CHAPTER 4

SOME DISTRIBUTIONS RELATED TO MEAN SQUARE SUCCESSIVE DIFFERENCE AND RADIAL ERROR
4.1 An approximation to the sampling distribution of mean square successive difference in a series of Laguerre polynomials:

Introduction:

In this section, we have derived an approximate sampling distribution of mean square successive difference, $\delta^2$ in a series of Laguerre polynomials for a sample sequence of $n$ observations drawn from any population. In addition to the normal theory function, we have obtained the expressions for the corrective terms of the cumulative distribution function involving the population values of $\lambda_4 = \beta_2 - 3$ and $\lambda_3 = \beta_1$. The numerical computation of these corrective terms has been done to obtain the actual probability corresponding to the upper 5 per cent and 1 per cent normal theory significant points. Such significant points are given by Cornish and Fisher's expansion for various values of the sample size exceeding 10.

The derivation of the sampling distribution of $\delta^2$:

We define a Laguerre polynomial (see Rainville (1965), Section 112, formula (3)) of degree $r$ in $x$, for a non-negative integer $r$, as
\( L_r(x) = \sum_{k=0}^{r} \frac{(-1)^k (1+m)_r x^k}{k! (r-k)! (1+m)_k} \), \hspace{1cm} \ldots \ (4.1.1) \]

where \((m)_k = m(m+1) \ldots (m+k-1), k \geq 1 \) and \((m)_0 = 1, m \neq 0\).

If we put \( g_m(x) = \frac{1}{\Gamma(m)} e^{-x} x^{m-1} \), \hspace{1cm} \ldots \ (4.1.2) \]

then the orthogonal properties of the Laguerre polynomials can be stated as

\[
\int_0^\infty L_r^{(m-1)}(x) L_s^{(m-1)}(x) g_m(x) \, dx = \frac{\Gamma(m+r)}{\Gamma(m) r!} \quad \text{for } r = s
\]

\[ = 0 \quad \text{for } r \neq s \]

where \( \text{Re}(m) > 0 \).

\[ \ldots \ (4.1.3) \]

Let \( x_i (i = 1, 2, \ldots, n) \) denote a sample sequence of \( n \) observations drawn from a population with a variable mean \( \mu_i \) and in which the cumulants, \( k_r \) of \( r^{th} \) order, \( r = 2, 3, \ldots \) exist and are constant. Let the variance \( \sigma^2 = k_2 \) and \( \lambda_r = k_r k_2^{-r/2} \). If \( \mu_i \) has slow moving trend then it is known that the statistic

\[
\frac{\hat{\sigma}^2}{\sigma^2} = \frac{\sum_{i=1}^{n-1} (x_i - x_{i+1})^2}{2(n-1)} \]

\[ \frac{\hat{\sigma}^2}{\sigma^2} = \frac{\sum_{i=1}^{n-1} (x_i - x_{i+1})^2}{2(n-1)} \]

\[ \ldots \ (4.1.4) \]

provides a consistent estimate of the population variance \( \sigma^2 \).

Here we investigate the sampling distribution of \( \hat{\sigma}^2 \) and for that
consider a function

\[
X = \frac{m\delta^2}{2\sigma^2}, \quad \text{where} \quad m = (n-1)/2. \quad \ldots (4.1.5)
\]

The first four moments of \( X \) about the mean are obtained from Moore (1955) by straightforward calculation as

\[
\mu(X) = m,
\]

\[
\mu_2(X) = \frac{1}{6} \left[ (4m-1) \lambda_4 + 2(6m-1) \right],
\]

\[
\mu_3(X) = \frac{1}{32} \left[ (8m-3) \lambda_6 + 6(4m-1) \lambda_3 + 48(3m-1) \lambda_4 + 16(10m-3) \right],
\]

\[
\mu_4(X) = \frac{1}{128} \left[ (16m-7) \lambda_8 + 24(16m-5) \lambda_5 \lambda_3 + 2(48m^2 + 336m - 157) \lambda_4 + 
\right.
\]

\[
48(12m-5) \lambda_6 + 16(90m-109) \lambda_3 + 24(24m^2 + 222m - 95) \lambda_4 + 
\]

\[
24(36m^2 + 128m - 57) \right]. \quad \ldots (4.1.6)
\]

We denote by \( f_m(X) \), the probability density function of \( X \).

Then it can be formally expanded in an infinite series of Laguerre polynomials as

\[
f_m(X) \sim g_m(X) \sum_{r=0}^{\infty} a_r L_r^{(m-1)}(X). \quad \ldots (4.1.7)
\]

Multiplying both the sides by \( L_r^{(m-1)}(X) \) and integrating over \( X \) in the range 0 to \( \infty \), we get the coefficients
\[ a_r = \frac{E \left[ L_r^{(m-1)}(X) \right]}{\frac{1}{\Gamma(m+r)} \left\{ \frac{1}{r!} \frac{1}{\Gamma(m)} \right\}} \]  \quad \ldots (4.1.8)

Evaluating \( E \left[ L_r^{(m-1)}(X) \right] \), for \( r = 0, 1, 2, 3 \) and \( 4 \), we get

\[ a_0 = 1, \quad a_1 = 0, \quad a_2 = \frac{(4m-1) \lambda_4 + 2(2m-1)}{8(m+1)m}, \]

\[ a_3 = -\frac{1}{32(m+2)(m+1)m} \left[ (8m-3) \lambda_4 + 6(4m-1) \lambda_3^2 + 24(2m-1) \lambda_4 \right] \]

and \( a_4 = \frac{1}{128(m+3)(m+2)(m+1)m} \left[ (16m-7) \lambda_5 + 24(16m-5) \lambda_5 \lambda_3 + \
2(48m^2+336m-157) \lambda_5^2 + 96(2m-1) \lambda_4 + 16(18m-91) \lambda_3^2 + \
24(8m^2+34m-23) \lambda_4 + 24(4m^2+8m-9) \right] \). \quad \ldots (4.1.9)

