Chapter 4

Centre of Inexact Groupoid

Introduction:

The notion of sub - inexact groupoid of an inexact groupoid \((E, \Delta)\) is introduced in this chapter. The Klein forms of the sub -inexact groupoid \(K_1, K_2, \ldots, K_m\) belonging to \((E, \Delta)\) are \(K_1, K_2, \ldots, K_m\) respectively. Among these Klein forms, there is an important Klein form \(K_v\), which can be expressed as \(K_v = \{p \in E : r \Delta p \Delta r = p \ \forall \ r \in E\}\) and this is called the centre of \((E, \Delta)\). The isomorphic relation among the Klein forms is determined. We have shown that the direct absolute difference of above mentioned Klein forms is a Boolean ring with respect to point wise absolute difference and fuzzy intersection. The Klein \(2^n\) - group action on a set of fuzzy sub-sets is established here. Material incorporated in this chapter was published in the Journal of Assam Science Society 37(2), 83-91, 1995 and the Bulletin, GUMA Vol 1, December 1994. The Klein \(2^n\) - group action on a set of fuzzy sub-sets will appear in the Journal of Fuzzy Mathematics.

4.1 Sub- Inexact groupoids.

There are different kinds of sub - inexact groupoids in \((E, \Delta)\). Before giving the formal definition of sub-inexact groupoid, we consider an example, which motivates the definition.

4.1.1 Example:

Let \(X = \{x_2, x_1, x_0\}\) and
\[A = \{0, 1, 2, h, 2h = 1\}\] then
\[E = A* = \{0, 1, 2, \ldots, 26\}\]
Here \( Y = \{x_0\} \) and \( Z = \{x_1, x_2\} \) are subsets of \( X \). Then the sets
\[
A^Y = \{0,1,2\}
\]
and \( A^Z = \{0,1,2,3,4,5,6,7,8\} \) are non-empty proper sub-sets of \( E \).

Again \( B = \{0\} \) and \( D = \{0, h\} \) are sub-sets of \( A \).

Then the sets \( B^x = \{0\} \),
\[
D^Y = \{0,1\}
\]
\[
D^Z = \{0,1,3,4\},
\]
\[
D^X = \{0,1,3,4,9,10,12,13\}
\]
are the proper subsets of \( E \).

If we take the proper subsets of \( A \) as \( D = \{0, h\} \) and \( T = \{0, 2h\} \), then the sets
\[
D^X = \{0,1,3,4,9,10,12,13\} \text{ and } T^X = \{0,2,6,8,18,20,24,26\}
\]
are sub-sets of \( E \). If a non-empty subset of \( E \) itself is an inexact groupoid, then it is called sub-inexact groupoid of \( E \).

4.1.2 Definition:
Let \((E, \Delta)\) be an inexact groupoid. A non-empty subset \( H \) of \( E \) is called sub-inexact groupoid of \( E \), if \( H \) itself is an inexact groupoid with respect to absolute difference, \( \Delta \).

4.2. Klein form of sub-inexact groupoid.

4.2.1. Definition:
Let \((E, \Delta)\) be an inexact groupoid and \( H \) be its sub-inexact groupoid. Then the complement of any element \( p \in H \) is
\[
C(p) = p \Delta \sup (H).
\]

For \( p, q \in H \), the equivalence relation \( p \equiv q \iff p \Delta C(p) = q \Delta C(q) \) partitions \( H \) into some equivalence classes.

The equivalence class \( \{p \in H : p \Delta C(p) = \sup (H)\} \) is the Klein form of \( H \).
If $K_r$ be the set of mappings from a finite set $X$ (given by 3.1.1.) to $\{0, r\}$, where $0 < r < m - 1$, then $K_r$ is a sub-inexact groupoid of $E$. The relation $p \equiv q \iff p \Delta C(p) = q \Delta C(q)$ for $p, q \in K_r$ shows that there is only one equivalence class in $K_r$. Hence $K_r$ is the Klein form of the sub-inexact groupoid $K_r$. Putting $r = 1, 2, 3, \ldots, m - 1$, we get Klein forms $K_1, K_2, \ldots, K_{m-1}$ of the sub-inexact groupoids $K_1, K_2, \ldots, K_{m-1}$ respectively. The Klein form $K_{m-1}$ is also the Klein form of inexact groupoid $(E, \Delta)$.

