CHAPTER 6

HIGHLY NONLINEAR ION ACOUSTIC WEAKLY RELATIVISTIC SOLITONS
IN MAGNETOPLASMA

(In communication)

6.1 Introduction

The existence of solitary waves under different physical situations in magnetized or unmagnetized plasmas have been investigated theoretically and experimentally by several authors using mostly reductive perturbation method. However, the fully nonlinear ion acoustic solitary waves in an unmagnetized plasma with cold ions have been investigated by Sagdeev (1966) without using the reductive perturbation technique. It is worthy to mention that the studies of solitary waves are greatly indebted to the works of Korteweg-de-Vries (1895) and Washimi and Taniuti (1966). Besides the usual ion-electron in the plasma, studies of ion acoustic waves in multispecies plasmas with negative ions, ion-electron beams, isothermal and non-isothermal electrons, electrons with various temperatures etc. are undertaken by many authors of the world. Davidson (1972), Tappert (1972) and Tagare (1973) have investigated ion-acoustic solitons in a simple composition of ion-electron plasma. In presence of negative ions, existence of solitary waves has been investigated by Das (1979), Watanabe (1984), Verheest (1988), Baboolal et al. (1989), Kalita and Kalita (1990), Kalita and Das (1998, 2002) besides others. Tagare (1986), Tagare and Reddy (1987) have also studied solitary waves in
presence of negative ions through the mKdV equations. These works were supplemented by experiments (not reported here) by many workers.

Shukla and Yu (1978) have shown the existence of finite amplitude ion acoustic solitons propagating obliquely to the external magnetic field with a density hump for $M > k_z$. Yu et al. (1980) have derived stationary soliton solutions of the complete system of equations of motion, for plane ion acoustic waves which propagate at an angle with respect to the magnetic field. Kalita et al. (1986) have investigated the drifting effect of electrons on fully non-linear ion acoustic waves in a magnetoplasma. They observed that the ion acoustic solitons are affected by the drift motion of the electrons along the direction of the magnetic field. Sen and Chatterjee (2006) have investigated large amplitude solitary waves in ion beam plasma (non-relativistic) with finite ion and ion-beam temperatures. In their investigation, they found that there exists a critical value of initial ion speed $u_0$, the value of $u$ at which $(u')^2 = 0$, beyond which the solitary waves cease to exist. This critical value is extremely sensitive to the parameters like soliton velocity, ion temperature, initial ion or ion-beam density. Recently, Abdelsalam et al. (2008) have investigated the fully non-linear propagation of ion acoustic solitary waves in collisionless dense / quantum electron – positron – ion plasma. They have obtained numerically finite amplitude solutions. Yinhua and Yu (1994) have investigated the fully non-linear ion acoustic solitary waves propagating obliquely to the external magnetic field in an impurity containing magnetized plasma. Chatterjee et al. (2009) have investigated IASWs and double layers in a two component dense magnetoplasma using Sagdeev’s pseudo-potential approach where Thomas-Fermi density distribution for
electrons is used. Besides, many workers have studied solitary waves in plasma in variety of ways which are abundant elsewhere.

The present trend of research to its end tends to include relativistic effect in some of the components and dust particles in the plasma. A lot of investigations have already been done with dust particles in the plasma (reported elsewhere) but in most of the investigations non-relativistic situations are adopted.

Relativistic effects play an important role in the formation of solitary waves when the speeds of particles are comparable to those of light. For example, ions with very high speed are frequently observed in the solar atmosphere and interplanetary space. Many workers like Chain and Clemmow (1975), Shukla et al. (1984) and Arons (1979) have investigated non-linear relativistic plasma waves in laser-plasma interaction and Astronomical models etc. Das and Paul (1985) have discussed relativistic solitons in a simple model of ion-electron plasma which based on initial streamings. Roychoudhury and Bhattacharyya (1988) first have used a non-perturbation technique to obtain large amplitude solitary wave solution in a relativistic plasma. Chatterjee and Roychoudhury (1994) have studied the effect of ion temperature in a relativistic plasma using Sagdeev's pseudo-potential approach with the consideration of electron inertia. They have shown that the ion temperature puts a restriction on the values of $V$, the soliton velocity. Roychoudhury et al. (1997) have studied the effect of ion and electron drifts on the existence of solitary waves using Sagdeev's pseudopotential approach in the relativistic plasma. Esfandyari et al. (2001) have investigated the effect of ion temperature and relativistic electron beam density on ion-acoustic solitons in a collisionless plasma consisting of warm ions. Singh et al. (2005) have established small amplitude relativistic
solitons effective for electron inertia and weak relativistic effect. In their investigation, presence of electron inertia is neglected and the soliton existence range between the electron-ion speeds is shown to contract under constant plasma pressure. Large amplitude solitary waves are investigated in a relativistic plasmas with finite ion-temperature with electron inertia by Das and Chatterjee (2006). In their investigation, they have found that there exists a critical value of $u_0$, the value of $u$ at which $(u')^2 = 0$, beyond which the solitary waves cease to exist.

