CHAPTER-FOUR

ijS* Connectedness

In this chapter we study the connectedness of a bitopological space and of a subset of a bitopological space in terms of ij-semi open as ijS* connected space and ijS* connected subset respectively. We also study preservation of ijS* connectedness of a bitopological space as the connectedness of a topological space. We also generalize these concepts in case of fuzzy bitopological spaces. Lastly we study some relationship between bitopological spaces and fuzzy bitopological spaces.

4.1. ijS* Connected Bitopological Spaces

Definition 4.1.1. A bitopological space \((X, \tau_1, \tau_2)\) is said to be ijS* disconnected space if \(\exists\) two ijSO sets \(U\) and \(V\) such that \(U\neq \emptyset\neq V\), \(U\cap V\) and \(X=U\cup V\), in this case \(X=U\cup V\) is said to be a ijS* disconnection of \((X, \tau_1, \tau_2)\) or simply of \(X\). If \((X, \tau_1, \tau_2)\) is not an ijS* disconnected space, \((X, \tau_1, \tau_2)\) is said to be an ijS* connected space.

Remark 4.1.2. The ijS* connectedness does not imply the jiS* connectedness, e.g.

We consider the bitopological space \((X, \tau_1, \tau_2)\) where \(X=\{x, y, z\}\), \(\tau_1=\{\emptyset, \{x\}, \{x, y\}, X\}\), \(\tau_2=\{\emptyset, \{x\}, \{y\}, \{x, y\}, X\}\). Now all 12SO sets in \((X, \tau_1, \tau_2)\) are \(\emptyset, \{x\}, \{x, z\}, \{x, y\}, X\) which shows that the bitopological space is 12S* connected. 21SO sets are \(\emptyset, \{x\}, \{y\}, \{x, y\}, \{x, z\}, \{y, z\}, X\) which shows that the bitopological space is 21S* disconnected.
Theorem 4.1.3. If a bitopological space \((X, \tau_1, \tau_2)\) is \(ijS^*\) connected then the topological space \((X, \tau)\) is connected.

**Proof:** Since every, \(\tau_i\)-open set is also a \(ijSO\) set, so if \((X, \tau_i)\) is disconnected topological space then the bitopological space \((X, \tau_1, \tau_2)\) becomes \(ijS^*\) disconnected, which is not possible, so \((X, \tau_i)\) is connected.

Remark 4.1.4. The converse of the above theorem does not hold, e.g.

We consider the bitopological space \((X, \tau_1, \tau_2)\) where \(X=\{x, y, z\}\), \(\tau_1=\{\emptyset, \{x\}, \{y\}, \{x, y\}, X\}\), \(\tau_2=\{\emptyset, \{y\}, X\}\). Now all \(12SO\) sets in \((X, \tau_1, \tau_2)\) are \(\emptyset, \{x\}, \{y\}, \{x, z\}, \{x, y\}, X\) which shows that the bitopological space is \(12S^*\) disconnected but the topological space \((X, \tau_1)\) is connected.

Remark 4.1.5. The \(ijS^*\) connectedness of a bitopological space \((X, \tau_1, \tau_2)\) does not depend on the connectedness of the topological space \((X, \tau_j)\) e.g.

In the example given in Remark 4.1.4 the bitopological space is \(12S^*\) disconnected but the topological space \((X, \tau_2)\) is connected.

We again consider the bitopological space \((X, \tau_1, \tau_2)\) where \(X=\{x, y, z\}\), \(\tau_1=\{\emptyset, \{x\}, \{x, y\}, X\}\), \(\tau_2=\{\emptyset, \{y\}, \{x, z\}, X\}\). Now all \(12SO\) sets are \(\emptyset, \{x\}, \{x, z\}, \{x, y\}, \{x, z\}, \{x, y\}, X\) which shows that the bitopological space is \(12S^*\) connected but the topological space \((X, \tau_2)\) is disconnected.

Theorem 4.1.6. A bitopological space \((X, \tau_1, \tau_2)\) is \(ijS^*\) connected iff \(X\) and \(\emptyset\) are the only subsets of \(X\) which are simultaneously \(ijSO\) set and \(ijSC\) set.

**Proof:** Let \((X, \tau_1, \tau_2)\) be a \(ijS^*\) connected space, if possible let \(\emptyset \neq A \neq X\) and \(A\) is simultaneously \(ijSO\) set and \(ijSC\) set then \(X=\emptyset \cup (X-A)\) is a \(ijS^*\) disconnection of the bitopological space, which is a contradiction, so \(X\) and \(\emptyset\) are the only subsets of \(X\) which are both \(ijSO\) set and \(ijSC\) set simultaneously.

