CHAPTER 3

STATISTICAL ANALYSIS

OF

TWO-DIMENSIONAL STATIONARY PROCESSES
STATISTICAL ANALYSIS OF TWO-DIMENSIONAL
STATIONARY PROCESSES

3.1 Introduction

The multidimensional processes play an important role in describing the behaviour of sea waves (Longuet-Higgins (1957), in the analysis of meteorological data, oceanography (Pherson and Tick 1957) and so on. Two-dimensional spectrum has been used to calculate the seismic energy by sea waves, where the directional distribution of energy is essentially involved (Longuet-Higgins 1950). Echtart (1953a) has used a two-dimensional analysis to calculate the scattering of sound from the sea surface. The increasing importance of multidimensional processes points to the need of extension of the results of the spectral theory of one-dimensional stationary processes to multidimensional processes.

In this chapter we consider the estimation of the spectral density of a two-dimensional stationary process. We propose a general class of estimate and obtain their variance and bias. We also investigate the optimum weight function.

Let \( X_{t,v} \ (t = 0, 1, 2, \ldots ; v = 0, 1, 2, \ldots) \) be a discrete two-dimensional wide sense stationary process. We assume \( E \{x_{t,v}\} = 0, \text{ all } t, v \).
Define the autocovariance function $R_{s,u}$ by

$$R_{s,u} = E[I_x(\tau)x_{t+\tau},x_{t+u}] \quad (3.1.1)$$

where $x^*_t$ denotes the complex conjugate of $x_t$.

There exists a spectral representation of $R_{s,u}$ in the form (Bartlett 1955 P 191)

$$R_{s,u} = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \rho^{\omega+i\omega} dF(\omega,\omega') \quad (3.1.2)$$

where $F(\omega,\omega')$ has the properties of a bivariate (probability) distribution function on the region $-\pi<\omega<\pi$, $-\pi<\omega'<\pi$.

The function $F(\omega,\omega')$ is termed as the "integrated spectrum".

The spectral function $f(\omega,\omega')$ is defined by

$$f(\omega,\omega') = \frac{\partial}{\partial \omega} F(\omega,\omega') \quad (3.1.3)$$

With $f(\omega,\omega')$ given by (3.1.3), the inverse of (3.1.2) can be written as

$$f(\omega,\omega') = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} R_{s,s'-\omega - \omega'} \quad (3.1.4)$$

For real process $R_{s,u} = R_{-s,-u}$ and $f(\omega,\omega') = f(-\omega,-\omega')$.

### 3.2 Estimation of the spectral density function

Let $X_{t,v}$ be a sample from the two-dimensional stationary process with $\mathbb{E}[x_{t,v}] = 0$ and spectral function $f(\omega,\omega')$.

Let the sample autocovariance function be given by
Consider an estimate

\[ \hat{f}(\omega, \tau) = \frac{1}{(\pi n)^{2}} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left( 1 - \frac{m}{n} \right) \left( 1 - \frac{\tau}{\rho} \right) k(m, n) e^{-i \omega \tau} \]

(3.2.2)

where \( k(Y, Z) \), defined for all \( Y \) and \( Z \) is even, bounded, square integrable and normalized so that \( k(0, 0) = 1 \) and has the Fourier transform

\[ k(Y, Z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i \omega Y - i \nu Z} k(\omega, \nu) d\omega d\nu \]

(3.2.3)

We define \( k(Y, Z) \) as "two-dimensional covariance averaging kernel". The constants \( \rho_n \) and \( \nu_m \) are sequences of positive numbers converging to zero but satisfying \( \nu_m, \rho_n \to 0 \).

Let \( q \) be the greatest integer (\( q > 0 \)) such that

\[ k_q = \lim_{Y \to 0} \lim_{\rho \to 0} \left( 1 - \frac{k(Y, Z)}{1 + Y + 1 + \rho} \right) \]

(3.2.4)

The greatest integer \( q \) is known as "characteristic exponent" and \( k_q \) the "characteristic coefficient".
We impose the conditions

\[(1) \sum_{t=0}^{\infty} \sum_{s=0}^{\infty} |1 + i \omega|^r \kappa_{t,s} < \infty \quad \text{for } \gamma > 0 \]

and

\[(2) \sum_{t=0}^{\infty} |1 + i \omega|^r \kappa_{t,s} < \infty \quad \text{for } \gamma > 0 \quad (3.2.5)\]

The following Lemma which is a direct extension of a result obtained for the uni-dimensional case by Hannan (1960 P.40) is needed for finding the variance of the estimate (3.2.2).