Very recently, Kalita and Das (2007) have investigated the higher and smaller order relativistic effects in the generation of compressive solitons of high amplitudes in a defined range $(u_0 - v_0)$ and rarefactive solitons of pretty small amplitudes in the small upper range of $|u_0 - v_0|$ subject to a sound mathematical condition; $u_0, v_0$ being the initial streaming speeds of electrons and ions respectively. It is reported to reasonably justify consideration of electron inertia in the plasma amenable to higher order relativistic effect.

In this Chapter, we investigate analytically the planar ion acoustic solitary waves in a magnetized cold weakly relativistic plasma with electron inertia. In this relativistic consideration, the energy integral unlike others has been derived in a weakly relativistic plasma in terms of Sagdeev potential. Both compressive and rarefactive subsonic solitary waves of interesting characteristics are shown to exist.
6.2 Dynamics of the motion

We consider a magnetized plasma consisting of unidirectional weakly relativistic ions and highly magnetized electrons of constant temperature $T_e$. The equations governing the dynamics of motion of such a plasma in the $zx$-plane are

\[
\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x} (n_i v_{ix}) + \frac{\partial}{\partial z} (n_i v_{iz}) = 0 \quad (6.1)
\]

\[
\left( \frac{\partial}{\partial t} + v_{ix} \frac{\partial}{\partial x} + v_{iz} \frac{\partial}{\partial z} \right) v_{ix} = -\frac{\partial \phi}{\partial x} + v_y \quad (6.2)
\]

\[
\left( \frac{\partial}{\partial t} + v_{ix} \frac{\partial}{\partial x} + v_{iz} \frac{\partial}{\partial z} \right) v_{iz} = -v_{ix} \quad (6.3)
\]

\[
\left( \frac{\partial}{\partial t} + v_{ix} \frac{\partial}{\partial x} + v_{iz} \frac{\partial}{\partial z} \right) v_y = -\frac{\partial \phi}{\partial z} \quad (6.4)
\]

for the ions and

\[
\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial z} (n_e v_{ez}) = 0 \quad (6.5)
\]

\[
\left( \frac{\partial}{\partial t} + v_{ez} \frac{\partial}{\partial z} \right) v_{ez} = \frac{1}{Q} \left( \frac{\partial \phi}{\partial z} - \frac{1}{n_e} \frac{\partial n_e}{\partial z} \right) \quad (6.6)
\]

for the electrons, where

\[
\gamma = \left[ 1 - \left( \frac{v_{ix}}{c} \right)^2 \right]^{-\frac{1}{2}} = 1 + \frac{v_{ix}^2}{2c^2}, \quad Q = \frac{m_e}{m_i}
\]