Conversely let \(X\) and \(\emptyset\) are the only subsets of \(X\) which are both \(ijSO\) set and \(ijSC\) set. Let if possible the bitopological space is \(ijS^*\) disconnected, so \(\exists\) a \(ijS^*\) disconnection \(X=A \cup B\) of the bitopological space, so \(A=X-B\) and \(B=X-A\), then \(A\) and \(B\) both are simultaneously \(ijSO\) set and \(ijSC\) sets and each of them
nether $X$ nor $\emptyset$, which is a contradiction so the bitopological space is $ijS^*$ connected.

**Theorem 4.1.7.** A bitopological space $(X, \tau_1, \tau_2)$ is $ijS^*$ connected iff no $ijS^*$ continuous mapping $f : (X, \tau_1, \tau_2) \to 2$ is surjective, where 2 is the topological space $\langle \{0, 1\}, \mathcal{P}(\{0, 1\}) \rangle$, i.e. the discrete topological space on $\{0, 1\}$.

**Proof:** Let $(X, \tau_1, \tau_2)$ be a $ijS^*$ connected space and let if possible $\exists$ $ijS^*$ continuous mapping $f : (X, \tau_1, \tau_2) \to 2$ which is surjective. Since $\{0\}$ and $\{1\}$ are both simultaneously open and closed in 2, so $A=f^{-1}(\{0\})$ and $B=f^{-1}(\{1\})$ are both simultaneously $ijSO$ set and $ijSC$ sets and both are non-empty since $f$ is surjective so $X=A \cup B$ is an $ijS^*$ disconnection of $X$, which is a contradiction, so there does not exists such mapping.

Conversely, let there is no surjective $ijS^*$ continuous mapping $f : (X, \tau_1, \tau_2) \to 2$. If possible let $(X, \tau_1, \tau_2)$ be an $ijS^*$ disconnected and $X=A \cup B$ be a $ijS^*$ disconnection of $X$. Let us define $f : (X, \tau_1, \tau_2) \to 2$ by $f(A)=0, f(B)=1$, then $f$ is $ijS^*$ continuous mapping and also surjective, which is a contradiction so $ijS^*$ connected.

**Theorem 4.1.8.** Let $(X, \tau_1, \tau_2)$ is a $ijS^*$ connected space and $(Y, \sigma)$ be a topological space, if $f : (X, \tau_1, \tau_2) \to (Y, \sigma)$ is surjective and $ijS^*$ continuous mapping then $(Y, \sigma)$ is a connected topological space.

**Proof:** Let if possible $(Y, \sigma)$ is disconnected and $Y=A \cup B$ a disconnection of $Y$, then $X=f^{-1}(A) \cup f^{-1}(B)$ becomes a $ijS^*$ disconnection of $X$, which is a contradiction so $(Y, \sigma)$ is a connected space.

**Remark 4.1.9.** The converse of the above theorem is not true. E.g.

We consider the bitopological space $(X, \tau_1, \tau_2)$ and the topological space $(Y, \sigma)$ where $X=\{x, y, z\}, \quad \tau_1=\{\emptyset, \{x\}, \{y\}, \{x, y\}, X\}, \quad \tau_2=\{\emptyset, \{y\}, X\}, \quad Y=\{a, b\}, \quad \sigma=\{\emptyset, \{a\}, Y\}$

Now all $12SO$ sets in $(X, \tau_1, \tau_2)$ are $X, \{x\}, \{y\}, \{x, z\}, \{y, z\}, \{x, y\}$, $X$ which shows that the bitopological space is $12SO$ disconnected and the
topological space \((Y, \sigma)\) is connected, but the mapping \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma)\) defined by \(f(x)=a\), \(f(z)=a\), \(f(z)=b\) is surjective and \(12S^*\) continuous mapping.

**Theorem 4.1.10.** Let \((X, \tau_1, \tau_2)\) is a \(ijS^*\) connected space and \((R, u)\) be the real line with usual topology \(u\), if \(f : (X, \tau_1, \tau_2) \rightarrow (R, u)\) is \(ijS^*\) continuous function then \(f(X)\) is an interval in \(R\).

**Proof:** Let \(f(X)\) is not an interval, then \(\exists a < r < b\), such that \(a, b \in f(X)\) but \(r \notin f(X)\), then \(X=f^{-1}(-\infty, r) \cup f^{-1}(r, +\infty)\) becomes a disconnection of \(X\), which is a contradiction so \(f(X)\) is an interval in \(R\).