4 2 2 Proposition:
Every Klein form of a sub-inexact groupoid is a Klein $2^n$-group with respect to absolute difference operation.

Proof is similar to 3.4.5.

4 2 3 Proposition:
Every Klein form of a sub-inexact groupoid is a Boolean ring with respect to the operations absolute difference and fuzzy intersection.

Proof is obvious.

4 3 Centre of an inexact groupoid:
4 3 1 Definition:
Let $K_1, K_2, \ldots, K_{m-1}$ be Klein forms contained in $(E, \Delta)$. The Klein form $K_1 = \{r \in E : r \Delta C(r) = \sup(K_1)\}$ can be expressed as $K_1 = \{p \in E : r \Delta p \Delta r = p \text{ for all } r \in E\}$ and this Klein form is called the centre of $(E, \Delta)$ and it is denoted by $Z(E)$.

4 3 2 Example:
From example 4.1.1, the Klein form of $D^4$ is $\{0, 1, 3, 4, 9, 10, 12, 13\} = K_1$, say. Here $K_1 = \{r \in E : r \Delta C(r) = 13\}$ is the centre of the inexact groupoid, $E = \{0, 1, 2, \ldots, 26\}$.

The centre of $(E, \Delta)$ is determined by the function given below:
Consider a function \( f : E \rightarrow E \), defined by

\[ f(p) = r \Delta p \Delta r \] for all \( p, r \in E \). Then there exists a set

\[ Z(E) = \{ p \in E \mid r \Delta p \Delta r = p \text{ for all } r \in E \} \].

This set \( Z(E) \) is equal to the Klein form \( K \) and called the centre of \( E \). The elements of \( Z(E) \) are determined from the following charts in which we use \( r \Delta p \Delta r \) in stead of \( r \Delta p \Delta r \).

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CHART - 2
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Chart - 3

From the above charts, we get \( Z(E) = \{0,1,3,9,10,12,13\} = K_1 \)
4.3.3 Proposition

Let $K$ be any Klein form contained in the inexact groupoid $(E, \Delta)$, then

centre $Z(E)$ and $K$ are isomorphic to each other.

Proof

Consider a mapping $f : Z(E) \rightarrow K$ defined by

$$f(p) = np \text{ for all } p \in Z(E) \text{ and } 0 < n \leq m-1.$$ 

$f$ is one-one, for

$$f(p) = f(q) \Rightarrow np = nq$$

$$\Rightarrow p = q \text{ for all } p, q \in Z(E)$$

Also corresponding to each element $np$ of $K$, there exists $p$ in $Z(E)$
such that $f(p) = np$. So, $f$ is onto.

For any two elements $p, q \in Z(E)$,

$$f(p \Delta q) = n(p \Delta q)$$

$$= np \Delta nq$$

$$= f(p) \Delta f(q),$$

which shows that $f$ is composition preserving.

Accordingly

$$(Z(E), \Delta) \cong (K, \Delta).$$

4.3.4 Proposition:

If $f : Z(E) \rightarrow K$ is an isomorphism,

then

i) the image of infimum of $Z(E)$ is the infimum of $K$ and

ii) the image of supremum of $Z(E)$ is the supremum of $K$ if and only if the

image of $C(p)$ in $Z(E)$ is the complement of $f(p)$ in $K$ i.e. $f(C(p)) = C(f(p))$. 