is the electron to ion mass ratio and $c$ is the speed of light. In obtaining the set of equations (6.1) to (6.6), we have normalized the densities by the equilibrium plasma density $n_0$, time by the inverse of the ion gyrofrequency $\Omega_i$, space by the ion gyroradius $\rho_s = \frac{C_s}{\Omega_i}$ and so speed by
C_s = \left( \frac{T_e}{m_i} \right)^{\frac{1}{2}} \] and potential by \( \frac{T_e}{e} \). Magnetized plasmas are anisotropic i.e., the properties parallel to the magnetic field are quite different from those in directions perpendicular to it. Further, in the perpendicular direction of motion, relativistically length (so the space) is not contracted (otherwise doesn’t change) and the proper time of the rest frame can be practically identical to time \( t' \) of the moving frame. Choosing \( \frac{v_y}{c} \) and \( \frac{v_z}{c} \) to be very small inherent early beginning of submission to propagation of solitary waves in one direction with constant speed will be rather more feasible along which, length contraction is mathematically justified. So, the components of velocity namely \( \frac{v_y}{c} \) and \( \frac{v_z}{c} \) in \( \gamma \) can be ignored. Besides, this approximation of proper time in perpendicular direction of motion during collisional stage before arriving at collisionless stage is always possible. Therefore, the relativistic effect considered in \( x \)-direction justifies ignorable effects in the perpendicular directions. Moreover, due to the above non-contraction of length in perpendicular direction of motion and higher thermal velocity \( \frac{v_{th}}{v_{eh}} \) for higher \( T_i \) in relativistic sense, than usual for high conductivity, unidirectional motion of the electrons is justified.

For a stationary solution, we consider a frame moving with the wave defined by

\[ \xi = k_x x + k_z z - Mt \] (6.7)
where \( M = \text{Mach number} \) (= \( V / C \), = pulse speed / ion sound speed) and \( k_x, k_z \) are the direction cosines such that \( k_x^2 + k_z^2 = 1 \). For the moving co-ordinate \( \xi \), we can write from (6.7)

\[
\frac{\partial}{\partial x} = k_x \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial z} = k_z \frac{\partial}{\partial \xi} \quad \text{and} \quad \frac{\partial}{\partial t} = -M \frac{\partial}{\partial \xi}
\]

Introducing the new coordinate \( \xi \) defined in (6.7), and using the boundary conditions \( v_\alpha = v_e = 0 \) at \( n = 1 \) as \( |\xi| \to \infty \) after integration, equation (6.1) reduces to

\[
k_x v_\alpha + k_z v_e = M \left( 1 - \frac{1}{n} \right)
\]

(6.8)

Using (6.7) and (6.8) equations (6.2) to (6.4) can be simplified to give

\[
\frac{M}{n_e} \frac{\partial}{\partial \xi} (v_\alpha v_e) = k_x \frac{\partial \phi}{\partial \xi} - v_\alpha
\]

(6.9)

\[
\frac{M}{n_e} \frac{\partial v_\alpha}{\partial \xi} = v_\alpha
\]

(6.10)

\[
\frac{M}{n_e} \frac{\partial v_e}{\partial \xi} = k_z \frac{\partial \phi}{\partial \xi}
\]

(6.11)

Again making use of (6.7) in (6.5) and (6.6) and integrating once, we get

\[
v_e = \frac{M}{k_z} \left( 1 - \frac{1}{n_e} \right)
\]

(6.12)

In deducing equation (6.12), we have used the boundary conditions \( v_e = 0 \), at \( n_e = 1 \) as \( |\xi| \to \infty \).

Employing the co-ordinate \( \xi \) and using (6.12), the equation (6.6) can be once integrated to give under the boundary conditions \( \phi = 0 \), at \( n_e = 1 \) as \( |\xi| \to \infty \).
\( n_e = e^t \exp \left[ \frac{Q M^2}{2 k_x^2} \left( 1 - \frac{1}{n_e^2} \right) \right] \)  

(6.13)

Making use of (6.13) and the charge neutrality condition \( n_i = n_e = n \), equation (6.11) can be integrated once to yield

\[
v_r = \frac{k_x}{M} (n-1) \left( 1 - \frac{Q M^2}{k_x^2} \right) \frac{1}{n} \]

(6.14)

With the use of (6.14), we can get from (6.8)

\[
v_a = \frac{M}{k_x} (n-1) \left( 1 - \frac{k_x^2}{M^2} \right) \left( 1 - \frac{Q M^2}{k_x^2} \right) \frac{1}{n} \]

(6.15)

Putting the value of \( v_a \) from (6.15) in (6.9), \( v_y \) can be determined as

\[
v_y = f(n) \frac{1}{n} \frac{dn}{d} \frac{dn}{d} \]