**Remark 4.1.11.** The converse of the above theorem is not true in general. *e.g.* We consider the bitopological space \((X, \tau_1, \tau_2)\) where \(X=[0,1]\), \(\tau_1=\)usual topology on \(X\) induced from \(R\), \(\tau_2=\)indiscrete topology on \(X\) then \((X, \tau_1, \tau_2)\) is \(12S^*\) disconnected (since but the under the \(12S^*\) continuous mapping \(f : (X, \tau_1, \tau_2) \rightarrow (R, u)\) defined by \(f(x)=x\), \(f(X)\) is an interval .
4.2. $ijS^*$ connectedness of a Subset of a Bitopological Space

**Definition 4.2.1.** Let $(X, \tau_1, \tau_2)$ be a bitopological space and $Y \subseteq X$, then $Y$ will said to be a $ijS^*$ disconnected subset if $\exists$ two $ijSO$ sets $U$ and $V$ such that $U \cap Y \neq \emptyset \neq V \cap Y$, $U \cap V \cap Y = \emptyset$ and $Y \subseteq U \cup V$, else $Y$ will said to be a $ijS^*$ connected subset.

**Theorem 4.2.2.** Let $\{U_a\}_{a \in \Lambda}$ be a family of $ijS^*$ connected subsets of a bitopological space $(X, \tau_1, \tau_2)$ such that $\bigcap_{a \in \Lambda} U_a \neq \emptyset$ then $\bigcup_{a \in \Lambda} U_a$ is also $ijS^*$ connected.

**Proof:** Let $x_0 \in \bigcap_{a \in \Lambda} U_a$ and if possible $Y = \bigcup_{a \in \Lambda} U_a$ is not $ijS^*$ connected, then $\exists$ two $ijSO$ sets $U$ and $V$ if $U \cap Y \neq \emptyset \neq V \cap Y$, $U \cap V \cap Y = \emptyset$ and $Y \subseteq U \cup V$, then let $x_0 \in U$ (other case is similar). Now $\exists \alpha \in \Lambda$ such that $U_\alpha \cap V \neq \emptyset$ also $U_\alpha \cap U \neq \emptyset$ (since $x_0 \in U_\alpha$) and also $U \cap V \cap U_\alpha = \emptyset$ and $U_\alpha \subseteq U \cup V$ (since $U_\alpha \subseteq Y$), which shows that $U_\alpha$ is $ijS^*$ disconnected subset and it is a contradiction, so $\bigcup_{a \in \Lambda} U_a$ is also $ijS^*$ connected.

**Theorem 4.2.3.** Let $U$ be a $ijS^*$ connected subset of a bitopological space $(X, \tau_1, \tau_2)$ then if $U \subseteq V \subseteq jjscl(U)$, then $V$ is also a $ijS^*$ connected subset.

**Proof:** Let if possible $V$ is not $ijS^*$ connected subset. Then $\exists$ two $ijSO$ sets $A$ and $B$ such that $A \cap V \neq \emptyset \neq B \cap V$, $A \cap B \cap V = \emptyset$ and $V \subseteq A \cup B$, so $A \cap B \cap U = \emptyset$ and $U \subseteq A \cup B$, since $U$ is $ijS^*$ connected subset so at least $A \cap U = \emptyset$ or $B \cap U = \emptyset$, let $A \cap U = \emptyset$ then $A \cap jjscl(U) = \emptyset$, i.e $A \cap V = \emptyset$ which is a contradiction, hence $V$ is $ijS^*$ connected subset.

**Corr 4.2.4.** Since $U \subseteq jjscl(U) \subseteq jjscl(U)$, hence if $U$ is a $ijS^*$ connected subset $jjscl(U)$ is also a $ijS^*$ connected subset.
4.3 Some Relationships with Fuzzy Bitopological Spaces

In this section we will generalize the concept of ij$S^*$ connectedness in fuzzy bitopological spaces as ij$FS^*$ connectedness and will find some relation between the connectedness of a bitopological space with the connectedness in fuzzy bitopological spaces.

**Definition 4.3.1.** A fuzzy bitopological space $(X, \delta_1, \delta_2)$ is said to be a ij$FS^*$ connected space if there does not exist any non-zero proper fuzzy set $\alpha$ of $X$ (proper means $0 \neq \alpha \neq 1$) which is simultaneously ij$FSO$ and ij$FSC$ set, else the space is said to be a ij$FS^*$ disconnected space.