**Lemma** Let \( \{x_{t,e}\} \) be a normal process with autocovariance function \( \{R_{s,u}\} \) and let the sample autocovariance function \( \{c_{s,u}\} \) be given by (3.2.1). Then

\[
\lim_{N,M \to \infty} \text{Cov}(c_{s,u};c_{l,v}) = \sum_{p} \sum_{q} \left( k_{p,v} R_{p+t,l+s} e^{i \omega (t-s)} + k_{p-t,l,v+s} \right)
\]

\[
\to (2\pi)^{-1} \int \int f^{*}(\omega,v) \{ \int e^{i \omega (t-s)} + e^{i \omega j} \} \int d\omega d\omega.
\]

\[(3.2.6)\]

Using (3.2.2), we can write the covariance of the spectral estimate as

\[
N^M \sum_{t,k,l,s} \text{Cov} \left[ \hat{f}(\omega_{k+1}^N), \hat{f}(\omega_{l+1}^N) \right] = \sum_{t,k,l,s} \left( \frac{N^M \kappa_{t,s}}{2 \pi} \right) \left\{ (1 - i \omega_{k+1}^N) (1 - i \omega_{l+1}^N) \right\}.
\]
The second term of (3.2.8) leads to

\[ k_n \sum_{t} k(\omega_m^t, \omega_m^s) k(\omega_m^t, \omega_m^s) \times \]
\[ \omega_1 + c^1, c^2 + \omega_1 + c^3, c^4 \]
\[ + \int_{-c^4}^{c^4} \int_{-c^3}^{c^3} f^2(\omega, \nu) \ d\omega \ d\nu \]

(3.2.9)

where

\[ f^2(\omega, \nu) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f^2(\omega, \nu) \ d\omega \ d\nu \]

(3.2.10)

Rearranging (3.2.9) we get

\[ \sum_{t} \sum_{s} k(\omega_m^t, \omega_m^s) \]
\[ \begin{cases} \omega_1 + c^1, c^2 + \omega_1 + c^3, c^4 \end{cases} \]
\[ k(\omega_m^t, \omega_m^s) \sum_{n} \sum_{m} \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f^2(\omega, \nu) \ d\omega \ d\nu \]

(3.2.11)
The term within braces of (3.2.11) converges to
\[
\lim_{m,n \to \infty} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} c_{l} \omega_{l}^{j} \frac{1}{\lambda_{k}} \left( \frac{1}{\lambda_{j}} \right) \left( \frac{1}{\lambda_{l}} \right) \delta_{k,l} \delta_{j,l} \delta_{l,l}
\]
\[
= \left\{ \begin{array}{ll}
\int \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{\lambda_{k}} \left( \frac{1}{\lambda_{j}} \right) \left( \frac{1}{\lambda_{l}} \right) \delta_{k,l} \delta_{j,l} \delta_{l,l} & \text{if } \omega_{k} = \omega_{j} = \omega_{l} \\
0 & \text{if } \omega_{k} \neq \omega_{j} \neq \omega_{l} \neq \omega_{k}
\end{array} \right. \quad (3.2.12)
\]

Hence (3.2.11) converges to
\[
\int \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{\lambda_{k}} \left( \frac{1}{\lambda_{j}} \right) \left( \frac{1}{\lambda_{l}} \right) \delta_{k,l} \delta_{j,l} \delta_{l,l}
\]
\[
= \left\{ \begin{array}{ll}
\int \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{\lambda_{k}} \left( \frac{1}{\lambda_{j}} \right) \left( \frac{1}{\lambda_{l}} \right) \delta_{k,l} \delta_{j,l} \delta_{l,l} & \text{if } \omega_{k} = \omega_{j} = \omega_{l} \\
0 & \text{if } \omega_{k} \neq \omega_{j} \neq \omega_{l} \neq \omega_{k}
\end{array} \right. \quad (3.2.13)
\]

The first term of (3.2.8) exists if \( \omega_{1} = -\omega_{2} = -\omega_{3} \).
Otherwise the term vanishes.

Now we calculate the bias in the estimate \( \hat{f}(\omega, v) \).

