. If this ratio is equal to one then the inexact groupoid $E$ is associative and otherwise it is non - associative.

2.2 Ordered subset of unit interval.

As a partition of a closed interval $[0, 1]$, we consider a finite ordered subset $A = \{0, h, 2h, \ldots, (m - 1)h = 1\}$, where $m$ is a positive integer.

Clearly $A$ is closed under the operations, absolute difference, maximum and minimum but it is not closed with respect to addition, subtraction and multiplication operations.

2.3. Inexact sets

2.3.1. Definition:

Let $X = \{x_0\}$ be an arbitrary singleton set and $A = \{0, h, 2h, \ldots, (m - 1)h = 1\}$ be an ordered sub-set of $[0, 1]$. Then the family of fuzzy subset $A^t$ denoted by $E$ is defined as inexact set. The elements of $E$ can be determined by the following mappings.

The co-efficient set of $h$ in $A$ is denoted by $A'$. So, $A' = \{0, 1, 2, \ldots, m - 1\}$. 
Taking \( m \) as base and elements of \( A' \) as digits, another set \( P = \{0, 1, 2, \ldots, m - 1\} \) is determined and the elements of \( P \) can be expressed as \( m \)-adic form in one place as given below:

\[
0 = (0)_{m}, 1 = (1)_{m}, 2 = (2)_{m}, \ldots, m - 1 = (m - 1)_{m}
\]

Consider the mappings \( f : A \rightarrow P \) and \( g : P \rightarrow E \) such that \( f(rh) = r \) and \( g(r) = \{(x_0, rh)\} \) for all \( rh \in A \).

Since \( f \) and \( g \) are one-one and onto mappings, so \( gof \) is also one-one and onto mapping. The composite mapping

\[
gofof (rh) = g (f (rh))
\]

\[
= g (r)
\]

\[
= \{(x_0, rh)\}
\]

The element \( \{(x_0, rh)\} \) i.e the fuzzy subset is denoted by \( U_r \).

Thus \( U_r = \{(x_0, rh)\} = \{(x_0, u_r (x_0))\} \). The image points of the elements of \( A \) are the elements of \( E \). The image points of the elements \( 0, 1h, 2h, \ldots, (m - 1)h \) are \( u_0, u_1, u_2, \ldots, u_{m-1} \) respectively.

Thus \( E = \{ u_0, u_1, u_2, \ldots, u_{m-1} \} \).

2.3.2. Definition:

Let \( U_p = \{(x_0, ph)\} \) and \( U_q = \{(x_0, qh)\} \) be two elements of inexact set \( E \), then \( U_p \) is said to be less than or equal to \( U_q \) denoted by \( U_p \leq U_q \) if \( p \leq q \) i.e. \( (p)_{m} \leq (q)_{m} \).

The elements of \( E \) can be written as \( u_0 < u_1 < u_2 < \ldots < u_{m-1} \). The elements \( u_0 = \{(x_0, 0)\} \) and \( u_{m-1} = \{(x_0, (m - 1)h = 1)\} \) are respectively infimum and supremum of \( E \).

2.3.3. Definition:

The degree of an inexact set is defined as the number of elements of the set \( X \). The degree of the inexact set \( E \) is denoted by \( d(E) \).
2.3.4 Definition:
The absolute difference denoted by $\Delta$ of two elements $u_r$ and $u_s$ of $E$ is defined as

$$u_r \Delta u_s = \{(x_0, |u_r(x_0) - u_s(x_0)|)\},$$

where $u_r = \{(x_0, rh)\}$ and $u_s = \{(x_0, sh)\}$.

2.4 Inexact groupoid of degree one.

2.4.1 Definition:
If $u_r \Delta u_s \in E$ for all $u_r \in E$, $u_s \in E$, then $(E, \Delta)$ is defined as inexact groupoid of degree one.