Thus we get the probability density function of \( X \) as

\[ f_m(X) = f_N(X) + \lambda_4 f_{\lambda_4}(X) + \lambda_3^2 f_{\lambda_3^2}(X) + \lambda_4 f_{\lambda_4}(X) + \lambda_5^2 f_{\lambda_5^2}(X) + \
\lambda_5 \lambda_3 f_{\lambda_5 \lambda_3}(X) + \lambda_8 f_{\lambda_8}(X) \], \quad \ldots (4.1.10)

where \( f_N(X) = g_m(X) \left[ 1 + \frac{(2m-1)}{4(m+1)m} L_2^{(m-1)}(X) + \
\frac{3(4m^2+8m-9)}{16(m+3)(m+2)(m+1)m} L_4^{(m-1)}(X) \right] \),
$f_{\lambda_4}(X) = g_m(X) \left[ \frac{(4m-1)}{8(m+1)m} L_2^{(m-l)}(X) - \frac{3(2m-1)}{4(m+2)(m+1)m} L_3^{(m-l)}(X) + \frac{3(8m^2+34m-23)}{16(m+3)(m+2)(m+1)m} L_4^{(m-l)}(X) \right],$

$f_{\lambda_3}(X) = g_m(X) \left[ - \frac{3(4m-1)}{16(m+2)(m+1)m} L_3^{(m-l)}(X) + \frac{(18m-91)}{8(m+3)(m+2)(m+1)m} L_4^{(m-l)}(X) \right],$

$f_{\lambda_2}(X) = g_m(X) \left[ - \frac{(8m-3)}{32(m+2)(m+1)m} L_3^{(m-l)}(X) + \frac{3(2m-1)}{4(m+3)(m+2)(m+1)m} L_4^{(m-l)}(X) \right],$

$f_{\lambda_2}(X) = g_m(X) \frac{(48m^2+336m-157)}{64(m+3)(m+2)(m+1)m} L_4^{(m-l)}(X),$

$f_{\lambda_3}(X) = g_m(X) \frac{3(16m-5)}{16(m+3)(m+2)(m+1)m} L_4^{(m-l)}(X)$

and $f_{\lambda_3}(X) = g_m(X) \frac{(16m-7)}{128(m+3)(m+2)(m+1)m} L_4^{(m-l)}(X).$

Here $f_{\lambda_4}(X)$ is the normal theory probability density function of $X$ and $f_{\lambda_4}(X), f_{\lambda_3}(X)$ etc. are corrective terms due to the population cumulants $\lambda_4, \lambda_3$ etc. respectively.

Thus the probability density function of $\delta^2$ is given by
\[ f(m, \delta^2) = \left( \frac{m}{2\sigma^2} \right) \left[ f_N \left( \frac{m \delta^2}{2\sigma^2} \right) + \lambda_4 f \chi_4 \left( \frac{m \delta^2}{2\sigma^2} \right) + \lambda_3^2 f \chi_3 \left( \frac{m \delta^2}{2\sigma^2} \right) + \lambda_4 f \chi_4 \left( \frac{m \delta^2}{2\sigma^2} \right) \right] \]

where \( m = \frac{(n-1)}{2} \). \hfill (4.1.11)

The cumulative distribution function of \( \delta^2 \) with population skewness and kurtosis only is given by

\[ F(\delta^2) = \int_0^{\delta^2} f(m, \delta^2) \, d\delta^2 = P_N(\delta^2) + \lambda_4 P_{\chi_4}(\delta^2) + \lambda_3^2 P_{\chi_3}(\delta^2), \]

\[ \ldots \hfill (4.1.12) \]

where \( f(m, \delta^2) = \left( \frac{m}{2\sigma^2} \right) \left[ f_N \left( \frac{m \delta^2}{2\sigma^2} \right) + \lambda_4 f \chi_4 \left( \frac{m \delta^2}{2\sigma^2} \right) + \lambda_3^2 f \chi_3 \left( \frac{m \delta^2}{2\sigma^2} \right) \right] \].

It may be computed for various values of \( n \) and \( (\delta^2/\sigma^2) \) directly from the algebraic expressions (using Karl Pearson's tables of Incomplete Gamma function). Now, we have

\[ P_N(\delta) = I \left[ \frac{X_0}{\sqrt{m}} , (m-1) \right] + (a+b) I \left[ \frac{X_0}{\sqrt{m}} , (m-1) \right] \]

\[ - 2(a+2b) I \left[ \frac{X_0}{\sqrt{(m+1)}} , m \right] + (a+6b) I \left[ \frac{X_0}{\sqrt{(m+2)}} , (m+1) \right] \]

\[ - 4b I \left[ \frac{X_0}{\sqrt{(m+3)}} , (m+2) \right] + b I \left[ \frac{X_0}{\sqrt{(m+4)}} , (m+3) \right], \]

\[ \ldots \hfill (4.1.13) \]

where \( X_0 = \frac{m \delta^2}{2\sigma^2} \), \( m = \frac{(n-1)}{2} \), \( a = \frac{(2m-1)}{8} \), \( b = \frac{(4m^2+8m-9)}{128} \).
and \( I(u, p) = \int_0^\infty \frac{e^{-u} v^p}{\Gamma(p+1)} \, dv \).

Using the recurrence relation

\[
I(u-n, p-n-1) = I(u_1, p) + \frac{e^{-u} v^p}{\Gamma(p+1)} \left\{ 1 + \frac{p}{\xi} + \frac{p(p-1)}{\xi^2} + \cdots + \frac{p(p-1) \cdots (p-n+1)}{\xi^n} \right\}
\]

where \( \xi = u_1 v^{(p+1)} = u_2 v^{(p+2)} = u_0 v^{(p+1)} = u_{-1} v^{(p+1)} \) and so on,

we get

\[
P_{\delta_0}(\frac{2}{m+3}) = I \left[ \frac{X_0}{\sqrt{(m+4)}}, (m+3) \right] + \frac{e^{-X_0}}{\Gamma(m+4)} \left[ (1-b) + (1+3b) \frac{(m+3)}{X_0} \right.
\]

\[
+ (1-a-3b) \frac{(m+3)(m+2)}{X_0^2} + (1+a+b) \frac{(m+3)(m+2)(m+1)}{X_0^3} \]

\[... \quad (4.1.14)\]

Similarly we have

\[
P_{\lambda_4}(\frac{2}{m+3}) = \frac{e^{-X_0}}{\Gamma(m+4)} \left[ \frac{1}{32} \left( \frac{1+18a+64b}{m+3} \right) \frac{(m+3)}{X_0} \right.
\]

\[
+ \frac{3+22a+192b}{32} \frac{(m+3)(m+2)}{X_0^2} + \frac{3+18a+64b}{32} \frac{(m+3)(m+2)(m+1)}{X_0^3} \]

\[... \quad (4.1.15)\]
and \( P_{\lambda_3} (\delta^2_o) = e^{-X_o (m+3) \over \Gamma (m+4)} \left[ - \frac{(36a-41)}{96} + \frac{(10a-21)}{16} \frac{(m+3)}{X_o} \right] \]

\[ \frac{(4a-45)}{32} \frac{(m+3)(m+2)}{X_o^2} + \frac{(3a+11)}{24} \frac{(m+3)(m+2)(m+1)}{X_o^3} \right]. \]

\[ \ldots (4.1.16) \]

**Tabulation of corrective functions:**

The values of \( P_{\lambda_4} (\delta^2_o) \) and \( P_{\lambda_3} (\delta^2_o) \) are calculated for the upper 5 per cent and 1 per cent normal theory significant points of \( (\delta^2_o / \sigma^2) \) when \( n = 10(5)20(10)50(25)100 \). Such significant points \( (\delta^2_o / \sigma^2) \) are given by Cornish and Fisher's (1937) expansion. The calculated values are given in Table IV.1 and Table IV.2.