(6.16)

where \( f(n) = \frac{A}{n^4} + \frac{B}{n^3} + \frac{C}{n^2} + D + E n + F n^2 \) with

\[
A = -\frac{3 M^4 (1 + Q)^3}{2 c^2 k_x^3} , \quad B = \frac{3 M^4 (1 + Q)^3}{c^2 k_x^3} + \frac{3 M^2 (1 + Q)^2 k_x^2}{c^2 k_x^3} ,
\]

\[
C = \left\{ \frac{M^2 (1 + Q)}{k_x} + \frac{3 M^4 (1 + Q)^3}{2 c^2 k_x^3} + \frac{9 M^2 (1 + Q)^2 k_x^2}{2 c^2 k_x^3} + \frac{3 (1 + Q) k_x^4}{2 c^2 k_x^3} + \frac{Q M^2 k_x^2}{k_x^2} \right\} ,
\]

\[
D = \frac{1}{k_x} + \frac{3 M^2 (1 + Q)^2 k_x^2}{2 c^2 k_x^3} + \frac{9 (1 + Q) k_x^4}{2 c^2 k_x^3} + \frac{3 k_x^6}{2 c^2 k_x^3 M^2} ,
\]

\[
E = -\frac{3 (1 + Q) k_x^4}{c^2 k_x^3} - \frac{3 k_x^6}{c^2 k_x^3 M^2} , \quad F = \frac{3 k_x^6}{2 c^2 k_x^3 M^2} .
\]
In deducing (6.16), we have used the equation (6.13). With the values of \( v_x \) and \( v_y \) from (6.15) and (6.16) respectively, one can obtain from (6.10) the following expression

\[
\frac{d}{d\xi} \left\{ f(n) \frac{1}{n} \frac{dn}{d\xi} \right\} = \frac{1}{k_x} (n-1) \left( 1 + Q - \frac{k_x^2}{M^2} n \right)
\]  

(6.17)

Multiplying both sides of (6.17) by the term in the parenthesis, it can be integrated to recover the following energy integral for the classical particles with Sagdeev potential \( \psi \)

\[
\frac{1}{2} \left( \frac{dn}{d\xi} \right)^2 + \psi(n, M, k_x) = 0
\]  

(6.18)

where

\[
\psi(n, M, k_x) = \lambda(n) \mu(n)
\]  

(6.19)

with \( \lambda(n) = \frac{n^2}{f(n)^2} \)  

(6.20)

and

\[
\mu(n) = -\frac{1}{k_x} \left[ \frac{A(1 + Q)}{4} \left( \frac{1}{n^4} - 1 \right) - \frac{1}{3} \left( 1 + Q + \frac{k_x^2}{M^2} \right) A - B(1 + Q) \left( \frac{1}{n^3} - 1 \right) \right.

- \frac{1}{2} \left( 1 + Q + \frac{k_x^2}{M^2} \right) B - C(1 + Q) - \frac{Ak_x^2}{M^2} \left( \frac{1}{n^2} - 1 \right) - \left( 1 + Q + \frac{k_x^2}{M^2} \right) C - \frac{Bk_x^2}{M^2} \left( \frac{1}{n} - 1 \right) - D(1 + Q) + \frac{Ck_x^2}{M^2} \right] \log n

+ \left( 1 + Q + \frac{k_x^2}{M^2} \right) D - E(1 + Q) \right] (n-1)
\]
and the boundary condition \( \frac{dn}{d\xi} = 0 \) at \( n = 1 \) has been used.

### 6.3 Conditions for the existence of solitary waves

The necessary conditions for the existence of localized solitary wave solutions can be obtained by studying the behaviour of \( \psi(n) \) near \( n = 1 \) and \( n = N \), where \( N \) is the maximum value of \( n \) i.e., the amplitude of the solitary wave pulse. For nonlinear dispersion relation, we are to set \( \psi(N) = 0 \) to yield the amplitude 'N' of the solitary wave pulse so that

\[
+ \frac{1}{2} \left( 1 + Q + \frac{k_z^2}{M^2} \right) E - F(1 + Q) - \frac{Dk_z^2}{M^2} \left( n^2 - 1 \right) \\
+ \frac{1}{3} \left( 1 + Q + \frac{k_z^2}{M^2} \right) F - \frac{Ek_z^2}{M^2} \left( n^3 - 1 \right) - \frac{Fk_z^2}{4M^2} \left( n^4 - 1 \right)
\]