The bias is given by
Let $A$ be the upper bound of $k(u_n, v_m)$, then

$$E(\hat{f}(w, \omega)) - f(w, \omega) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left(1 - \frac{1}{n} \right) \left(1 - \frac{1}{m} \right) R_{k, s}(u_n, v_m) e^{j\theta}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left(1 - \frac{1}{n} \right) \left(1 - \frac{1}{m} \right) R_{k, s}(u_n, v_m) e^{j\theta}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left(1 - \frac{1}{n} \right) \left(1 - \frac{1}{m} \right) R_{k, s}(u_n, v_m) e^{j\theta}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left(1 - \frac{1}{n} \right) \left(1 - \frac{1}{m} \right) R_{k, s}(u_n, v_m) e^{j\theta}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left(1 - \frac{1}{n} \right) \left(1 - \frac{1}{m} \right) R_{k, s}(u_n, v_m) e^{j\theta}$$

$$(3.2.15)$$

The right hand side of (3.2.16) tends to zero because of the conditions (3.2.5). Hence the second term of (3.2.15) tends to zero for large $N$ and $M$. Similarly it can be shown that the third term of (3.2.15) tends to zero. It is obvious
that the fourth term tends to zero.

Hence

\[ B_l \left( \hat{f}(\omega, \nu) \right) \rightarrow \frac{1}{(2\pi)^2} \sum_{-N+1}^{N-1} \sum_{-M+1}^{M-1} \left[ l_{\nu M} t_{\nu}^{(v)} + l_{\nu N} t_{\nu}^{(n)} \right] u_{\nu} e^{i \omega \xi} \]

\[ R_{k, s} e^{i \omega \xi} \]

(3.2.17)

which follows from (3.2.4).

Alternatively, (3.2.17) can be written as

\[ B_l \left( \hat{f}(\omega, \nu) \right) = k_{\nu} \left[ l_{\nu M} t_{\nu}^{(v)} f_{\nu}^{(w)}(\omega, \nu) + l_{\nu N} t_{\nu}^{(n)} f_{\nu}^{(w)}(\omega, \nu) \right] \]

(3.2.18)

where

(i) \[ f_{\nu}^{(w)}(\omega, \nu) = \frac{1}{(2\pi)^2} \sum_{k} \sum_{s} l_{\nu M} R_{k, s} e^{i \omega \xi} \]

(ii) \[ f_{\nu}^{(w)}(\omega, \nu) = \frac{1}{(2\pi)^2} \sum_{k} \sum_{s} l_{\nu N} R_{k, s} e^{i \omega \xi} \]

3.3 A relation for the truncation points

As in the uni-dimensional case we consider the mean square percentage error \( \eta \) as a figure of merit of the kernels. \( \eta \) is defined as

\[ \eta = \frac{\text{var}(\hat{f}(\omega, \nu)) + B_l(\hat{f}(\omega, \nu))}{\hat{f}(\omega, \nu)} \]

(3.3.1)

Substituting the expressions for the variance and the bias,
we get

\[ \eta^{2} = \frac{1}{\nu_{n} \nu_{m}} \int \int k^{2}(k, s) \, dk \, ds + k_{n}^{2} \left[ \frac{\nu_{n}^{2} - 1}{\chi^{(n)}(\omega, \nu)} \right. \\
+ \left. \frac{\nu_{m}^{2} - 1}{\chi^{(m)}(\omega, \nu)} \right] \]

(3.3.2)

where

\[ \chi^{(n)}(\omega, \nu) = \left| \frac{\int \frac{f(\omega, s)}{f^{(n)}(\omega, \nu)}}{f(\omega, s)} \right|^{2} \]

(3.3.3)

\[ \chi^{(m)}(\omega, \nu) = \left| \frac{\int \frac{f(\omega, s)}{f^{(m)}(\omega, \nu)}}{f(\omega, s)} \right|^{2} \]

The relation (3.3.2) can also be used to express the product of the sample (N M) in terms of \( \eta^{2} \) and \( \nu_{n}, \nu_{m} \). We get

\[ (N M) = \frac{1}{\nu_{n} \nu_{m}} \int \int k^{2}(k, s) \, dk \, ds \]

(3.3.4)

\[ \eta^{2} = \left[ \frac{k_{n}^{2} \nu_{n}^{2} - 1}{\chi^{(n)}(\omega, \nu)} + \frac{k_{m}^{2} \nu_{m}^{2} - 1}{\chi^{(m)}(\omega, \nu)} \right]^{-1} \]

The product (N M) will be a minimum if the denominator of (3.3.4) is a maximum.