For brevity, we shall omit $u$, thus $u_r$ and $u_s$ become $r$ and $s$ respectively. Then $E = \{0, 1, 2, \ldots, m-1\}$

Let $r = \{(x_0, r(x_0))\}$ and $s = \{(x_0, s(x_0))\}$ be any two arbitrary elements of $E$, then

$$r \Delta s = \{(x_0, |r(x_0) - s(x_0)|)\} \in E,$$

since $r(x_0) \in A$, $s(x_0) \in A \Rightarrow |r(x_0) - s(x_0)| \in A$ by 2.2.

Thus $r \in E, s \in E \Rightarrow r \Delta s \in E$. Hence $E$ is an inexact groupoid of degree one and it is denoted by $(E, \Delta)$.

A composite table of inexact groupoid $(E, \Delta)$ is as follows:

<table>
<thead>
<tr>
<th>$\Delta$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>$\ldots$</th>
<th>m - 4</th>
<th>m - 3</th>
<th>m - 2</th>
<th>m - 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>$\ldots$</td>
<td>m - 4</td>
<td>m - 3</td>
<td>m - 2</td>
<td>m - 1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>$\ldots$</td>
<td>m - 5</td>
<td>m - 4</td>
<td>m - 2</td>
<td>m - 1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$\ldots$</td>
<td>m - 6</td>
<td>m - 5</td>
<td>m - 3</td>
<td>m - 2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>$\ldots$</td>
<td>m - 7</td>
<td>m - 5</td>
<td>m - 3</td>
<td>m - 2</td>
</tr>
</tbody>
</table>

$\ldots$

| m - 4 | m - 5 | m - 6 | m - 7 | $\ldots$ | 0    | 1    | 2    | 3    |
| m - 3 | m - 4 | m - 5 | m - 6 | $\ldots$ | 1    | 0    | 1    | 2    |
| m - 2 | m - 3 | m - 4 | m - 5 | $\ldots$ | 2    | 1    | 0    | 1    |
| m - 1 | m - 2 | m - 3 | m - 4 | $\ldots$ | 3    | 2    | 1    | 0    |

Table - 1
From the above table it is clear that

i) \((E, \Delta)\) is commutative i.e. \(r \Delta s = s \Delta r\), for all \(r, s \in E\).

ii) \((E, \Delta)\) is not associative i.e. for some \(r, s, t \in E\), \((r \Delta s) \Delta t \neq r \Delta (s \Delta t)\).

2.5. Measuring Associativity in a Groupoid of Natural Numbers

The absolute difference \(*\) of natural number \(a, b\) is defined by
\[ a * b = |a - b|. \]
Then (for all integers \(n \geq 1\)) \(*\) is a closed law of composition on the set \(N_n = \{0, 1, 2, \ldots, n - 1\}\) and \((N_n, *)\) is a commutative but (for \(n > 2\)) not associative: e.g. \((1 * 1) * 2 = 0 * 2 = 2\) while \(1 * (1 * 2) = 1 * 1 = 0\). So \((N_n, *)\) is a commutative non-associative groupoid. It is interesting [see(1)] to consider the measure of its associativity, defined to be \(K(N_n) / n^3\), where \(K(N_n)\) is the number of associative triples among the total of \(n^3\) that can be formed from elements of \(N_n\). Clearly \(0 < K(N_n) / n^3 < 1\) when \(n > 2\).

For illustration we first consider \((N_3, *)\) whose composition table is Table 2. From this we can construct Table 3 giving \((i * j) * k\) for all \(i, j, k\) in \(N_3\).

When \((i * j) * k \neq i * (j * k)\) we bracket the entry for \((i * j) * k\). Because the commutativity of \(*\) gives \(i * (j * k) = (k * j) * i\), which can read off from Table 3 together with \((i * j) * k\), the comparison is easily made. We see that 25 of the 27 triples are associative, so \(K(N_3) / 8 = 25 / 27\) is the associativity measure for \((N_3, *)\).