**Table IV.1**

Values of \( P_{\lambda_4} (\delta^2_o) \) and \( P_{\lambda_3} (\delta^2_o) \) for the upper 5 per cent normal theory significant points \( (\delta^2_o / \sigma^2) \) in different sample sizes.

<table>
<thead>
<tr>
<th>( n )</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>75</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta^2_o / \sigma^2 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.17</td>
<td>3.71</td>
<td>3.46</td>
<td>3.16</td>
<td>2.99</td>
<td>2.88</td>
<td>2.71</td>
<td>2.61</td>
<td></td>
</tr>
<tr>
<td>( P_{\lambda_4} (\delta^2_o) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.0141</td>
<td>-0.0187</td>
<td>-0.0218</td>
<td>-0.0253</td>
<td>-0.0272</td>
<td>-0.0284</td>
<td>-0.0300</td>
<td>-0.0308</td>
<td></td>
</tr>
<tr>
<td>( P_{\lambda_3} (\delta^2_o) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0431</td>
<td>0.0358</td>
<td>0.0316</td>
<td>0.0291</td>
<td>0.0284</td>
<td>0.0280</td>
<td>0.0290</td>
<td>0.0307</td>
<td></td>
</tr>
</tbody>
</table>
Table IV.2

Values of $P_{\chi^4}^2(\delta^2_0)$ and $P_{\chi^2}^2(\delta^2_0)$ for the upper 1 per cent normal theory significant points ($\delta^2_0/\sigma^2$) in different sample sizes

<table>
<thead>
<tr>
<th>n</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>75</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta^2_0/\sigma^2$</td>
<td>5.66</td>
<td>4.80</td>
<td>4.33</td>
<td>3.82</td>
<td>3.53</td>
<td>3.34</td>
<td>3.07</td>
<td>2.91</td>
</tr>
<tr>
<td>$P_{\chi^4}^2(\delta^2_0)$</td>
<td>-0.0126</td>
<td>-0.0137</td>
<td>-0.0141</td>
<td>-0.0141</td>
<td>-0.0141</td>
<td>-0.0141</td>
<td>-0.0134</td>
<td>-0.0134</td>
</tr>
<tr>
<td>$P_{\chi^2}^2(\delta^2_0)$</td>
<td>0.0036</td>
<td>0.0026</td>
<td>0.0023</td>
<td>0.0022</td>
<td>0.0023</td>
<td>0.0025</td>
<td>0.0028</td>
<td>0.0032</td>
</tr>
</tbody>
</table>

Discussion on the results:

It appears from the values of $P_{\chi^4}^2(\delta^2_0)$ and $P_{\chi^2}^2(\delta^2_0)$ that the increase in the size of the sample does not tend to reduce the effects of non-normality due to $\chi^4$ and $\chi^2_3$. It is to be noted that the actual level of significance is given by

$$\left[1 - P_N^2(\delta^2_0)\right] - \lambda_4 P_{\chi_4^4}^2(\delta^2_0) - \lambda_3^2 P_{\chi_3^2}^2(\delta^2_0)$$

Table IV.3 gives the actual level of significance for the upper 5 per cent and 1 per cent normal theory significant points ($\delta^2_0/\sigma^2$) and $n = 50$ in different non-normal situations for $\chi^4 = -0.5, 0, 2$ and $\chi^2_3 = 0, 0.2, 0.4$. 
Table IV.3

Actual level of significance for the upper 5 per cent and 1 per cent normal theory significant points ($\frac{t_n}{\sigma}$) when $n = 50$ in different non-normal situations

<table>
<thead>
<tr>
<th>$\lambda_4$</th>
<th>$P_N(t^2) = 0.95$, $\frac{\delta^2}{\sigma^2} = 2.88$</th>
<th>$P_N(t^2) = 0.99$, $\frac{\delta^2}{\sigma^2} = 3.34$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_3$</td>
<td>$\lambda_3^2$</td>
<td>$\lambda_3^2$</td>
</tr>
<tr>
<td>0</td>
<td>0.05</td>
<td>0.0029</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.0388</td>
<td>0.0024</td>
</tr>
<tr>
<td>0</td>
<td>0.0444</td>
<td>0.0095</td>
</tr>
<tr>
<td>2</td>
<td>0.1068</td>
<td>0.0372</td>
</tr>
</tbody>
</table>

From the results of the above table, it is seen that the use of the normal theory significant points ($\frac{t_n}{\sigma}$) may not sometimes be appropriate in the non-normal situations. If we use such values, then the level of significance increases for leptokurtic population and decreases for the platykurtic population compared to that based on the normal theory assumption. The effect of $\lambda_3$ is to reduce the normal theory level of significance.

For a comparative study of the forms of the normal theory frequency curves of ($\frac{\delta^2}{\sigma^2}$) and those of the corrective functions for $\lambda_4$ and $\lambda_3^2$, diagrams have been constructed for $n = 10, 20, 50$ and 100. They are shown in figures 1-4.
Fig. 1. Showing the terms in the distribution

\[ f_m(\delta^2) = (m/2\sigma^2) \left[ f_N(m\delta^2/2\sigma^2) + \lambda_4 f_{\lambda_4} \left( m\delta^2/2\sigma^2 \right) + \lambda_3 f_{\lambda_3} \left( m\delta^2/2\sigma^2 \right) \right] \]

for \( n = 2m + 1 = 1 \).