Proof

i) Let $0$ and $0'$ be the infimums of $Z(E)$ and $K$ respectively. For $p \in Z(E)$,
we get
\[ p \Delta 0 = p \]
\[ \Rightarrow f(p \Delta 0) = f(p) \]
\[ \Rightarrow f(p) \Delta f(0) = f(p) \] \( \text{(1)} \)
Again \( f(p) \Delta 0' = f(p) \) \( \text{(2)} \)

(1) and (2) imply that
\[ f(p) \Delta f(0) = f(p) \Delta 0' \]

Here \( K \) is an Abelian group and cancellation law holds in \( K \). Hence we get \( f(0) = 0' \) i.e. the image of the infimum of \( Z(E) \) is the infimum of \( K \).

1) Let \( s \) and \( s' \) be the supremums of \( Z(E) \) and \( K \) respectively. If \( f(s) = s' \), then we require to show that \( f(C(p)) = C(f(p)) \).

By definition
\[ p \Delta C(p) = s \]
\[ \Rightarrow f(p \Delta C(p)) = f(s) \]
\[ \Rightarrow f(p) \Delta f(C(p)) = f(s) \] \( \text{(3)} \)
Again \( f(p) \Delta C(f(p)) = s' = f(s) \), by hypothesis. \( \text{(4)} \)

(3) and (4) imply that
\[ f(p) \Delta f(C(p)) = f(p) \Delta C(f(p)) \]
\[ \Rightarrow f(C(p)) = C(f(p)) \]

Conversely, if \( f(C(p)) = C(f(p)) \), then
\[ f(p \Delta s) = f(p) \Delta s' \]
or, \( f(p) \Delta f(s) = f(p) \Delta s' \)
or, \( f(s) = s' \)
Hence proved
4 3 5 Proposition.

Let $Z(E)$ be the centre of the inexact groupoid $(E, \Delta)$ and $Z_i$ be the subset of $E$ with a composition $\Delta$ such that there exists one-one and onto mapping $f : Z(E) \rightarrow Z_i$ such that $f(p \Delta q) = f(p) \Delta f(q)$ for all $p, q \in Z(E)$, then $Z_i$ is a Klein-$2^n$ group.

Proof

The binary operation $\Delta$ on $Z_i$ satisfies the following properties:

i) Let $p'$ and $q'$ be any two arbitrary elements of $Z_i$, then $f$ being one-one and onto, there exist elements $p$ and $q$ in $Z(E)$ such that

$$f(p) = p' \text{ and } f(q) = q'.$$

Again $p \in Z(E), q \in Z(E) \implies p \Delta q \in Z(E)$

$$\implies f(p \Delta q) \in Z_i$$
$$\implies f(p) \Delta f(q) \in Z_i$$
$$\implies p' \Delta q' \in Z_i \text{ for } p', q' \in Z_i$$

ii) Let $p', q'$ and $r'$ be any three arbitrary elements of $Z_i$, then $f$ being one-one and onto there exist elements $p, q, r$ in $Z(E)$ such that

$$f(p) = p', f(q) = q' \text{ and } f(r) = r'.$$

So

$$(p' \Delta q') \Delta r' = (f(p) \Delta f(q)) \Delta f(r)$$
$$= f(p \Delta q) \Delta f(r)$$
$$= f((p \Delta q) \Delta r)$$
$$= f(p \Delta (q \Delta r)), \text{ since } (p \Delta q) \Delta r = p \Delta (q \Delta r)$$
$$= f(p) \Delta f(q \Delta r)$$
$$= f(p) \Delta [f(q) \Delta f(r)]$$
$$= p' \Delta (q' \Delta r')$$

iii) Let $p'$ be an arbitrary element of $Z_i$, then $f$ being one-one and onto, there exists $p$ in $Z(E)$ such that $f(p) = p'$. 


Now if 0 be the identity element in \( Z(E) \) then \( f(0) \in Z \) and \( p \Delta 0 = p \) for all \( p \in Z(E) \).

But \( p \Delta 0 = p \Rightarrow f(p \Delta 0) = f(p) \)

\[ \Rightarrow f(p) \Delta f(0) = f(p) \]

\[ \Rightarrow p' \Delta f(0) = p' \]

Similarly, \( f(0) \Delta p' = p' \)

Hence \( f(0) = 0 \) is the identity element of \( Z_t \).

iv) Let \( p' \) be an arbitrary element of \( Z_t \), then \( f \) being one-to-one and onto, there exists an element \( p \) in \( Z(E) \) such that \( f(p) = p' \).