Differentiating the denominator with respect to \( \nu_{n} \) and \( \nu_{m} \), we get
3.3.5

\[ \eta = k^L_y \left[ - \frac{\Lambda_{(\omega, v)}^{(w)}}{\chi_{(\omega)}^{(w, \nu)}} + \frac{\Lambda_{(\omega, v)}^{(w, \nu)}}{\chi_{(\omega)}^{(w, \nu)}} - \frac{\Lambda_{(\omega)}^{(w, \nu)}}{\chi_{(\omega)}^{(w, \nu)}} \right] \]

and

\[ \eta = k^L_y \left[ - \frac{\Lambda_{(\omega, v)}^{(w)}}{\chi_{(\omega)}^{(w, \nu)}} + \frac{\Lambda_{(\omega, v)}^{(w, \nu)}}{\chi_{(\omega)}^{(w, \nu)}} - \frac{\Lambda_{(\omega)}^{(w, \nu)}}{\chi_{(\omega)}^{(w, \nu)}} \right] = 0 \]

(3.3.5)

(3.3.6)

From (3.3.5) and (3.3.6), we get

\[ \frac{\chi_{(\omega)}^{(w, \nu)}}{\chi_{(\omega)}^{(w, \nu)}} = \frac{n}{\chi_{(\omega)}^{(w, \nu)}} \]

(3.3.7)

The relation (3.3.7) shows that \( n \) or \( m \) can be obtained provided \( \chi_{(\omega)}^{(w, \nu)} \) and \( \Lambda_{(\omega)}^{(w, \nu)} \) are known. Since the exact values of \( \chi_{(\omega)}^{(w, \nu)} \) and \( \Lambda_{(\omega)}^{(w, \nu)} \) are seldom known we have to use their approximate values.

3.4 An "uncertainty relation" for the estimate

To estimate \( f(\lambda, \mu) \), we also consider the estimate given by

\( f(\lambda, \mu) = \sum_{\lambda} \sum_{\mu} \frac{1}{(2\pi)^2} \langle \psi | \phi_{\lambda, \mu} \rangle \)
where \( c_{s,v}(\omega,v) \) satisfy the conditions

\[
(1) \quad c_{s,v}(\omega,v) = \omega_s, v_s, \ldots, v_L \quad \forall s, v, v_s, v_L
\]

\[
(11) \quad W_{s,v}(\omega,v) = \frac{1}{N^2} \sum_{\omega_s,v_s} c_{s,v}(\omega,v) \quad (3.4.2)
\]

\[
(iii) \quad \int \int W_{s,v}(\omega,v) \, d\omega \, dv < \infty \quad s,v = \ldots, v_L
\]

(iv) For any \( t_1, t_2 > 0 \)

\[
W_{s,v}(\omega,v) \to 0 \quad \omega, v \to \infty
\]

uniformly for \( |x| > t, \quad |w| > t_L \).

The relation (3.4.1) can also be written as

\[
\mathcal{F}(x,v) = \int_{-N}^{N} \int_{-N}^{N} I_{N,M}(\omega,v) \, W_{s,v}(\omega,v) \, d\omega \, dv
\]

where \( I_{N,M}(\omega,v) \) is given by

\[
I_{N,M}(\omega,v) = \frac{1}{(2\pi)^N} \left| \sum_{\omega_s,v_s} c_{s,v}(\omega,v) \right| \quad (3.4.4)
\]

\( I_{N,M}(\omega,v) \) is known as "two-dimensional periodogram".

Under the conditions (3.4.2), it can be shown that

\[
E \left[ \hat{f}(x,v) \right] = \int \int I_{N,M}(\omega,v) \, W_{s,v}(\omega,v) \, d\omega \, dv \quad (3.4.5)
\]

and

\[
\hat{f}(x,v) = \hat{f}(x,v)
\]
Let us find the optimum weight function $W_{x, \omega}(\omega, \nu)$ which minimises the variance of the estimate subject to the condition of unbiasedness. We minimise

$$\phi = \text{Var}\left(\hat{f}(x, \omega)\right) = \mathbb{E}\left[\int \int f(x, \omega) W_{x, \omega}(\omega, \nu) d\omega d\nu\right]$$

$$\phi = \mathbb{E}\left[\int f(x, \omega)\right]$$

with respect to $W_{x, \omega}(\omega, \nu)$. It follows easily that the optimum weight function is

$$W_{x, \omega}(\omega, \nu) = \frac{\int f(x, \omega) W_{x, \omega}(\omega, \nu) d\omega d\nu}{\int \int f(x, \omega) W_{x, \omega}(\omega, \nu) d\omega d\nu}$$

(3.4.7)

The optimum weight function $W_{x, \omega}(\omega, \nu)$ given by (3.4.7) shows that for all values of $x$ and $\omega$, the weight function is stochastically equivalent and this is unacceptable. Hence a relation has to be found to measure this "uncertainty".