(6.21)
\[ + \frac{1}{3} \left\{ \left( 1 + Q + \frac{k_z^2}{M^2} \right) F - \frac{E_{k_z^2}}{M^2} \right\} (N^3 - 1) - \frac{F_{k_z^2}}{4M^2} (N^4 - 1) = 0 \]  
(6.22)

Besides, the required conditions for the existence of solitary waves are

\[ \psi(1) = \psi(N) = \psi'(1) = 0 \]  
(6.23)

and \[ \psi(n) < 0 \]  
(6.24)

between \( n = 1 \) and \( n = N \).

Now, to arrive at the mathematical conditions, we consider

\[ \mu'(n) = -\frac{1}{k_x} \left[ \frac{5A(1+Q)}{n^6} - 4 \left\{ \left( 1 + Q + \frac{k_z^2}{M^2} \right) A - B(1+Q) \right\} \frac{1}{n^5} \right. \]
\[ + \left\{ \left( 1 + Q + \frac{k_z^2}{M^2} \right) B - C(1+Q) - \frac{Ak_z^2}{M^2} \right\} \frac{1}{n^4} \]
\[ + \left\{ \left( 1 + Q + \frac{k_z^2}{M^2} \right) C - \frac{Bk_z^2}{M^2} \right\} \frac{1}{n^3} - \left\{ (1+Q)D + \frac{Ck_z^2}{M^2} \right\} \frac{1}{n^2} \]
\[ + \left\{ (1+Q)E - F(1+Q) - \frac{Dk_z^2}{M^2} \right\} \frac{1}{n} \]
\[ + \left\{ (1+Q)F - \frac{Ek_z^2}{M^2} \right\} n^2 - \frac{Fk_z^2}{M^2} n^3 \]  
(6.25)

and

\[ \mu''(n) = -\frac{1}{k_x} \left[ \frac{5A(1+Q)}{n^6} - 4 \left\{ \left( 1 + Q + \frac{k_z^2}{M^2} \right) A - B(1+Q) \right\} \frac{1}{n^5} \right. \]
\[ - 3 \left\{ \left( 1 + Q + \frac{k_z^2}{M^2} \right) B - C(1+Q) - \frac{Ak_z^2}{M^2} \right\} \frac{1}{n^4} \]
\[
- \left( \frac{1 + Q + \frac{k^2_z}{M^2}}{1 + Q + \frac{k^2_z}{M^2}} \right) C - \frac{Bk_z^2}{M^2} \left\{ 1 + \frac{Q + \frac{k^2_z}{M^2}}{1 + Q + \frac{k^2_z}{M^2}} \right\} \left( \frac{1}{n^3} + \frac{1}{n^2} \right) + \left\{ 1 + Q + \frac{Ck_z^2}{M^2} \right\} \left( \frac{1}{n^3} \right) \\
+ \left[ \left( 1 + Q + \frac{k^2_z}{M^2} \right) E - F(1 + Q) - \frac{Dk_z^2}{M^2} \right] \\
+ 2 \left[ \left( 1 + Q + \frac{k^2_z}{M^2} \right) F - \frac{Ek_z^2}{M^2} \left( n - 3 \frac{Fk_z^2}{M^2} n^2 \right) \right] \tag{6.26}
\]

where prime denotes differentiation with respect to \( n \).

It is seen from equations (6.21), (6.25) and (6.26) that at \( n = 1 \)
\[
\mu(1) = 0, \mu'(1) = 0 \text{ and} \\
\mu''(1) = \frac{1 + Q - \frac{k^2_z}{M^2}}{M^2} \left[ Q + k^2_z - \frac{k^2_z}{M^2} \right] \frac{M^2}{k^2_z k^2_z} \tag{6.27}
\]

Using these values and equation (6.20) at \( n = 1 \), we get
\[
\psi(1) = 0, \psi'(1) = 0 \text{ and} \\
\psi''(1) = \frac{k^2_z \left( 1 + Q - \frac{k^2_z}{M^2} \right)}{M^2 \left[ Q + k^2_z - \frac{k^2_z}{M^2} \right]} \tag{6.28}
\]