Now the elements of \( Z(E) \) have self inverses, so \( p \Delta p = 0 \).

But \( p \Delta p = 0 \Rightarrow f(p \Delta p) = f(0) \)

\[ \Rightarrow f(p) \Delta f(p) = f(0) \]

\[ \Rightarrow p' \Delta p' = f(0) \]

i.e. every element in \( Z_t \) has self inverse in \( Z_t \).

v) For \( p' \) and \( q' \in Z_t \),

\[ p' \Delta q' = f(p) \Delta f(q) \]

\[ = f(p \Delta q) \]

\[ = f(q \Delta p) \]

\[ = f(q) \Delta f(p) \]

\[ = q' \Delta p' \]

vi) For all \( p' \in Z_t \), there exists \( C(p') \) in \( Z_t \) such that

\[ p' \Delta C(p') \]

\[ = f(p) \Delta C(f(p)) \]
\[ f(p) \triangleq f(C(p)) \text{, since } f(C(p)) = C(f(p)). \]
\[ = f(p \triangleq C(p)) \]
\[ = f(\sup(Z(E))) \]
\[ = \text{Supremum of } Z_i \text{ by 4.3.4(ii)} \]

Hence \( (Z, \Delta) \) is a Klein 2\textsuperscript{n} group.

Putting \( i = 1, 2, 3, \ldots, m-1 \), we get \( Z_1, Z_2, \ldots, Z_{m-1} \), which are isomorphic images of \( Z(E) \) and these are Klein 2\textsuperscript{n} groups contained in \( (E, \Delta) \).

4.3.6 Proposition:

The relation of isomorphism in the set of Klein 2\textsuperscript{n} groups contained in an inexact groupoid \( (E, \Delta) \) is an equivalence relation.

Proof is obvious.

4.4 Direct absolute difference of Klein forms:

In this section, we briefly introduce the construction of new Klein forms from one or more given Klein forms. One of the most important methods of construction is direct absolute difference of Klein forms. The external direct absolute difference of Klein forms is defined as follows:

4.4.1 Definition:

Let \( K_1, K_2, \ldots, K_{m-1} \) be the Klein forms contained in an inexact groupoid \( (E, \Delta) \). Let \( K \) be the cartesian product set

\[ K_1 \times K_2 \times \ldots \times K_{m-1}, \text{ that is, the set of all ordered (m-1) tuples } \]

\[ (p_1, p_2, \ldots, p_{m-1}), p_i \in K_i, i=1,2, \ldots, m-1. \]

Here \( (p_1, p_2, \ldots, p_{m-1}) = (q_1, q_2, \ldots, q_{m-1}) \)

\[ \Rightarrow p_i = q_i \text{ for } i=1,2, \ldots, m-1. \]

The operation absolute difference, \( \Delta \) and fuzzy intersection \( \cap \) are defined on \( K \) in the following way:
4.4.2. Remarks:

i) The set $K = K_1 \times K_2 \times \cdots \times K_m$ is closed under absolute difference operator. So $(K, \Delta)$ is a groupoid.

ii) If $p = (p_1, p_2, \ldots, p_m) \in K$, then $C(p) = (C(p_1), C(p_2), \ldots, C(p_m))$.

iii) There is only one equivalence class in $K$. So $K$ is the Klein form of $(K, \Delta)$.

4.4.3 Proposition:

If $K_1, K_2, \ldots, K_m$ be Klein forms in an inexact groupoid $(E, \Delta)$, then $K = K_1 \times K_2 \times \cdots \times K_m$ is a Boolean ring with respect to pointwise absolute difference $\Delta$ and fuzzy intersection $\cap$.

Proof is obvious.

4.5 Sub-Klein $2^r$-groups.

4.5.1 Definition:

Let $K$ be a Klein $2^n$-group contained in an inexact groupoid $(E, \Delta)$. A non-empty subset $L$ of $K$ is called a sub-Klein group, simply S.K. group if $L$ is itself a Klein $2^r$-group, where $r \leq n$. 