Let the total mass of the weight function of $(\omega, \nu)$ plane be projected on the $\omega$-axis and we shall call it as "marginal weight function". $W_x(\omega)$ given by

$$W_x(\omega) = \int_\nu W_{x, \omega}(\omega, \nu) d\nu$$

Similarly

$$W_\omega(\nu) = \int_\omega W_{x, \omega}(\omega, \nu) d\omega$$

(3.4.8)

Then we assume

$$W_{x, \omega}(\omega, \nu) = W_x(\omega) W_\omega(\nu)$$

(3.4.9)
which implies that the weight functions along $C_0$ and $\phi$ axis are independent.

As a measure of resolvability, we take

$$U_{1,1} = \left[ \int_0^\mu (1 - v) W_p(u) \, du \right]^{1/2}$$

and

$$U_{1,2} = \left[ \int_0^\mu (u - v)^2 W_p(u) \, du \right]^{1/2}$$

As a measure of sampling variability, we consider

$$U_L = \int_0^\mu \int_0^\mu f(u, v) W_p(u) W_p(v) \, du \, dv$$

Define

$$u_{1, n} = \int_0^\mu t^n \int_0^\mu W_p(u) \, du \, dp$$

where

$$u_{1, n} = \int_0^\mu W_p(u) \, du$$

We write

$$U_{1,1} = \frac{\zeta_{1,1}}{(k_{1,1})^{1/2}}$$

and

$$U_{1,2} = \frac{\zeta_{1,2}}{k_{1,0}}$$

where

$$\zeta_{1,1} = \left[ \int_0^\mu (1 - v) W_p(u) \, du \right]^{1/2}$$

and

$$\zeta_{1,2} = \left[ \int_0^\mu (u - v)^2 W_p(u) \, du \right]^{1/2}$$
\[ \Delta_{1,2} = \left[ \int_{0}^{\beta} k_{1} \frac{w_{\lambda}(\mu)}{k_{1,0}} \, d\mu \right]^{\frac{1}{2}} \] (3.4.16)

Let us take \( \left( \frac{w_{\lambda}(\mu)}{k_{1,0}} \right) \) to be the "frequency function" of the variable \( k \) along \( \omega \) axis and similarly \( \left( \frac{w_{\lambda}(\mu)}{k_{2,0}} \right) \) to be the frequency function of the variable \( \mu \) along \( \nu \) axis. Then

\[ E(\{k_{1,2}\}) = \int_{0}^{\nu} k_{1,2} \frac{w_{\lambda}(\mu)}{k_{1,0}} \, d\mu \leq \left[ \int_{0}^{\nu} \frac{w_{\lambda}(\mu)}{k_{1,0}} \, d\mu \right]^{\frac{1}{2}} \] (3.4.17)

\[ E(\{k_{1,2}\}) = \int_{0}^{\nu} \frac{w_{\lambda}(\mu)}{k_{2,0}} \, d\mu \leq \left[ \int_{0}^{\nu} \frac{w_{\lambda}(\mu)}{k_{2,0}} \, d\mu \right]^{\frac{1}{2}} \]

From Chebyshev's inequality, it follows that

\[ \Pr \{ |k_{1} - \lambda| < \varepsilon \} > 1 - \frac{1}{\varepsilon^{2}} \] (3.4.18)

and

\[ \Pr \{ |(m - \mu)| < \varepsilon \} > 1 - \frac{1}{\varepsilon^{2}} \]

From Schwarz's inequality

\[ \int_{0}^{\nu} \frac{w_{\lambda}(\mu)}{k_{1,0}} \, d\mu \cdot \int_{0}^{\nu} \frac{w_{\lambda}(\mu)}{k_{2,0}} \, d\mu \geq \int_{0}^{\nu} \left( \frac{w_{\lambda}(\mu)}{k_{1,0}} \right) \left( \frac{w_{\lambda}(\mu)}{k_{2,0}} \right) \, d\mu \]

\[ > \frac{1}{\varepsilon^{2}} \cdot \left( \int_{0}^{\nu} \frac{w_{\lambda}(\mu)}{k_{1,0}} \, d\mu \right)^{2} \]

\[ > \frac{1}{\varepsilon^{2}} \cdot \left( \int_{0}^{\nu} \frac{w_{\lambda}(\mu)}{k_{2,0}} \, d\mu \right)^{2} \] (3.4.19)