**Definition 4.3.2.** A fuzzy bitopological space $(X, \delta_1, \delta_2)$ is said to be a strongly ij$FS^*$ ($s$-ij$FS^*$) connected space if there do not exist non-zero proper ij$FSO$ sets $\alpha$ and $\beta$ of $X$ such that $\alpha + \beta \geq 1$, i.e. $\forall x \in X, \alpha(x) + \beta(x) \geq 1$, else the space is said to be a $s$-ij$FS^*$ disconnected space.

**Remark 4.3.3.** ij$FS^*$ connectedness does not imply the ji$FS^*$ connectedness e.g., We consider the fuzzy bitopological space $(X, \delta_1, \delta_2)$ where $X = \{x, y\}$, $\delta_1 = \{0, 1, \{x, 3\}\}$, $\delta_2 = \{0, 1, \{x, 7\}, \{y, 3\}, \{x, 7, y, 3\}\}$

$\delta_1$ closed sets are $0, 1, \{x, 7, y\}$ and $\delta_2$ closed sets are $0, 1, \{x, 3, y\}, \{x, 1, y, 7\}, \{x, 3, y, 7\}$ so all $12$FSO sets are $0, 1, \{x, 3, y\} \forall 0 \leq l \leq .7$ and all $21$FSO sets are $0, 1, \{x, 7, y\} \forall 0 \leq l \leq .7, \{x, 7, y, 3\} \forall 0 \leq l \leq .7$.

Since there is no non-zero proper fuzzy set of $X$ which simultaneously $12$FSO and $12$FSC so the fuzzy bitopological space is $12$FS* connected but $\{x, 7, y\}$ is simultaneously $21$FSO and $21$FSC set, so the fuzzy bitopological space is not $21$FS* connected space.

**Remark 4.3.4.** s-ij$FS^*$ connectedness does not implies the s-ji$FS^*$ connectedness
In the example of Remark 4.3.3 the fuzzy bitopological space is $s$-$12FS^*$ connected but not $s$-$21FS^*$ connected space.

**Theorem 4.3.5.** $s$-$ijFS^*$ connectedness implies the $ijFS^*$ connectedness.

**Proof:** If a fuzzy bitopological space $(X, \delta_1, \delta_2)$ is not $ijFS^*$ connected then $\exists$ a non-zero proper fuzzy set $\alpha$ of $X$ which is simultaneously $ijFSO$ and $ijFSC$ set i.e. if $\beta=1-\alpha$ then $\beta$ is also a non-zero proper fuzzy $ijFSO$ set and $\alpha+\beta=1$ which contradicts the $s$-$ijFS^*$ connectedness of the space, so the space should be $ijFS^*$ connected.

**Remark 4.3.6.** The converse of the Theorem 4.3.5 is not true. *e.g.*

We consider the fuzzy bitopological space $(X, \delta_1, \delta_2)$ where $X=\{x, y\}$, $\delta_1=\{0, 1, \{x_4, y_6\}\}$, $\delta_2=\{0, 1, \{x_7\}, \{x_7, y_3\}\}$, $\delta_2$ closed sets are $0, 1, \{x_3, y_1\}, \{x_1, y_7\}, \{x_3, y_7\}$ so all $12FSO$ sets are $0, 1, \{x_4+k, y_6+l\} \forall 0\leq l \leq .6$ and $0\leq l \leq .1$.

Since there is no non-zero proper fuzzy set which is simultaneously $12FSO$ and $12FSC$, so the space is $12FS^*$ connected but $\alpha=\{x_5, y_6\}$ and $\beta=\{x_6, y_7\}$ are $12FSO$ sets and $\alpha+\beta\geq 1$, so the space is not $s$-$12FS^*$ connected.

**Theorem 4.3.7.** A fuzzy bitopological space $(X, \delta_1, \delta_2)$ is $s$-$ijFS^*$ connected iff $\exists$ no non-zero proper $ijFSC$ sets $\alpha$ and $\beta$ of $X$ such that $\alpha+\beta\leq 1$.

**Proof:** Since complement of a $ijFSC$ set is $ijFSO$ set and conversely, so the proof follows directly from Definition 4.3.2.

**Theorem 4.3.8.** Let $(X, \delta_1, \delta_2)$ be a fuzzy bitopological space and $(Y, \eta)$ be a fuzzy topological space, $f: X \to Y$ be a surjective $ijFS^*$ continuous mapping then if $(X, \delta_1, \delta_2)$ is $ijFS^*$ connected space then $(Y, \eta)$ is a fuzzy connected space.

**Proof:** Let $(Y, \eta)$ is not a fuzzy connected space, so $\exists$ non-zero proper fuzzy set $\alpha$ in $Y$ which is simultaneously fuzzy open and fuzzy closed, then $f^{-1}(\alpha)$ is non-zero proper fuzzy set in $X$ which is simultaneously $ijFSO$ and $ijFSC$ set, a contradiction and the proof follows.