4.5.2 Proposition:
A necessary and sufficient condition for a non-empty set $L$ of a Klein $2^n$-group $K$ to be a S.K. group is that

$$p \in L, q \in L \Rightarrow p \Delta q \in L$$

Proof is obvious.

4.5.3 Proposition:
The absolute difference of two S.K. groups $L_1$ and $L_2$ of a Klein $2^n$-group $K$ is a S.K. group.

Proof.
The absolute difference of $L_1$ and $L_2$ is given by

$$L_1 \Delta L_2 = \{p_1 \Delta q_1 : p_1 \in L_1 \text{ and } q_1 \in L_2\}$$

Let $p_1 \Delta q_1 \in L_1 \Delta L_2$ and $p_2 \Delta q_2 \in L_1 \Delta L_2$,

then

$$(p_1 \Delta q_1) \Delta (p_2 \Delta q_2)$$

$= p_1 \Delta (q_1 \Delta p_2) \Delta q_2$$

$= p_1 \Delta (p_2 \Delta q_1) \Delta q_2$, since $q_1 \Delta p_2 = p_2 \Delta q_1$

$= (p_1 \Delta p_2) \Delta (q_1 \Delta q_2) \in L_1 \Delta L_2$$

Since $p_1, p_2 \in L_1 \Rightarrow p_1 \Delta p_2 \in L_1$

$q_1, q_2 \in L_2 \Rightarrow q_1 \Delta q_2 \in L_2$.

Again $p_1 \Delta q_1 \in L_1 \Delta L_2$ and $C(p_1 \Delta q_1) \in L_1 \Delta L_2$

imply that $(p_1 \Delta q_1) \Delta C(p_1 \Delta q_1)$

$= (p_1 \Delta q_1) \Delta (p_1 \Delta q_1) \Delta \sup (L_1 \Delta L_2)$

$= \sup (L_1 \Delta L_2)$

Therefore $L_1 \Delta L_2$ is a S.K. group of $K$. 
4.6 Internal direct absolute difference

4.6.1 Definition

A Klein $2^n$ - group $K$ is said to be an internal direct absolute difference of its $S.K.$ groups $K_1, K_2, \ldots, K_m$ if each element of $K$ is uniquely expressible as

$$p = p_1 \Delta p_2 \Delta \ldots \Delta p_r, \quad p_i \in K_i \quad \text{for } i = 1, 2, \ldots, r$$

in the sense that if

$$p = q_1 \Delta q_2 \Delta \ldots \Delta q_r, \quad q_i \in K_i \quad \text{for } i = 1, 2, \ldots, r$$

then for all $i$,

$$p_i = q_i.$$

4.6.2 Proposition.

If a Klein $2^n$ - group $K$ is the internal direct absolute difference of its $S.K.$ groups $K_1, K_2, \ldots, K_m$, then $K \cong K_1 \times K_2 \times \ldots \times K_m$.

Proof:

Consider a mapping

$$f : K \to K_1 \times K_2 \times \ldots \times K_m$$

defined by

$$f(p) = (p_1, p_2, \ldots, p_m) \quad \text{for all } p = p_1 \Delta p_2 \Delta \ldots \Delta p_m \in K.$$

For $p = p_1 \Delta p_2 \Delta \ldots \Delta p_m$ and

$q = q_1 \Delta q_2 \Delta \ldots \Delta q_m$ be any two elements of $K$, where

$p_i, q_i \in K_i \quad \text{for } i = 1, 2, \ldots, m.$

Here $f$ is one - one, for

$$f(p) = f(q) \Rightarrow (p_1, p_2, \ldots, p_m) = (q_1, q_2, \ldots, q_m)$$

$$\Rightarrow p_1 = q_1, p_2 = q_2, \ldots, p_m = q_m$$

$$\Rightarrow p_1 \Delta p_2 \Delta \ldots \Delta p_m = q_1 \Delta q_2 \Delta \ldots \Delta q_m$$