Similarly

\[ \int_{0}^{\nu} \frac{w_{\lambda}(\mu)}{k_{1,0}} \, d\mu \geq \frac{1}{\varepsilon^{2}} \cdot \left( \int_{0}^{\nu} \frac{w_{\lambda}(\mu)}{k_{1,0}} \, d\mu \right)^{2} \] (3.4.20)
Hence
\[\int_0^\infty \frac{W^2_\lambda(t)}{k_{1,0}} dt \int_0^\infty \frac{W^2_\mu(m)}{k_{2,0}} dm > \frac{1}{16 (\theta_{\lambda,\mu})^q} (1, 1, 1) (3.4.21)\]

Consider
\[U_2 = \int_0^\infty \int_0^\infty W^2_{\lambda,\mu}(t, m) dt dm > F^2 \left( \int_0^\infty \int_0^\infty W^2_{\lambda,\mu}(t, m) dt dm \right) \frac{\mu^2 \lambda_{1,0} \mu_{2,0}}{16 (\theta_{\lambda,\mu})^q} (1, 1, 1) (3.4.22)\]

\[k_{0,0} = \int_0^\infty \int_0^\infty W^2_{\lambda,\mu}(t, m) dt dm > \int_0^\infty \int_0^\infty W^2_{\lambda,\mu}(t, m) dt dm > \frac{F^2}{\theta_{\lambda,\mu}} (1, 1, 1) (3.4.23)\]

where \(F\) and \(G\) are respectively lower and upper bounds of \(f(\cdot, \cdot, \cdot)\).

\[W = \left[ \lambda_{0,1} \lambda_{1,1} \lambda_{2,1} / f^2(t, m) \right] \text{ which is always positive can be taken to measure the uncertainty of the estimates.} \]

3.5 Investigation of the optimum weight function

Now we investigate the optimum weight function \(W_\lambda, \mu(t, m) = \lambda_{\lambda}(t), \mu_{\mu}(m)\) from assumption 3.4.9) for which \([\lambda_{0,1}, \lambda_{1,1}, \lambda_{2,1}]\) has a minimum.
We choose \( r = 2 \) on the assumption that the common measure of dispersion is a suitable measure of resolvability as well.

Let \( W_x(u) > 0, \ W_u(m) > 0 \) Let \( W_x(u) = s^2(u) \)
and \( W_u(m) = u^2(m) \) where \( s(u) \) and \( u(m) \) are the functions corresponding to which

\[
\frac{\Delta_{1,1}^{(2)} \Delta_{1,1}^{(3)} u_1}{(k_{1,0} k_{2,0})^{1/2}} \tag{3.5.1}
\]

has a minimum.

For the optimum weight functions \( W_x(u) \) and \( W_u(m) \)

\[
k_{1,v} = \left[ \int_0^u v(u) \, du \right] ; \quad k_{2,v} = \left[ \int_0^u p^m u^2(p) \, dp \right] \tag{3.5.2}
\]

\[
\Delta_{1,1}^{(2)} = \left[ \left( u^2 + v^2 \right) u_{1,1} \right]^{1/2} = \left[ k_{1,0} + k_{1,1} + k_{1,2} \right]^{1/2} \tag{3.5.3}
\]

and

\[
\Delta_{1,1}^{(3)} = \left[ \int_0^u \left( m - v \right) u^2(m) \, dm \right]^{1/2} = \left[ k_{1,0} + k_{1,1} + k_{1,2} \right]^{1/2} \tag{3.5.4}
\]

We can write

\[
U_1 = \int_0^u \int_0^u f_{1,1}(u, m) s(u) u^2(m) \, du \, dm \quad \tag{3.5.5}
\]

and the condition for asymptotic unbiasedness

\[
\int_0^u \int_0^u f_{1,1}(u, m) s(u) u^2(m) \, du \, dm = f(u, m) \tag{3.5.6}
\]
Let us take the functions $\xi_1 V(t) + \xi_2 U(t)$ and $\xi_1 V(t) + \xi_2 U(t)$ where $\xi_1$ and $\xi_2$ are small and $V(t)$ and $U(t)$ are arbitrary.