Explanation of symbols for all figures (Figs. 1-4):

\[
\begin{align*}
\text{---} & \quad (m/2) f_N(m\delta^2/2\sigma^2) \\
\text{-----} & \quad (m/2) f_{\lambda_4} \left( m\delta^2/2\sigma^2 \right) \\
\text{----} & \quad (m/2) f_{\lambda_3} \left( m\delta^2/2\sigma^2 \right)
\end{align*}
\]
\[ f(\delta^2) = \left( \frac{m}{2\sigma^2} \right) [ f_N(\delta^2/2\sigma^2) + \lambda f_4(\delta^2/2\sigma^2) + \lambda^2 f_3(\delta^2/2\sigma^2) ] \]

for \( n = 2m + 1 = 20 \).
FIG. 3. SHOWING THE TERMS IN THE DISTRIBUTION

\[ f_m(\delta^2) = \frac{m}{2\sigma^2} \left[ f_N(m\delta^2/2\sigma^2) + \lambda_4 f_{\lambda_4}(m\delta^2/2\sigma^2) + \lambda_3^2 f_{\lambda_3^2}(m\delta^2/2\sigma^2) \right] \]

for \( n = 2m + 1 = 50 \).
FIG. 4. SHOWING THE TERMS IN THE DISTRIBUTION

\[ f_m(s^2) = \left(\frac{m}{2\sigma^2}\right) \left[ f_n(m\delta^2/2\sigma^2) + \lambda f_{4\lambda 4}(m\delta^2/2\sigma^2) + \lambda^2 f_{3\lambda 3}(m\delta^2/2\sigma^2) \right] \]

for \( n = 2m + 1 = 100 \).
4.2 On the sampling distribution of the weighted sum of mean square successive difference and sample variance:

Introduction:

This problem deals with a particular combination of some of the estimates which are already used for estimating the population variances as a measure of radial dispersion, the sample being drawn from a bivariate universe under the assumption that the variables are identically and independently distributed. Moranda (1959) and Kamat (1962) have discussed some similar combinations in connection with the study of circular probable error.

Here we investigate the properties of

\[
\frac{s^2}{\sigma^2_M} = \frac{\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2 + 2 \sum_{i=1}^{n} (y_i - \bar{y})^2}{2(n-1)} \quad \quad \ldots (4.2.1)
\]

where \((x_i, y_i), i = 1, 2, \ldots, n\) are \(n\) observations with means \(\mu_i\) and \(\sigma\) for \(x_i\) and \(y_i\) respectively.

Its relative efficiency is compared with those of \(s^2\), the radial sample variance with constant means and \(\frac{s^2}{2}\), the radial mean square successive difference subject to gradual shifts in both the population means defined as follows:

\[
s^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2 + \sum_{i=1}^{n} (y_i - \bar{y})^2}{(n-1)} \quad \quad \ldots (4.2.2)
\]
and \( \frac{\sum_{i=1}^{n-1} (x_i - x_{i+1})^2 + \sum_{i=1}^{n-1} (y_i - y_{i+1})^2}{2(n - 1)} \). \hfill \ldots \ (4.2.3)

The study of \( (\delta^2) \) will be useful in situations where an observation is recorded by two co-ordinates, but due either to some exogenous or endogenous factors or to both there may have an effective gradual shift in one of the population means, while the effect of the shift in the other one may be negligible. Such situations often arise in Ballistics, Astronomy, Electric Circuit Theory etc.

Moments:

Let \( \mu_r (r = 2, 3, 4, \ldots) \) be the moments about the respective means for both the variables. The first four moments of \( \delta^2_M \) about the mean are, then, given by

\[
(n - 1) \mu'_1(\delta^2_M) = 4(n - 1) \mu_2,
\]

\[
(n - 1)^2 \mu'_2(\delta^2_M) = \frac{2(4n^2 - 7n + 2)}{n} \mu_4 - \frac{2(2n^2 - 9n + 6)}{n} \mu_2^2,
\]

\[
(n - 1)^3 \mu'_3(\delta^2_M) = \frac{2(8n^3 - 19n^2 + 12n - 4)}{n^2} \mu_6 - \frac{8(13n^3 - 31n^2 + 22n - 10)}{n^2} \mu_3^2
\]

\[
+ \frac{6(23n^2 - 44n + 20)}{n^2} \mu_4 \mu_2 - \frac{4(4n^3 + 41n^2 - 108n + 60)}{n^2} \mu_2^3,
\]

\[
(n - 1)^4 \mu'_4(\delta^2_M) = \frac{2(16n^4 - 47n^3 + 48n^2 - 32n + 8)}{n^3} \mu_8.
\]
The first four moments of $s^2$ about the mean are given by

$$
\begin{align*}
\mu'_1(s^2) &= 2 \mu_2, \\
\mu_2(s^2) &= \frac{2\mu_4}{n} - \frac{2(n-3)}{n(n-1)} \mu_2^2, \\
\mu_3(s^2) &= \frac{2\mu_6}{n^2} - \frac{4(3n^2 - 6n + 5)}{n^2(n-1)^2} \mu_3 - \frac{6(n-5)}{n^2(n-1)^2} \frac{\mu_4}{n} \mu_2 + \frac{4(n^2 - 12n + 15)}{n^2(n-1)^2} \mu_2^3, \\
\mu_4(s^2) &= \frac{2\mu_8}{n^3} - \frac{8(n-7)}{n^3(n-1)^2} \mu_4 \mu_2 - \frac{16(3n^2 - 6n + 7)}{n^3(n-1)^2} \frac{\mu_4}{n^2} \mu_3 \mu_2.
\end{align*}
$$
\[ + \frac{2(6n^4 - 21n^3 + 51n^2 - 63n + 35)}{n^3(n-1)^3} \mu_4 \]

\[ + \frac{24(-n^4 + 6n^3 - 28n^2 + 54n - 35)}{n^3(n-1)^3} \mu_4 \mu_2 \]

\[ + \frac{12(n^4 - 8n^3 + 54n^2 - 132n + 105)}{n^3(n-1)^3} \mu_4 \]

\[ + \frac{32(6n^3 - 27n^2 + 50n - 35)}{n^3(n-1)^3} \mu_3 \mu_1. \quad \ldots (4.2.5) \]

And the first four moments of \( \delta \) about the mean are given by

\[ (n-1) \mu_4(\delta^2) = 4(n-1) \mu_2, \]

\[ (n-1)^2 \mu_2(\delta^2) = 4(2n-3)\mu_4 + 4 \mu_2^2, \]

\[ (n-1)^3 \mu_3(\delta^2) = 4(4n-7)\mu_6 - 16(7n-13)\mu_3^2 + 12(4n-5)\mu_4 \mu_2 - 8(8n-11)\mu_2^3, \]

\[ (n-1)^4 \mu_4(\delta^2) = 4(8n-15)\mu_8 + 16(16n-27)\mu_4 \mu_2 - 64(16n-33)\mu_5 \mu_3 + 4(48n^2 - 64n - 47)\mu_4^2 - 144(8n-15)\mu_4 \mu_2^2 \]

\[ + 24(28n-51)\mu_2^4 + 64(25n-154)\mu_3^2 \mu_1. \quad \ldots (4.2.6) \]
Relative efficiency of estimators:

Relative efficiency (R.E.) of \( \frac{\delta}{{\bar{\Omega}}} \) to \( s^2 \) is defined by the ratio of variance of \( s^2 \) to variance of \( \frac{\delta}{{\bar{\Omega}}} \) and is equal to

\[
\frac{4(n-1) \{ (n-1) \beta_2 - (n-3) \}}{[4n^2 - 7n + 2] \beta_2 - (2n^2 - 6n + 6)}
\]

... (4.2.7)

where kurtosis, \( \beta_2 \) refers to the population sampled. R.E. increases with the increase in the value of \( \beta_2 \), for any fixed \( n \). But for a fixed \( \beta_2 \) below 5, it decreases and above 5 increases asymptotically to a limit. For \( \beta_2 = 5 \), the value remains more or less steady. It is to be noted that the R.E. tends to \( 2(\beta_2 - 1)/(2\beta_2 - 1) \), as \( n \) increases indefinitely.

R.E. of \( \frac{\delta}{{\bar{\Omega}}} \) to \( \frac{\delta}{{\bar{\Omega}}} \) is defined by the ratio of variance of \( \frac{\delta}{{\bar{\Omega}}} \) to variance of \( \frac{\delta}{{\bar{\Omega}}} \) and is equal to

\[
\frac{[(4n^2 - 7n + 2) \beta_2 - (2n^2 - 6n + 6)]}{[2n \{ (2n - 3) \beta_2 + 1 \}]}
\]

... (4.2.8)

It follows that for a fixed value of \( n \), the R.E. increases with \( \beta_2 \). The R.E. is more or less constant for \( \beta_2 = 5 \). But it decreases or increases to a limit asymptotically according as \( \beta_2 \) is less or greater than 5. It is seen here that the R.E. tends to \( (2\beta_2 - 1)/2\beta_2 \), as \( n \) tends to infinity.
We observe from the above results that R.E. of \( \hat{J}^2_M/2 \) compared to \( s^2 \) is greater than that of \( \hat{J}^2/2 \) to \( \hat{J}^2_M/2 \). The use of \( \hat{J}^2_M/2 \) as a measure of radial dispersion will not be seriously inefficient in situations where the sample is of moderate or large size and the parent population is leptokurtic. It is seen that R.E. is independent of skewness, \( \beta_1 \) and that for a highly leptokurtic population, the increase in the sample size has got little effect on the corresponding increase in the R.E. It is interesting to note here that the effect of trend values on bias to standard error of the statistic \( \hat{J}^2_M/2 \) compared to \( \hat{J}^2/2 \) is somewhat less than that of \( s^2/\sigma^2 \) as considered by many authors in similar situations.

**Exact distributions of** \( \delta^2_M \) **for** \( n = 2, 3 \):

(i) Let \((x_1, y_1), (x_2, y_2)\) be a sample sequence of two observations from an uncorrelated bivariate normal population with means zero and a common variance \( \sigma^2 \). The frequency function of \( \delta^2_M \) is given by the following simple form

\[
f(\delta^2_M) = \left(\frac{1}{4\sigma^2}\right) e^{-\left(\frac{\delta^2_M}{4\sigma^2}\right)}.
\]

... (4.2.9)

It is worth-noting that the definition of \( \delta^2_M \) holds for \( n \) greater than 2. For \( n \) equal to 2, it reduces to that of \( s^2 \).

(ii) Let \((x_1, y_1), (x_2, y_2), (x_3, y_3)\) be a sample sequence of three observations from an uncorrelated bivariate
normal population with means zero and a common variance \( \sigma^2 \). The characteristic function of \( \delta_M^2 \) is, then, given by

\[
\phi(t) = \frac{1}{(1-2\sigma^2 it) \sqrt{(1-\sigma^2 it)(1-3\sigma^2 it)}}
\]

and the probability density function of \( \delta_M^2 \) is given by

\[
f(\delta_M^2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-it\delta_M^2}}{(1-2\sigma^2 it) \sqrt{(1-\sigma^2 it)(1-3\sigma^2 it)}} dt.
\]

Applying contour integration, we find

\[
f(\delta_M^2) = e^{-\frac{2\delta_M^2}{3\sigma^2} - \frac{\delta_M^2}{2\sigma^2}} \sum_{r=0}^{\infty} \sum_{k=1}^{\infty} \frac{(2r)!}{(r!)^2} \frac{(\delta_M^2/6\sigma^2)^{2r+k}}{(2r+k)!}.
\]

An alternative form of \( f(\delta_M^2) \) involving terms in Laguerre polynomials (see Rainville (1965), Section 115, formula (3)) is given by

\[
f(\delta_M^2) = \frac{e^{-\frac{2\delta_M^2}{2\sigma^2} - \frac{\delta_M^2}{2\sigma^2}}}{(2\sigma^2)} \sum_{k=0}^{\infty} \frac{(2k)!}{2^{4k} (k!)^2} L_{2k}(\frac{\delta_M^2}{2\sigma^2}).
\]

where

\[
n! e^{-x} x^k L_n(x) = \frac{d^n}{dx^n} \left( e^{-x} x^{n+k} \right).
\]

From the expression of the characteristic function \( \phi(t) \) given in (4.2.10), it can be easily deduced that
\[ \mu'_s = 2^s s! \sum_{r=0}^{s} 2^r \left\{ \sum_{k=0}^{s-r} \frac{(1/2)^k (1/2)^{s-r-k}}{k! (s-r-k)!} \right\} \] ... (4.2.15)

where \((a)_k = a(a+1) \ldots (a+k-1)\).

From the expression of \(f(\delta^2_M)\) given in (4.2.12), we find

\[ \mu'_s = \frac{3^{3/2} (3s^2)^s}{2^{s+1}} \sum_{r=0}^{\infty} \sum_{k=1}^{\infty} \frac{(2r)!}{(r!)^2} \frac{(2r+k+s)!}{(2r+k)!} \frac{1}{4^{2r+k}} \] ... (4.2.16)

And from the expression of \(f(\delta^2_M)\) given in (4.2.13), we get

\[ \mu'_s = (2\pi^2)^s \frac{s!}{(s+1)!} \sum_{k=0}^{\lfloor s/2 \rfloor} \frac{1}{2^{4k} (k!)^2 (2k+1) (s-2k)!} \] ... (4.2.17)

where \([s/2]\) denotes the greatest integer contained in it.