$$\Rightarrow p = q.$$

Also $f$ is onto, for if

$$(p_1, p_2, \ldots, p_m) \in K_1 \times K_2 \times \ldots \times K_m,$$

then $p_1 \Delta p_2 \Delta \ldots \Delta p_m = p \in K$ such that

$$f(p) = (p_1, p_2, \ldots, p_m).$$
Moreover, \( f \) is composition preserving, for if \( p \) and \( q \) be any two elements of \( K \), then

\[
f(p \Delta q) = f[(p_1 \Delta q, p_2 \Delta q, \ldots, p_m \Delta q_m)] = f(p_1 \Delta q_1, p_2 \Delta q_2, \ldots, p_m \Delta q_m) = (p_1, p_2, \ldots, p_m) \Delta (q_1, q_2, \ldots, q_m) = f(p) \Delta f(q)
\]

Hence \( f \) is an isomorphism.

4.7. Klein \( 2^n \) - group action on a set of fuzzy sub sets.

An action of a field \((K, +, \cdot)\) on an Abelian group \((V, +)\) is a map

\[
V \times K \to V
\]

\((p, a) \to p^a\) such that for every \( p, q \in V \) and \( a, b \in K \)

i) \((p + q)^a = p^a + q^a\)

ii) \(p^{a+b} = p^a + p^b\)

iii) \(p^{ab} = (p^b)^a\) and

iv) \(1^1 = p\), where \(1\) is the identity element of \( K \). If \( K \) acts on \( V \) as above, then \( V \) is called a linear space over the field \( K \). If \( K \) be a group and \( V \), a set then the action of \( K \) on \( V \) is a map

\[
V \times K \to V,
\]

which satisfies the last two axioms (iii) and (iv).

This action is a group action. With the help of this group action, we shall introduce Klein \( 2^n \) - group action on a set of fuzzy sub sets.

4.7.1 Definition.

Let \( K \) be a Klein \( 2^n \) - group of an inexact groupoid \((E, \Delta)\) and a set of fuzzy sub sets of \( E \) be \( V \).
A Klein \(2^n\) - group action of \(K\) on \(V\) is a map
\[
V \times K \rightarrow V
\]
\[(a, p) \rightarrow a^p, \text{ such that}\]
i) for every \(p, q \in K\) and \(a \in V\),
\[((a, p), q) \rightarrow (a^p)^q = a^{p \cdot q}\) and

ii) for every \(a^0 = a\, , \text{ where } 0 \text{ is the identity element of} K\).

If a Klein \(2^n\) - group \(K\) acts on \(V\), we say that \(V\) is a Klein \(2^n\) - group space.

Here we shall use \(K\) - action and \(K\) - space in place of Klein \(2^n\) - group action and Klein \(2^n\) - group space respectively.

4.7.2 Examples

i) Let \(K\) be a Klein \(2^n\) - group and let \(V = K\), then \(K\) acts on \(V\). If \(a \in V\) and \(p \in K\), then the action
\[
V \times K \rightarrow V, \text{ defined by}
\]
\[a^p = a \cdot p \text{ is a } K\text{- action and } V \text{ is a } K\text{- space.}\]

ii) Let \(K\) be a sub-Klein group of a Klein \(2^n\) - group = \(V\), then the action
of \(K\) on \(V\) is the map
\[
V \times K \rightarrow V, \text{ defined by}
\]
\[a^p = a \cdot p \text{ for } a \in V \text{ and } p \in K. \text{ This action is a } K\text{- action and } V \text{ is a } K\text{- space.}\]

iii) Let \(H\) be a sub - Klein group of the Klein \(2^n\) - group \(K\), then the action
of \(K\) on \(V = K/H\) is
\[
V \times K \rightarrow V, \text{ defined by}
\]
\[(H \cdot a)^p = (H \cdot a) \cdot (H \cdot p) \text{ for } H \cdot a \in V \text{ and } p \in K. \text{ This action is a } K\text{- action and } V \text{ is a } K\text{- space.}\]

iv) The Klein \(2^n\)-group \(K\) of an inexact groupoid \((E, \Delta)\) acts on the centre \(V = Z(E)\), then the action
\[ V \times K \to V, \text{ defined by } \]
\[ a^p = p \Delta a \Delta p = a \quad \text{for all } a \in V \text{ and } p \in K \text{ is a } K\text{-action and } V \text{ is a } K\text{-space.} \]