Let
\[
\delta \left[ \frac{\Delta_{i,1} \Delta_{i,2} U - \Delta_{i,1} \Delta_{i,2} U}{(k_{i,0} k_{i,0})^{1/2}} \right]
\]
denote the variation in
\[
\left( \frac{\Delta_{i,1} \Delta_{i,2} U - \Delta_{i,1} \Delta_{i,2} U}{(k_{i,0} k_{i,0})^{1/2}} \right)
\]

We have
\[
\delta \left[ \frac{\Delta_{i,1} \Delta_{i,2} U}{(k_{i,0} k_{i,0})^{1/2}} \right] = \frac{1}{k_{i,0}} \left\{ U \delta \left( \frac{\Delta_{i,1} \Delta_{i,2} U}{(k_{i,0} k_{i,0})^{1/2}} \right) + \left( \frac{\Delta_{i,1} \Delta_{i,2} U}{(k_{i,0} k_{i,0})^{1/2}} \right) \delta (U) \right\}
\]
\[
- \left( \frac{\Delta_{i,1} \Delta_{i,2} U}{(k_{i,0} k_{i,0})^{1/2}} \right) \delta (k_{i,0} k_{i,0})^{1/2} \right\} \quad (3.5.7)
\]

Now
\[
\delta (k_{i,m}) = 2 \xi_1 k_{i,m} + \xi_2 \int_0^1 \ln V(t) \, dt \quad (3.5.8)
\]
\[
\delta (k_{i,m}) = 2 \xi_1 k_{i,m} + \xi_2 \int_0^p \int U^+ (p) \, dp \quad (3.5.9)
\]

where
\[
k_{i,m} = \int_0^1 \ln V(t) V(t) \, dt \quad (3.5.10)
\]
\[
k_{i,m} = \int_0^1 \ln U(t) U(t) \, dt \quad (3.5.11)
\]

We have
\[
\delta (\Delta_{i,1}) = \delta \left[ \int_0^1 \left( t - M \right) \ln U(t) \, dt \right] \quad (3.5.12)
\]
\[
\begin{align*}
\delta(\Delta_{1,2}) &= \delta \left[ \int_0^n (u - u')^2 u'(m) \, dm \right]^{1/2} \\
&= k_{2,0} - \omega k_{2,1} + \omega k_{2,0} \\
& \quad - \sqrt{u_{1,0} - 2 \omega k_{1,1} + \omega^2 k_{1,0}} \\
& + \int_0^n (u - u')^2 u'(m) \, dm \\
& \quad - \sqrt{u_{1,0} - 2 \omega k_{1,1} + \omega^2 k_{1,0}} \\
& \quad - \omega k_{2,0} \\
(3.5.12)
\end{align*}
\]

\[
\begin{align*}
\delta(\Delta_{1,2}) &= \Delta_{1,1} \delta(\Delta_{1,2}) + \Delta_{1,1} \delta(\Delta_{1,2}) \\
&\quad + \delta(\Delta_{1,2}) \\
(3.5.13)
\end{align*}
\]

\[
\begin{align*}
\delta(u) &= \delta \left[ \int_0^n f(k, m) \, u_k^* (k, m) \, dk \, dm \right]^{1/2} \\
&= \int_0^n f(k, m) \, u_k^* (k, m) \, u'(k) \, V(k) \, dk \, dm \\
& + \int_0^n f(k, m) \, u_k^* (k, m) \, u_k^3 (m) \, u(m) \, dk \, dm \\
& + \int_0^n f(k, m) \, u_k^3 (m) \, u_k^3 (m) \, u(m) \, dk \, dm \\
& + \delta(u) \\
(3.5.14)
\end{align*}
\]

and
\[
\begin{align*}
\delta(\zeta_{1,2}) &= \zeta_{1,0} \delta(\zeta_{1,2}) + \zeta_{1,0} \delta(\zeta_{1,2}) \\
&\quad + \delta(\zeta_{1,2}) \\
\end{align*}
\]
Substituting the expressions from (3.5.8), (3.5.9), ...
(3.5.14) in (3.5.7) and neglecting higher powers of \( \tau \), \( \Lambda \), and \( \gamma \), we get