Next we look at \((N_4, *)\). The reader may construct the composition table and then check Table 4 for \((i * j) * k\). We find 54 associative triples among the 64, so the measure is \(54/64 = 27/32\).

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2
Let us now consider the number \( l(N_n) \) of non-associative triples of elements from \( N_n \).

From Tables 3, 4 we have seen that \( l(N_3) = 2 \) and \( l(N_4) = 10 \) - an increase of 8; i.e. \( l(N_4) - l(N_3) = 8 \). Thus \( l(N_4) = 2 + 8 = 2(1^2 + 2^2) \). Likewise, the reader may check that from \( N_4 \) to \( N_5 \) there is an increase of 18; i.e. \( l(N_5) - l(N_4) = 18 \), so \( l(N_5) = 2(1^2 + 2^2 + 3^2) \). By considering systematically the triples introduced when we pass from \( N_n \) to \( N_{n+1} \) we will prove similarly the following generalisation of these results

\[
 l(N_{n+1}) - l(N_n) = 2(n - 1)^2 \quad (n > 1) \tag{1}
\]

Recall that \( N_{n+1} = \{0, 1, 2, \ldots, n\} \). For brevity we shall omit *; thus \((i * j) * k, i * (j * k)\) become \((ij)k, i(jk)\) respectively.
By commutativity of * we have \((ij)k = (ji)k = k (ji)\), and similarly (as noted earlier) \((jk) = (kj)\), so \((ij)k \neq i(jk)\) if and only if \((kj)i \neq k (ji)\). Thus \((i, j, k)\) is non-associative if and only if its "reverse", \((k, j, i)\) is non-associative so we need consider only those \((i, j, k)\) with \(i \leq k\). We can exclude immediately the case \(k = i\) because (from above) \((ij)i = i(ji)\); and then also the case \(k = j\) because \(i < k = j\) gives \((ij)i = (j - i)j = j - (j - i) = i = i0 = i(ij)\). If at least one of \(i, j, k\) is 0 then \((i, j, k)\) is associative: for example, \((0)j = jk = 0(jk)\). So henceforth we can confine our search for non-associative triples to those with \(i, j, k\) non-zero and \(i < k, j \neq k\).

Passing from \(N_n\) to \(N_{n+1}\) (\(n > 1\)) introduces new triples, of which (by the preceding paragraph) only those of types \((i, j, k)\) with \(i, j, k\) non-zero in \(N_{n+1}\) need consideration. Moreover \(i \neq n\), and \(j = n\) or \(k = n\), but not both. We have

\[
(i)n = \begin{cases} 
(j - i)n & \text{if } j > i \\
(i - j)n & \text{if } j \leq i
\end{cases} = \begin{cases} 
\begin{align*}
n - (j - i) & = n \pm (i - j), \\
n - (i - j) &
\end{align*}
\end{cases}
\]

while

\[
i(n) = \begin{cases} 
(n - j)^-i & \text{if } n - j > i \\
(n - j)^-(n - j) & \text{if } n - j \leq i
\end{cases} = n \pm (i - j).
\]

So all these \((i, j, n)\) are non-associative. There are \((n - 1) \times (n - 1) = (n - 1)^2\) of them. Similarly,

\[
(n)k = \begin{cases} 
(n - i) - k & \text{if } n - i > k \\
k - (n - i) & \text{if } n - i \leq k
\end{cases} = \begin{cases} 
\begin{align*}
n - i - k & = i + k - n, \\
i + k - n &
\end{align*}
\end{cases}
\]

while,

\[
i(nk) = \begin{cases} 
(n - k)^-i & \text{if } n - k > i \\
(n - k)^-(n - k) & \text{if } n - k \leq i
\end{cases} = \begin{cases} 
\begin{align*}
n - i - k & \text{if } n - i > k \\
i + k - n & \text{if } n - i \leq k
\end{align*}
\end{cases}
\]