\( v) \text{ Let } K \text{ be a Klein } 2^n \text{- group of inexact groupoid } (E, \Delta) \text{ and } V \text{ be the centre of } K, \text{ then the map} \]
\[ V \times K \to V \text{ defined by} \]
\[ a^p = p \Delta a \Delta p = a \quad \text{for all } a \in V \text{ and } p \in K \text{ is } K\text{-action and } V \text{ is a } K\text{-space.} \]

4.7.3. Definition:

Let \( V \) be a \( K\)-space. The orbit of \( a \in V \) is the set
\[ a^K = \{ a^p : p \in K \}. \]
If \( a^K = V \) for the \( K\)-action on \( V \), then \( K \) acts on \( V \) transitively and \( V \) is transitive \( K\)-space.

From examples 4.7.2., the \( K\)-spaces No (i) and (iii) are transitive.
The \( K\)-spaces No (ii), (iv) and (v) are not transitive.

4.7.4 Remark:

If \( K \) be a sub Klein group of \( V \), then the distinct orbits of the elements belonging to \( K\)-space are the distinct cosets of \( K \) in \( V \).

4.7.5 Proposition:

If \( V \) be a \( K\)-space of the examples 4.7.2. (iv) or (v), then the orbit of any element belonging to \( V \) is singleton.

Proof is obvious.
Proposition:

Any two orbits in $K$-space $V$ are either disjoint or identical.

Proof

Let $V$ be a $K$-space, then the orbits of two distinct elements $a$ and $b$ of $V$ are $a^K$ and $b^K$ respectively.

Case 1:

If $a^K$ and $b^K$ be two singleton orbits, then $a^K = a$ and $b^K = b$ are distinct. Hence $a^K \cap b^K = \emptyset$.

Case 2:

If $a^K$ and $b^K$ are not singleton, then we have to show that $a^K \cap b^K = \emptyset$ or $a^K = b^K$.

Let $a^K$ and $b^K$ are not disjoint. Then they possess an element say $c$ in common and $c$ can be written as $c = a \Delta p_1$ and $c = b \Delta p_2$, where $p_1, p_2 \in K$.

Therefore, $a \Delta p_1 = b \Delta p_2$.

$\Rightarrow a = b \Delta p_2 \Delta p_1$ for $p_1, p_2 \in K$.

$\Rightarrow a \in b^K$.

$\Rightarrow a^K \subseteq b^K$. ............ (1)

In the same way, we can show that $b \in a^K$ and $b^K \subseteq a^K$. ............... (2)

From (1) and (2) we get $a^K = b^K$.

Therefore two orbits of two distinct elements of a $K$-space $V$ are identical if they are not disjoint.

Thus $a^K \cap b^K = \emptyset$ or $a^K = b^K$. 
4.7.7 Proposition:

Every $K$-space $V$ is equal to the union of its all orbits i.e.,

$$V = a^K \cup b^K \cup c^K \cup \ldots \ldots$$

where $a, b, c, \ldots \in V$.

Proof:

Let $d$ be any element of $K$-space $V$, then $d \in V$ implies $d \in d^K$.

$$\Rightarrow d \in a^K \cup b^K \cup c^K \cup d^K \cup \ldots \ldots$$

Hence $V \subseteq a^K \cup b^K \cup c^K \cup d^K \cup \ldots \ldots \ldots (1)$

where $a, b, c, d, \ldots \in V$.