\[
\begin{align*}
\mathcal{S} \left[ \frac{\Delta^{(r)}_{1,1} \Delta^{(r)}_{1,2}}{(k_{1,0} k_{2,0})^L} \right] & \left[ \begin{array}{c} \lambda_{1,1} \ U_{1,2} k_{1,0} k_{2,0} \\ \lambda_{1,2} \ U_{1,2} k_{1,0} k_{2,0} \\ \lambda_{1,3} \ U_{1,2} k_{1,0} k_{2,0} \\ \vdots \\ \lambda_{L,1} \ U_{1,2} k_{1,0} k_{2,0} \\ \lambda_{L,2} \ U_{1,2} k_{1,0} k_{2,0} \\ \lambda_{L,3} \ U_{1,2} k_{1,0} k_{2,0} \\ \vdots \\ \lambda_{L,L} \ U_{1,2} k_{1,0} k_{2,0} \\ \end{array} \right] \begin{array}{c} k_{1,0} k_{2,0} \\ \lambda_{1,1} \ U_{1,2} k_{1,0} k_{2,0} \\ \lambda_{1,2} \ U_{1,2} k_{1,0} k_{2,0} \\ \lambda_{1,3} \ U_{1,2} k_{1,0} k_{2,0} \\ \vdots \\ \lambda_{L,1} \ U_{1,2} k_{1,0} k_{2,0} \\ \lambda_{L,2} \ U_{1,2} k_{1,0} k_{2,0} \\ \lambda_{L,3} \ U_{1,2} k_{1,0} k_{2,0} \\ \vdots \\ \lambda_{L,L} \ U_{1,2} k_{1,0} k_{2,0} \\ \end{array} 
\end{align*}
\]

The condition of unbiasedness gives

\[
\int \int f(k,m) \left( \frac{(k_1 + \tau, \nu(k))}{\nu(k)} \right)^L u(k,m) \nu(k,m) \ d k d m
\]

Neglecting higher powers of \( \tau \) and \( \nu \), we have from

(3.5.16)

\[
2 \hat{\gamma} \int \int f(k,m) \gamma^L(k) \nu(k,m) \ d k d m
\]

Neglecting higher powers of \( \tau \), \( \Lambda \), and \( \gamma \), we have from

(3.5.17)
**Case 1**

We put \( \eta_2 = 0 \) in (3.5.15) and (3.5.17). Because


V(\( \eta \)) and U(\( \eta \)) give minimum expressions (3.5.15) and (3.5.17) must be positive for small values of \( \eta \), positive or negative. This implies that the coefficient of \( \eta \) in the expressions (3.5.15) and (3.5.17) must vanish.

Hence

\[
\left[ k_{c,0} U_0 (k_{r,0} - \kappa k_{r,1} + \kappa k_{r,0}) + \xi k_{r,0} (k_{r,0} - \kappa k_{r,1} + \kappa k_{r,0}) \right] \\
\int \int f(k, m) v(k, m) \phi^2 V(k) \, dl \, dm
\]

From (3.5.17) we get

\[
\int \int f(k, m) \phi(k) V(k) \, dl \, dm = 0
\]

We choose

\[
V(k) \phi k = \frac{1}{A} \left( -\frac{t(k - x, m - \cdot)}{f(x, m) \phi(x, m)} - \frac{t(k - y, m - \cdot)}{f(y, m) \phi(y, m)} \right)
\]

where

\[
t(k, m) = \begin{cases} 
0 & (k, m) \neq (0, 0) \\
1 & \text{Otherwise}
\end{cases}
\]

Substituting the right hand expression of (3.5.20) for \( V(k) \phi k \) in (3.5.18) and simplifying, we get...
\[ \psi_k(x) = \frac{A}{f(x,s)} + \frac{B}{f(x,s)} \left[ 1 - \frac{(\gamma-s)^2}{C} \right] \]

where \( B = U_2/4 \psi_{1,0} \) and \( A \) is independent of \((\alpha, \chi)\).

**Case (i1)**  
We put \( \psi = 0 \) in (3.5.15) and (3.5.17).

Proceeding as before, we can show that

\[ \psi_k(x) = \frac{C}{f(x,s)} + \frac{D}{f(x,s)^2} \left[ 1 - \frac{(\gamma-s)^2}{U_{1,2}} \right] \]

where \( D = U_2/4 \psi_{1,0} \) and \( C \) is independent of \((\gamma, s)\).

As in the unidimensional case, the optimum weight functions \( \psi_k(x) \) and \( u_{\gamma, \omega} \) given by (3.5.21) and (3.5.22) are theoretical in nature since the spectral density must be known to compute these optimum weight functions.