**Approximations to the distribution law of \(\delta^2_M\):**

As the exact distribution law of \(\delta^2_M\) for an arbitrary value of \(n\) could not be obtained, so an approximation to it may be attempted by one of the following well-known methods:

(a) Fitting of the Charlier differential series,

(b) Fitting of a series involving terms in Laguerre polynomials,

(c) The Toeplitz approximation,

(d) Approximation by the method of minimum distance.
The distribution of \( \delta_M^2 \) is positively skewed and leptokurtic for moderate sample sizes from an uncorrelated bivariate normal population. To obtain the approximate normal theory significant points of \( \delta_M^2 \) for the various sample sizes the following two methods have been used:

(a) the Cornish-Fisher's technique

and

(b) the Log-normal transformation of \( \delta_M^2 \).

The approximate upper 5 per cent and 1 per cent (in parenthesis) normal theory significant points of \( (\delta_M^2/2\sigma^2) \) are given in the table IV.4 below.

**Table IV.4**

The upper 5 per cent and 1 per cent normal theory significant points of \( (\delta_M^2/2\sigma^2) \)

<table>
<thead>
<tr>
<th>Sample size</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>75</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cornish-Fisher's technique</td>
<td>3.36</td>
<td>3.07</td>
<td>2.91</td>
<td>2.73</td>
<td>2.63</td>
<td>2.55</td>
<td>2.45</td>
<td>2.38</td>
</tr>
<tr>
<td>(4.16)</td>
<td>(3.67)</td>
<td>(3.40)</td>
<td>(3.10)</td>
<td>(2.94)</td>
<td>(2.83)</td>
<td>(2.66)</td>
<td>(2.56)</td>
<td></td>
</tr>
<tr>
<td>Log-normal transformation</td>
<td>3.38</td>
<td>3.09</td>
<td>2.93</td>
<td>2.75</td>
<td>2.64</td>
<td>2.57</td>
<td>2.46</td>
<td>2.39</td>
</tr>
<tr>
<td>(4.31)</td>
<td>(3.77)</td>
<td>(3.45)</td>
<td>(3.16)</td>
<td>(2.93)</td>
<td>(2.86)</td>
<td>(2.68)</td>
<td>(2.58)</td>
<td></td>
</tr>
</tbody>
</table>
4.3 On the distribution and moments of radial error:

Introduction:

The distribution of radial error has been considered by many authors because of its importance in the field of ballistics and other technological problems. Here we derive the probability density function, the cumulative distribution function and moments of radial error for the bivariate normal distribution. Certain particular cases are obtained.

Let \( X_1 \) and \( X_2 \) follow bivariate normal distribution with means \( \mu_{11} \) and \( \mu_{22} \), variances \( \sigma_1 \) and \( \sigma_2 \) respectively and covariance \( \sigma_{12} \). The radial error is defined by

\[
R = \sqrt{(X_1^2 + X_2^2)}.
\]

Using the transformation

\[
x_1 = X_1 \cos \theta + X_2 \sin \theta,
\]
\[
x_2 = X_1 \sin \theta - X_2 \cos \theta,
\]

we get \( \sqrt{(x_1^2 + x_2^2)} = R = \sqrt{(x_1^2 + x_2^2)} \), where \( x_1 \) and \( x_2 \) are normally distributed. For \( \theta = \tan^{-1} \left( \frac{\sigma_{12}}{\sigma_{11} - \sigma_{22}} \right) \) they are also independently distributed with means

\[
\mu_1 = \mu_{11} \cos \theta + \mu_{22} \sin \theta,
\]
\[
\mu_2 = \mu_{11} \sin \theta - \mu_{22} \cos \theta
\]

... (4.3.3)
and variances

\[
\sigma_t^2 = \frac{1}{2} \left[ (\sigma_{11} + \sigma_{22}) + \left\{ (\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2 \right\}^{1/2} \right],
\]

\[
= \frac{1}{2} \left[ (\sigma_{11} + \sigma_{22}) - \left\{ (\sigma_{11} - \sigma_{22})^2 + 4\sigma_{12}^2 \right\}^{1/2} \right].
\]

(4.3.4)

**Derivation of probability density function of \( R \):**

The characteristic function of \( R^2 \) is given by

\[
\phi(t) = e^{\frac{1}{2} \left( \mu_1^2/\sigma_1^2 + \mu_2^2/\sigma_2^2 \right)} \left[ \frac{\mu_1^2}{2\sigma_1^2(1-2it\sigma_1^2)} + \frac{\mu_2^2}{2\sigma_2^2(1-2it\sigma_2^2)} \right]\times
\sqrt{\frac{1}{(1-2it\sigma_1^2)(1-2it\sigma_2^2)}}.
\]

(4.3.5)

The frequency function of \( R^2 \) is

\[
\text{Prob}\{ x \leq R^2 \leq x + dx \} = f_{R^2}(x) \, dx.
\]

Assuming that the term by term integration is valid, we get

\[
f_{R^2}(x) = e^{-a^2/2} \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} A(r_1,r_2) \, I(x,r_1,r_2) \quad \ldots \quad (4.3.6)
\]

where \( a^2 = \mu_1^2/\sigma_1^2 + \mu_2^2/\sigma_2^2 \),

(4.3.7)

\[
A(r_1,r_2) = \frac{(\mu_1^2/2\sigma_1^2)^{r_1-r_2}(\mu_2^2/2\sigma_2^2)^{r_2}}{(r_1-r_2)! \, r_2!}
\]

(4.3.8)
\[ I(x; r_1, r_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-itx}}{(1-2it\sigma_1^2)^{r_1-r_2+1/2} (1-2it\sigma_2^2)^{r_2+1/2}} \, dt. \]  

... (4.3.9)

Assuming \( \sigma_2^2 < \sigma_1^2 \), we have (see Erdélyi (1954), Section 3.2, formula (10))

\[ I(x; r_1, r_2) = \frac{1}{(2\sigma_1^2)^{r_1-r_2+1/2} (2\sigma_2^2)^{r_2+1/2}} e^\frac{x}{2\sigma_2^2} \Gamma^\frac{x^2}{2\sigma_1^2} \frac{\sigma_2^2 - \sigma_1^2}{2\sigma_1^2 \sigma_2^2} \, x \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) 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Particular cases:

(i) When $\mu_1 = \mu_2 = 0$ which implies $\mu_{11} = \mu_{22} = 0$ and using Kummer's second formula (see Rainville (1965), Section 70, formula (9))

$$e^{-z} F_1(\lambda; 2\lambda; 2z) = \Phi(-; \lambda+1/2; z^2/4),$$

provided $2\lambda$ is not an odd integer $< 0$, we get

$$f(R) = \left(\frac{R}{\sigma_1^2}\right) e^{-\frac{1}{2}R^2/4\sigma_1^2} I_0(\beta R^2/4\sigma_1^2) \quad (0 \leq R < \infty)$$

... (4.3.14)

where $I_0$ is the Modified Bessel function of the first kind and order zero (see Rainville (1965), Section 65, formula (1) ) and $\delta = (\lambda-2)/(\lambda-1)$.