Again $e \not\in a^K \cup b^K \cup c^K \cup d^K \cup \ldots \ldots$  

$$\Rightarrow e \in e^K \text{ for some } e \in V.$$

$$\Rightarrow e \in V$$

Thus $a^K \cup b^K \cup c^K \cup d^K \cup \ldots \ldots \subseteq V \ldots \ldots (2)$

From (1) and (2) we get

$$V = a^K \cup b^K \cup c^K \cup d^K \cup \ldots \ldots$$

4.7.8 Definition:

For $a \in V$, the set $K_a = \{ p \in K : a^p = a \}$ is a sub group of $K$ and is called the stabilizer of $a$ in $K$.

4.7.9 Proposition:

If $V$ be a $K$-space of the examples 4.7.2. (iv) or (v), then the stabilizer of any element $a \in V$ is the Klein $2^n$- group $K$ i.e., $K_a = K$.

Proof:

By definition of $K_a$, the stabilizer of $a \in V$, we get
\[ K_a \subseteq K \quad \dot{\ldots} \ldots \ldots \quad (1) \]
Now \( p \in K \) implies \( a^p = a \)
\[ \Rightarrow p \in K_a \]
Hence \( K \subseteq K_a \quad \dot{\ldots} \ldots \ldots \quad (2) \]
From (1) and (2), we get \( K_a = K \).

4.8 K - morphism:

4.8.1 Definition:
Let \( V \) and \( V' \) be two \( K \)-spaces, then the map
\[ f : V \rightarrow V' \]
is a \( K \)-morphism, if for all \( a \in V \) and \( p \in K \),
\[ f(a^p) = [f(a)]^p \]
If \( f \) is bijective, then \( f \) is \( K \)-isomorphism.

4.8.2 Proposition:
Let \( V \) and \( V' \) be two \( K \)-spaces, where \( V = \) centre of inexact groupoid \( E \), \( K \) is a subset of \( E \) and \( V' = \) endomorphic image of \( V' \), then the map
\[ f : V \rightarrow V' \]
is \( K \)-morphism.

Proof
Consider the map
\[ f : V \rightarrow V' \]
defined by
\[ f(a) = r \cap a \]
where \( r, a \in V \).
Now for \( a \in V \) and \( p \in K \), we get
\[ f(a^p) = r \cap a^p \]
\[ = p \Delta r \cap a^p \Delta p \]
\[ = p \Delta r \cap a \Delta p \Delta p \]
\[ = p \Delta r \cap a \]
\[ = p \Delta f(a) \]
\[ = [f(a)]^p \]
Hence \( f \) is a \( K \)-morphism.
4.8.3. Proposition:

Let $V$ and $V'$ be two $K$-spaces, where $V = \text{centre of inexact groupoid } E$, $K$ is a sub set of $E$, $V' = \text{centre of } K$ and $|V| = |V'|$. Then the map $V \rightarrow V'$ is a $K$-isomorphism.

Proof:

Consider a map $f: V \rightarrow V'$ defined by $f(a) = ra$, where $a \in V$ and $0 < r \leq m - 1$. Clearly $f$ is one-one and onto.

Now for $a \in V$ and $p \in K$, we have

\[
f(a^p) = f(p \Delta a \Delta p) = f(a) = ra = p \Delta ra \Delta p = p \Delta f(a) \Delta p = [f(a)]^p.
\]

Hence $f$ is a $K$-isomorphism.

4.8.4. Proposition:

Let $H_r$ and $H_s$ be two endomorphic images of a Klein $2^n$-group $K$ such that $H_r \subset H_s \subset K$, then the mapping from $H_r$-space $K$ into $H_r$-space $H_s$ is a $H_r$-morphism.
Proof
Consider an endomorphic map
\[ f : K \to H_s \] defined by
\[ f(p) = p \cap s \] for \( p \in K, s \in K \).
Now for \( p \Delta h = p^h \in K \) and \( h \in H_r \),
we get \( f(p^h) = p^h \cap s \)
\[ = p \Delta h \cap s \]
\[ = h \Delta p \cap s \]
\[ = h \Delta f(p) \]
\[ = [f(p)]^h \]

Hence the mapping \( f \) is a \( H_r \)-morphism.