Chapter - 1

Essentials

In this chapter, we have reproduced some known definitions and results as ready references for easy understanding of the material incorporated in the subsequent chapters. For details we refer to [ ], [ ] etc.

1 1. Special Divisibility.

1.1.1 Congruence [ 47 ]

Let \( n \) be a fixed positive integer. Two integers \( a \) and \( b \) are said to be congruent modulo \( n \), symbolized by

\[
a = b \pmod{n}
\]

if \( n \) divides the difference \( a - b \) that is \( a - b = kn \) for some integer \( k \).

1.1.2 \( p \)-adic number system [ 47 ]

Let \( p \) be a positive integer greater than one. Then every positive integer \( N \) can be written uniquely in the form

\[
N = r_{m-1}p^{m-1} + r_{m-2}p^{m-2} + \ldots + r_1p + r_0
\]

where \( m \geq 0 \), \( r_i \) is an integer, \( 0 \leq r_i < p \), \( r_{m-1} \neq 0 \), \( p \) is called the base of \( N \), which is denoted by \( ( r_{m-1} r_{m-2} \ldots r_1 r_0 )_p \) and known as the representation of \( N \) and the resulting system of enumeration is called the \( p \)-adic number system.
1.2 The Joseph measure of commutativity for a group.


1.2.1 Joseph Measure [12]

The Joseph measure for a finite group $G$ is denoted by $J(G)$ and defined as

$$J(G) = \frac{\text{number of commutative order pairs}}{|G|^2}$$

where $|G|$ denotes the order of the group $G$. The group $G$ is commutative if $J(G) = 1$ and non commutative if $J(G) < 1$.

1.2.2 Theorem [12]

If $K(G)$ be the number of conjugate classes of a finite group $G$, then

$$J(G) = \frac{K(G)}{|G|}$$

1.2.3 Klein 4-group [63]

The alternating group $A_4$, contained in the symmetric group $S_4$, whose elements are the permutations of $X = \{a, b, c, d\}$, has a normal sub-group $V = \{I, (ab)(bc), (ac)(bd), (ad)(bc)\}$

This sub-group is called Klein 4-group after the name of German mathematician Felix Klein.
1.3 Wreath product

The wreath product of a finite group by a cycle of order two was constructed in a short note by Cayley (1878). The concept of the wreath product of groups came into view through a number of published research papers. Dr B.K. Tamuli introduced the wreath product as a holomorph of a group. A number of results concerning some important subgroups of the wreath product are obtained alongwith some properties of homomorphisms and automorphisms of wreath products in his thesis [78], submitted to Guwahati University for the Degree of Doctor of Philosophy.

1.3.1 Wreath Product [18]

Let $G$ and $H$ be permutation groups on sets $A$ and $B$ respectively. The wreath product of $G$ by $H$ written $G \wr H$ is defined in the following way:

$G \wr H$ is the group of all permutations $\theta$ on $A \times B$ of the following kind:

$$(a,b)\theta = (a \gamma_b, b \eta),$$

where for each $b \in B$, $\gamma$ is a permutation of $G$ on $A$, but for different $b$'s the choices of permutations $\gamma_b$ are independent. The permutation $\eta$ is a permutation of $H$ on $B$. The permutation $\theta$ with $\eta = 1$ form a normal subgroup $G^*$ isomorphic to the direct product of $n$ copies of $G$, where $n$ is the number of letters in the set $B$. The factor group $G \wr H / G^*$ is isomorphic to $H$ and the permutations $\theta$ with all $\gamma_b = 1$ form a subgroup isomorphic to $H$, whose elements may be taken as co-set representation of $G^*$ in $G$.

1.4. Boolean rings

1.4.1 Definition [54]

An element $a$ of a ring $R$ is said to be idempotent if $a^2 = a$.

A ring $R$ in which every element is an idempotent is called a Boolean ring.
1.4.2 Proposition [45]

Every Boolean ring is isomorphic to a ring of subsets of some set.

1.5 Fuzzy Subsets:

In the real world elements are perturbed by imperfection. This is the central idea of Platonic Philosophy. If we look for a perfectly round element, then we see that there exists no element that is perfectly round. "perfect notations" or "exact concepts" correspond to the sort of things discussed in pure Mathematics, while "inexact structures" prevail in our every day lives. Perhaps the environment is full of inexact structures. A mathematical formulation within which these various types of inexact structures can be properly characterized and investigated is available in terms of the theory of Fuzzy Sets introduced by Zadeh (1965).