(ii) When $\sigma_1 = \sigma_2 = \sigma$ which implies $\sigma_{12} = 0$,

$$\sigma_{11} = \sigma_{22} = \sigma^2,$$

we have

$$f(R) = \left(\frac{R}{\sigma^2}\right) e^{-\frac{1}{2}(a^2+R^2)/\sigma^2} I_0(aR/\sigma) \quad (0 \leq R < \infty)$$

... (4.3.15)

where $a = \sqrt{(\mu_1^2+\mu_2^2)/\sigma}$.

(iii) When $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2 = \sigma$, we have

$$f(R) = \left(\frac{R}{\sigma^2}\right) e^{-R^2/2\sigma^2} \quad (0 \leq R < \infty).$$

... (4.3.16)

The expressions in (4.3.14-16) are found in various literature.
Method of evaluation of cumulative distribution function of $R$:

The cumulative distribution function of $R$ is given by

$$F(\sqrt{2} \sigma R_0) = \text{Prob} \{0 \leq R \leq \sqrt{2} \sigma R_0\} = \int_0^{\sqrt{2} \sigma R_0} f_R(x) \, dx$$

where $R_0$ is a particular value of $R$.

$$= e^{-a^2/2} \sum_{k=0}^{\infty} \sum_{\nu=0}^{r_1} A(r_1, r_2) \gamma^{-(r_2+1/2)} I(r_2, r_1, \beta, R_0^2)$$

where

$$I(r_2, r_1, \beta, R_0^2) = \int_0^{r_1} \frac{x}{\Gamma(r_1+1)} \gamma^{-(r_2+1/2)} F_1(r_2+1/2; r_1+1; \beta R_0^2) x^{r_2} \, dx.$$  \hfill (4.3.17)

Integrating by parts, we get the following recurrence relation for all integral values of $r_1$ and $r_2$:

$$I(r_2, r_1, \beta, R_0^2) = - e^{-R_0^2/2} \frac{2r_1}{\Gamma(r_1+1)} I(r_2+1/2; r_1+1; \beta R_0^2) + I(r_2, r_1-1, \beta, R_0^2).$$  \hfill (4.3.18)

By repeated applications of the above relation for $r_2+1 \leq r_1$, we have

$$I(r_2, r_1, \beta, R_0^2) = - e^{-R_0^2/2} \sum_{i=0}^{r_1-1} \frac{2(r_1-1)}{\Gamma(r_1-1)} I(r_2+1/2; r_1-i+1; \beta R_0^2) +$$

$$I(r_2, r_2, \beta, R_0^2).$$  \hfill (4.3.20)
Now with the application of Kummer's first formula (see Rainville (1965), Section 69, formula (2))

\[ {}_1F_1(\lambda; \mu; z) = e^z {}_1F_1(\mu - \lambda; \mu; z), \]

we get

\[ I(r_2, r_2, \beta, R_0^2) = \gamma^{(r_2+1)}(r_2, \beta, R_0^2/\gamma) (r, 0, 0, \beta, R_0^2/\gamma). \]

... (4.3.21)

Hence using the similar recursion formula as given in (4.3.19) for \( 1 \leq r_2 \), we get

\[
I(r_2, r_2, \beta, R_0^2) = \gamma^{(r_2+1)}(r_2, \beta, R_0^2/\gamma) \left[ e^{-R_0^2/\gamma} \sum_{j=0}^{R_0^2/\gamma} \frac{R_0^2/\gamma}{(r_2-j+1)!} \right] .
\]

... (4.3.22)

Thus the only basic integral which is to be calculated before we proceed to compute the value of the cumulative distribution function in (4.3.17) is

\[
I(0, 0, 0, \gamma, R_0^2/\gamma) = \sum_{k=0}^{\infty} \left( \frac{\Gamma(2k)}{2^{2k-1} \Gamma(k+1)} \right)^k I(u, k) \quad \ldots \quad (4.3.23)
\]

where the incomplete gamma function, \( I(u, k) \) is given by

\[
I(u, k) = \int_0^{\frac{u \gamma}{\Gamma(k+1)}} e^{-v} v^k \frac{dv}{\Gamma(k+1)} \quad \text{and} \quad u = \frac{R_0^2/\gamma}{\Gamma(k+1)} .
\]

It may be pointed out that the computation of (4.3.17) involves incomplete gamma function, confluent hypergeometric function and
the series (4.3.23). But this may not be prohibitive when the needed values are available in the tables, since the series (4.3.23) converges very rapidly. Even for the values of the argument not covered in the tables, one could use a quick and fairly accurate method found in Grubbs (1964). The series (12) of the Gilliland (1962) paper is also quite easy to compute with. However if the needed values be not readily available, it may be difficult to compute (4.3.17), then its importance is only theoretical in nature.

**Error analysis:**

In order to compute the value of the cumulative distribution function, if the first \((n+1)\) terms of (4.3.17) are used, then an upper bound on the truncation error is obtained as follows:

\[
|R_n| = e^{-a^2/2} \sum_{i=n+1}^{\infty} \sum_{r=0}^{\gamma_i} A(r_1,r_2) \gamma_i (r_2 + 1/2) \left| I(r_2,r_1,\beta, R_0^2) \right|.
\]

... (4.3.24)

Now we have

\[
\left| I(r_2,r_1,\beta, R_0^2) \right| \leq (1 - e^{-R_0^2}) \frac{2r_1}{R_0} \frac{R_0}{\Gamma(r_1+1)} \left| F_1(r_2+1/2; r_1+1; |\beta| R_0^2) \right|
\]

... (4.3.25)

and for \(r_2 \leq r_1\)

\[
\left| F_1(r_2+1/2; r_1+1; |\beta| R_0^2) \right| \leq e^{\beta R_0^2}
\]

... (4.3.26)
Using (4.3.25) and (4.3.26) in (4.3.24), we get an upper bound on the truncation error as

\[ |R_n| \leq \frac{-\frac{1}{2}(a^2-2|\beta| R_0^2) - R_0^2}{\gamma^2} (1-e^{-\gamma}) \frac{W^{n+1}}{\Gamma(n+2)^2} \left[ 1 - \frac{W}{(n+2)^2} \right] \]

... (4.3.27)

where \( W = R_0^2(\mu_1^2/2\sigma_1^2 + \mu_2^2/2\sigma_2^2) \)

and \( 0 \leq \frac{W}{(n+2)^2} \leq 1 \).