The nonlinear dispersion relation (6.22) is deduced by setting
\[
\psi(N) = \lambda(N) \mu(N) = 0 \text{ for which } \mu(N) = 0, \text{ since } \lambda(N) \neq 0 \text{ so that} \\
\mu'(N) = - \left( \frac{N - 1}{k_x} \right) \left[ 1 + Q - \frac{k^2_z}{M^2} \right] \frac{3(N - 1)^2}{2c^2 k_x} \frac{1}{M^2 N^4} \left\{ N^2 k^2_z - (1 + Q) M^2 \right\} \\
\left\{ Nk_z^2 - (1 + Q)M^2 \right\} \right) + \left( \frac{N^2 - M^2}{{k}_z M^2} \frac{k_z^2 - QM^2}{N^2 k_z M^2} \right]
\]
and

\[ \psi'(n) = \frac{2c^2k_z^2N^5(N-1)[Nk_z^2-(1+Q)M^2]}{3k_z^2(N-1)^2\{N^2k_z^2-(1+Q)M^2\}^2}[Nk_z^2-(1+Q)M^2] + 2c^2k_z^2M^2N^2\left\{N^2-M^2\right\}k_z^2-QM^2 \]

(6.29)

The set of conditions (6.23) are satisfied due to (6.28) and the nonlinear dispersion relation (6.22). But for the second set of conditions (6.24), we expand \( \psi(n) \) by Taylor’s series near \( n \approx 1 \) and \( n \approx N \) to give

\[ \psi(n \approx 1) = \psi(1) + (n-1)\psi'(1) + \frac{(n-1)^2}{2!}\psi''(1) + \ldots \]

and

\[ \psi(n \approx N) = \psi(N) + (n-N)\psi'(N) + \frac{(n-N)^2}{2!}\psi''(N) + \ldots \]

With the help of (6.28), (6.22) and (6.29), these can be written as

\[ \psi(n \approx 1) = \frac{(n-1)^2k_z^2\left(1+Q-\frac{k_z^2}{M^2}\right)}{2M^2\left(Q + k_z^2 - \frac{k_z^2}{M^2}\right)} \]

(6.30)

which is reducible from the works of Kalita et al. (1986) exactly for \( \nu' = 0 \) in non-relativistic case. As there is no initial streaming in this consideration, therefore at equilibrium stage where \( n = 1 \), there is no relativistic effect and so the above condition is justified.

and \( \psi(n \approx N) \)

\[ \psi(n \approx N) = \frac{2c^2k_z^2(N-N)N^5(N-1)[Nk_z^2-(1+Q)M^2]}{3k_z^2(N-1)^2\{N^2k_z^2-(1+Q)M^2\}^2}[Nk_z^2-(1+Q)M^2] + 2c^2k_z^2M^2N^2\left\{N^2-M^2\right\}k_z^2-QM^2 \]

(6.31)
From (6.30) and (6.31), the following conditions finally can be derived for \( \psi(n) < 0 \) between \( n = 1 \) and \( n = N \) to represent solitary waves;

- near \( n = 1 \), \( Q + k_z^2 < \frac{k_z^2}{M^2} < 1 + Q \) \( (6.32) \)
- near \( n = N \), \( 1 > M \geq N \), \( M \geq k_z \) when \( N < 1 \) \( (6.33) \)
- and \( N > \frac{M^2}{k_z} \), \( M \geq k_z \) when \( N > 1 \) \( (6.34) \)

6.4 Discussion

In the magnetized weakly relativistic plasma of our consideration, both compressive \((N > 1)\) and rarefactive \((N < 1)\) subsonic \((M < 1)\) solitary waves are found to exist depending on wave speeds in various directions of propagation. The amplitude of the compressive soliton is found to diminish rapidly [Fig. 6.1 (a)] with \( k_z^2 \) for all soliton speeds \( M = .8, .85, .9 \). It is worthwhile to mention that, in the vicinity where the values of the direction of wave propagation \( k_z \) tend to attain the upper limit of subsonic soliton speeds \((k_z \to M)\), the relativistic compressive solitons appear to move almost with constant amplitudes. On the other hand, they attain various high amplitudes for small \( k_z < M \) [Fig. 6.1].