Thus all these \((i, n, k)\) are associative. Consequently the increase in the number of non-associative triples when we pass from \(N_n\) to \(N_{n+1}\) is (allowing for the reverse of those just determined)

\[
2 (n - 1)^2; \text{ ie. (since } I(N_2) = 0\text{) we have proved (1).}
\]
It follows from (1) that,
\[ l(N_n) = 2 \sum_{r=1}^{n-2} r^2 = 1/3 (n - 2) (n - 1) (2n - 3) \quad (n > 2). \]

(This also holds trivially for \( n = 1, 2 \).) The number \( K(N_n) \) of associative triples is thus
\[ n^3 - l(N_n) = 1/3 (n^3 + 9 n^2 - 13n + 6), \]
so the associativity measure of our groupoid \((N_n, \ast)\) is (for \( n > 2 \))
\[ K(N_n)/n^3 = 1/3 (1 + 9/n - 13/n^2 + 6/n^3) = f(n) \quad \text{say}. \]

Because \( f(n) - f(n + 1) = \ldots = (n - 1) \left( 9n^3 + n^2 - 11n - 6 \right)/n^3 \left( n + 1 \right)^3 > 0 \) for \( n > 2 \), the sequence \( f \) is strictly decreasing and converges to \( 1/3 \) when \( n \to \infty \).

Hence \( 1/3 < K(N_n)/n^3 < 1 \quad (n > 2) \)

In other words, more than one-third of the triples for \( N_n \) are associative. Our groupoid, with neutral element \( 0 \) and each element self-inverse, narrowly misses being an Abelian group.

2.6. Measure of associativity for inexact groupoid of degree one.

The measure of associativity for \((E, \Delta)\) has been discussed here.

2.6.1. Definition:

A triple \((a, b, c)\) of the elements of \( E \) is said to be associative if
\[ (a \Delta b) \Delta c = a \Delta (b \Delta c). \]
2.6.2 Definition.

The measure of associativity for inexact groupoid \((E, \Delta)\) may be defined to be \(N(E)/m^3\), where \(N(E)\) is the number of associative triples, among the total of \(m^3\) that can be formed from elements of \(E\). Clearly \(0 < \frac{N(E)}{m^3} < 1\), where \(m > 2\).

A chart of some inexact groupoid of degree one is given below:

<table>
<thead>
<tr>
<th>SI No.</th>
<th>Singleton set ({x_0})</th>
<th>Ordered subset of ([0, 1])</th>
<th>Inexact groupoids (E)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>({x_0})</td>
<td>({0, h = 1})</td>
<td>({0, 1}) = (E_2)</td>
</tr>
<tr>
<td>2</td>
<td>({x_0})</td>
<td>({0, h, 2h = 1})</td>
<td>({0, 1, 2}) = (E_3)</td>
</tr>
<tr>
<td>3</td>
<td>({x_0})</td>
<td>({0, h, 2h, 3h = 1})</td>
<td>({0, 1, 2, 3}) = (E_4)</td>
</tr>
<tr>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
</tr>
<tr>
<td>(m - 1)</td>
<td>({x_0})</td>
<td>({0, h, 2h, \ldots, (m - 1)h = 1})</td>
<td>({0, 1, 2, \ldots, m - 1}) = (E_{m-1})</td>
</tr>
<tr>
<td>(m)</td>
<td>({x_0})</td>
<td>({a, h, 2h, \ldots, mh = 1})</td>
<td>({0, 1, 2, \ldots, m}) = (E_m)</td>
</tr>
</tbody>
</table>

Chart - 1

2.6.3. Proposition:

The associative measure of inexact groupoid \((E_m, \Delta)\), (for \(m > 2\)) is \(N(E_m)/m^3 = 1/3 (1 + 9/m - 13/m^2 + 6/m^3)\), where \(N(E_m)\) is the number of associative triples among the total of \(m^3\). This can be established as given by 2.5.