To compare precisely the upper bound of (4.3.27) on the truncation error with that of (19) of the Gilliland (1962) paper, it may be worth while to note in actual practice how many summands make up the first \((n+1) r_{1\text{-terms}}\) of (4.3.17) and how many make up the first \((n+1) m\text{-terms}\) of the corresponding series (12) of the Gilliland (1962) paper.

**Moments:**

The \( s\text{-th} \) moment of \( R \) about the origin is given by

\[ \mu_s = e^{-a^2/2} \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{r_1} A(r_1, r_2) \gamma^{-(r_2+1/2)} \int_0^\infty x^{s/2} g(x/2\sigma_1^2) \, d(x/2\sigma_1^2) \]

... (4.3.28)

where \( g(x/2\sigma_1^2) \) is defined in (4.3.12).

Now we get (see Erdélyi (1954), Section 6.9, formula (9))
\[ \mu_s' = e^{-\frac{a^2}{2}} \left( \frac{2\pi^2}{2} \right)^{s/2} \sum_{\gamma_1=0}^{\infty} A(r_1, r_2) \frac{\Gamma(s/2+r_1+1)}{\Gamma(r_1+1)} \left( \frac{r_2+1/2}{r_2+1} \right)^{\gamma_1} \]  

\[ _2F_1(r_2+1/2, s/2+r_1+1; r_1+1; \beta) . \]  

... (4.3.29)

Using the following relation between two hypergeometric functions (see Rainville (1966), Section 38, formula (4))

\[ _2F_1\left(\lambda, \frac{s}{2}; 1-z \right) = (1-z)^{-\lambda} _2F_1\left(\lambda, s - \mu; -\frac{s}{2}; z/(1-z) \right), \]

we get

\[ \mu_s' = e^{-\frac{a^2}{2}} \left( \frac{2\pi^2}{2} \right)^{s/2} \sum_{\gamma_1=0}^{\infty} \sum_{\gamma_2=0}^{\infty} A(r_1, r_2) \frac{\Gamma(s/2+r_1+1)}{\Gamma(r_1+1)} \left( \frac{r_2+1/2}{r_2+1} \right)^{\gamma_1} \left( \frac{s/2+r_1+1}{s/2+r_1+1} \right)^{\gamma_2} \]  

\[ _2F_1(-s/2, r_2+1/2; r_1+1; \beta) . \]  

... (4.3.30)

An alternative method for the derivation of the above moment relation is as follows:

\[ \mu_s' = \frac{e^{-\frac{a^2}{2}}}{2\pi r_1 r_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_1^2+x_2^2)^{s/2} e^{-\frac{1}{2} (x_1^2/r_1^2 + x_2^2/r_2^2)} \cosh(x_1^2/M_1^2) \]  

\[ \cosh(x_2^2/M_2^2) \, dx_1 \, dx_2 , \]  

... (4.3.31)

since \( R \) is an even function of \( x_1 \) and \( x_2 \).

Let \( x_1 = \sigma_1 \, v \cos \phi \)

\[ x_2 = \sigma_2 \, v \sin \phi . \]
Then we get

\[ \mathcal{M} = e^{-a^2/2} \sigma_i \frac{s}{2\pi} \int_0^\infty \int_0^\infty e^{-v^2/2} v^{s+1}(1 - \sin^2 \phi)^{s/2} \cosh(\frac{\nu \mu_i \cos \phi}{\sigma_i}) \cos(h(\nu \mu_i \sin \phi)) \, dv \, d\phi \]

\[ = e^{-a^2/2} \sigma_i \sum_{r=0}^\infty \int_0^\infty e^{-v^2/2} v^{s+2r_1+1} dv \sum_{r_2=0}^\infty \left[ \frac{(\nu/\sigma_i)^{2r_1-2r_2}(\mu_i/\sigma_i)^{2r_2}}{(2r_1-2r_2)! (2r_2)!} \right] \]

\[ \sum_{t=0}^\infty \binom{s/2}{t} (-x)^t \frac{1}{2\pi} \int_0^{2\pi} \sin^{2t+2r_2} \cos^{2r_1-2r_2} \, d\phi \]

\[ = e^{-a^2/2} \sigma_i \sum_{r=0}^\infty \Gamma(s/2+r_1+1) 2^{s/2+r_1} \sum_{r_2=0}^\infty \left[ \frac{(\nu/\sigma_i)^{2r_1-2r_2}(\mu_i/\sigma_i)^{2r_2}}{(2r_1-2r_2)! (2r_2)!} \right] \]

\[ \sum_{t=0}^\infty \binom{s/2}{t} (-x)^t \frac{1}{\pi} \frac{\Gamma(t+r_2+1/2) \Gamma(r_1-r_2+1/2)}{\Gamma(t+r_1+1)} . \]

... (4.3.32)

Now using Legendre's duplication formula for gamma function ,

we get the expression in (4.3.30).

**Particular cases:**

(i) When \( \mu_1 = \mu_2 = 0 \), we have

\[ \mathcal{M}' = (2\sigma_i^2)^{s/2} \Gamma(s/2+1) \betaF{-s/2, 1/2; 1; x}. \quad \ldots \quad (4.3.33) \]
(ii) When \( \sigma_1 = \sigma_2 = \sigma \), we get

\[
\lambda'_s = e^{-(\mu_1^2 + \mu_2^2)/2\sigma^2} (2\sigma^2)^{s/2} \Gamma(s/2+1) \; \text{I}_F(s/2+1; 1, (\mu_1^2 + \mu_2^2)/2\sigma^2).
\]

... (4.3.34)

The result of (4.3.33) is reported by Scheuer (1962) and that of (4.3.34) is a particular case of widely discussed moment relation of non-central chi-square.