It is essential to report, that the potential depths (dot curves) of compressive relativistic solitons [Fig. 6.2] are greater than those of non-relativistic solitons (full curves) for all \( k_z \) accommodating more dense plasma particles in the potential well. Further, it demonstrates the increase of compressive soliton amplitude as the propagation direction approaches the direction of the magnetic field.
Unlike compressive solitons, the depths of potential wells (1) and (3) of rarefactive relativistic solitons [Fig. 6.3(a)] for $M = 0.2$ and [Fig. 6.3(b)] for $M = 0.3$ are found to be considerably smaller than those of non-relativistic ones when $k_z = 0.10$ and $0.15$. This clearly reveals that the plasma particles (lighter) moving with relativistic speeds are not constrained to be in deeper potential wells in case of relativistic but rarefactive solitons ascertaining submission to relativistic effects. The rarefactive soliton widths being dependent on potential depths are observed to decrease with the mach number $M$ [Fig. 6.3 (a), 6.3(b)] as it increases slowly.

Further, the potential depths of relativistic rarefactive solitons are seen to increase with $k_z$ for fixed higher values of $M = 0.7$ (Fig. 6.4). Otherwise, as the direction of propagation deviates from that of the magnetic field, the potential depths of rarefactive solitons tend to be smaller and smaller.

Interestingly, the amplitudes of the compressive solitons reflect uniform increase [Fig. 6.5(a)] with subsonic wave speed $M$ for all $k_z = .3, .35, .4$. On the other hand, amplitudes of the rarefactive solitons which appear to exist far away from the direction of the magnetic field [Fig. 6.5 (b)] i.e., for all (small)$k_z = .05, .1, 15$ are observed to increase almost linearly with $M$. But the amplitudes of compressive solitons grow nonlinearly with $M$ and at increasing difference at the step-up increase of $k_z$ [Fig. 6.5 (a)]. To the contrary, those of rarefactive solitons grow almost linearly with $M$ maintaining almost regular difference at the step-up increase of $k_z$ except for smaller $k_z$.

The higher and nearly constant amplitudes of the rarefactive solitons are found to decrease slowly initially [Fig. 6.6(a)] for small $k_z$ for all $M = .6, .7, .8$ which gradually increase with $k_z$. Besides, higher is the wave speed, higher is the corresponding
rarefactive soliton amplitude. The corresponding widths of the rarefactive solitons [Fig. 6.6 (b)] are seen to increase almost uniformly (rapidly) for higher (smaller) values of $M$ with $k_z$ but for fixed $N = .85$. 
References


D. J. Korteweg and De-Vries, Philos. Mag. 39, 422 (1895).


Fig. 6.1 Variation of subsonic compressive soliton amplitude with $k_z$ for different wave speeds $M = 0.8, 0.85, 0.9$ for $c = 300$.

Fig. 6.2. The potential wells characterized by $\psi(n)$ of the energy integral are reflected, showing the relativistic compressive (dot curves) and non-relativistic (full curves) solitons' amplitudes and corresponding depths for $k_z = 0.3, 0.35$, for $c = 300$ and $M = 0.7$. 
Fig. 6.3. The potential wells characterized by $\psi(n)$ of the energy integral are exhibited the relativistic rarefactive soliton (dot curves) and non-relativistic (full curves) solitons' amplitudes and depths for $k_z = .10, .15$, for $c = 300$ $M = .2$ (a), $M = .3$ (b).

Fig. 6.4. The potential wells characterized by $\psi(n)$ of the energy integral are shown to reflect the relativistic rarefactive solitons' amplitudes and depths for $k_z = .15, .20, .25$ for $c = 300$ and $M = .7$. 
Fig. 6.5 Variation of subsonic compressive soliton (a) and rarefactive soliton (b) amplitudes with $M$ for different $k_z = .3, .35, .4$ (a) and $k_z = .05, .10, .15$ (b) for $c = 300$.

Fig. 6.6 Variation of subsonic rarefactive soliton amplitude (a) and width (b) for $N = .85$ with $k_z$ for different wave speeds $M = .6, .7, .8$ for $c = 300$. 