Goguen (1967) explained the representation of inexact concepts by Fuzzy Sets in a theorem. This theorem states that any system satisfying certain axioms is equivalent to a system of Fuzzy Sets. Since the axioms are apparently pleasing by immediate inspection for the system of all inexact concepts, the theorem allows to conclude that inexact concepts can be represented by Fuzzy Sets. Some of the concepts connected with ambiguity are nonspecificity - one to many relation, variety, generality, diversity and divergence and some of the concepts connected with vagueness are fuzziness, hainess, cloudiness, unclearness, indistinctiveness and sharplessness. If we describe ambiguity, vagueness and ambivalence in an inexact structure then fuzzy sets arise as a generalisation of characteristic function.

1.5.1 Definition [70]

Let $S$ be a non-empty set. Then the mapping $u : S \rightarrow [0,1]$ is said to be a fuzzy subset of $S$. We write $u=\{(x_i, u(x_i))\}$, where $x_i \in S$. 
15.2 Set theoretic operations [25]

Let \( u = \{(x, u(x))\} \) and \( v = \{(x, v(x))\} \) be two fuzzy sub sets.

Then

i) \( u \) and \( v \) are said to be equal and written as \( u = v \) if \( u(x) = v(x) \).

ii) \( u \) is contained in \( v \) written as \( u \subseteq v \) if and only if \( u(x) \leq v(x) \).

iii) The union of \( u \) and \( v \) is denoted by \( u \cup v \) and defined as

\[
  u \cup v = \{(x, \max(u(x), v(x)))\}
\]

iv) The intersection of \( u \) and \( v \) is denoted by \( u \cap v \) and defined as

\[
  u \cap v = \{(x, \min(u(x), v(x)))\}
\]

v) The complement of \( u \) is denoted by \( u^c \) and defined as

\[
  u^c = \{(x, 1 - u(x))\}
\]

1.6 Fuzzy subgroups

The concept of a fuzzy subgroup with respect to the t-norm minimum was introduced by A. Rosenfeld (1971). Anthony and Sherwood redefined a fuzzy subgroup with respect to general t-norm. Subsequently Sherwood, Abuosman, P. Das studied fuzzy subgroup with respect to general t-norm and showed most of the results obtained by Rosenfeld.

1.6.1 Fuzzy subgroups [62]

Let \( G \) be a multiplicative group.

A fuzzy subset \( u \) of \( G \) is said to be a fuzzy subgroup of \( G \) if

i) \( u(x, y) \geq \min(u(x), u(y)) \) for all \( x, y \in G \)

ii) \( u(x) = u(x^{-1}) \) for all \( x \in G \)
1.6.2. Supremum property [62].

A fuzzy subset $u$ of $G$ is said to have the supremum property if there exists $x_0$ in $G$ such that $u(x_0) = \sup \{ u(x) : x \in G \}$

1.6.3 Proposition [62]

$u$ is a fuzzy subgroup of $G$ if and only if $u(xy^{-1}) \geq \min( u(x), u(y) )$ for all $x, y, \in G$

1.6.4 Definition [60]

Let $u$ be a fuzzy subgroup of $G$ then $u(e)$ is called the tip of $u$, where $e$ is the identity of $G$.

1.6.5 Definition [62]

Let $f : X \rightarrow Y$ be a function from a set $X$ into a set $Y$. If $u$ and $v$ are fuzzy subsets of $X$ and $Y$ respectively,

then fuzzy sub-sets $f^{-1}(v)$ and $f(u)$ are defined as follows:

a) $f^{-1}(u)(x) = v(f(x))$ for every $x \in X$

b) $f(u)(y) = \begin{cases} \sup u(z), f(z) = y, z \in X \\ 0 \text{ if there is no } z. \end{cases}$
1.6.6 Proposition [62]

A homomorphic image or pre-image of a fuzzy sub-group is a fuzzy sub-group provided sup. property holds.

1.6.7 Definition [23]

Let $f$ be a homomorphism (isomorphism) from the group $G$ onto the group $G'$. $A$ and $A'$ are two fuzzy subgroups of $G$ and $G'$ respectively. If $A' = f(A)$, then $f$ is said to be a homomorphism (isomorphism) from $A$ onto $A'$.

1.6.8 Definition [37]

Let $g$ be any function from a set $S$ to a set $S'$. A fuzzy sub-set $\alpha$ of $S$ is called $g$-invariant if $g(x) = g(y) \Rightarrow \alpha(x) = \alpha(y)$ where $x, y \in S$.

1.6.9 Definition [42]

Let $A$ and $B$ be fuzzy subsets of the non-empty sets $G$, $G'$. The direct product $A \times B$ is the fuzzy subset of $G \times G'$ defined by

$$A \times B(x, y) = \min \{ A(x), B(y) \}$$

for all $(x, y) \in G \times G'$. 