**Remark 4.3.9.** The converse of the above theorem is not true. *e.g.*
We consider the fuzzy bitopological space \((X, \delta_1, \delta_2)\) where \(X=\{x, y\}\), \(\delta_1=\{0, 1, \{x.4\}\}, \delta_2=\{0, 1, \{x.5\}\}\) so all \(12FSO\) sets are \(0, 1, \{x.5, y\}\) so all \(12FSO\) sets are \(0, 1, \{x.4+k, y\}\) \(\forall 0\leq s\leq 1\) and \(0\leq t\leq 1\). Since \(\{x.5, y.5\}\) is non-zero proper fuzzy set which is simultaneously \(12FSO\) and \(12FSC\) so the space is not \(12FS^*\) connected.

Now we consider the fuzzy topological space \((Y, \sigma)\) where \(Y=\{a, b\}\) and \(\sigma=\{0, 1, \{a.5, b.4\}\}\), since there are no non-zero proper fuzzy set in \(Y\) which is simultaneously fuzzy open and fuzzy closed so the space is fuzzy connected , we consider the mapping \(f : X \to Y\) defined as \(f(x)=a, f(y)=b\) then the mapping is surjective \(12FS^*\) continuous.

**Theorem 4.3.10.** Let \((X, \delta_1, \delta_2)\) be a fuzzy bitopological space and \((Y, \eta)\) be a fuzzy topological space, \(f : X \to Y\) be a surjective \(ijFS^*\) continuous mapping then if \((X, \delta_1, \delta_2)\) is \(s-ijFS^*\) connected space then \((Y, \eta)\) is a strongly fuzzy connected space.

**Proof:** If \((Y, \eta)\) is not a strongly fuzzy connected space, so \(\exists\) non-zero proper fuzzy open set \(\alpha\) and \(\beta\) in \(Y\) such that \(\alpha+\beta\geq 1, \therefore f^{-1}(\alpha)+f^{-1}(\beta)\geq 1\), since for any \(x\in X, (f^{-1}(\alpha)+f^{-1}(\beta))(x)=\alpha(f(x))+\beta(f(y))\geq 1\), since \(f^{-1}(\alpha)\) and \(f^{-1}(\beta)\) are non zero proper \(ijFSO\) sets, the space \((X, \delta_1, \delta_2)\) becomes a \(s-ijFS^*\) disconnected, which is a contradiction and hence the theorem follows.

**Remark 4.3.11.** The converse of the above theorem is not true. e.g.

We consider the fuzzy bitopological space \((X, \delta_1, \delta_2)\) where \(X=\{x, y\}\), \(\delta_1=\{0, 1, \{x.4\}\}, \delta_2=\{0, 1, \{x.5\}\}\) \(\delta_2\) closed sets are \(0, 1, \{x.5, y\}\) so all \(12FSO\) sets are \(0, 1, \{x.4+k, y\}\) \(\forall 0\leq s\leq .6\) and \(0\leq t\leq 1\). Since \(\alpha=\{x.5, y.6\}\) and \(\beta=\{x.6, y.7\}\) are non-zero proper \(12FSO\) set such that \(\alpha+\beta\geq 1\), so the space is not \(s-12FS^*\) connected.

Now we consider the fuzzy topological space \((Y, \sigma)\) where \(Y=\{a, b\}\) and \(\sigma=\{0, 1, \{a.5\}, \{a.4, b.2\}\}\), since there are no non-zero proper fuzzy open set \(\gamma\) and \(\lambda\) in \(Y\) so that \(\gamma+\lambda\geq 1\), so space is strongly fuzzy connected , we consider the mapping \(f : X\to Y\) defined as \(f(x)=a, f(b)\) then the mapping is surjective \(12FS^*\) continuous.
Theorem 4.3.12. For a bitopological space \((X, \tau_1, \tau_2)\) if \((X, \omega(\tau_1), \omega(\tau_2))\) ijFS* connected then \((X, \tau_1, \tau_2)\) is ijS* connected.

Proof: If \((X, \tau_1, \tau_2)\) is not ijS* connected there exists a ijS* disconnection \(X=A \cup B\), so \(\chi_A + \chi_B = 1\) and \(\chi_A\) is a non-zero proper fuzzy set which is simultaneously ijFSO set and ijFSC set, which is a contradiction so \((X, \tau_1, \tau_2)\) is ijS* connected.

Remark 4.3.13. The converse of the above theorem is not true. e.g.

We consider the bitopological space \((X, \tau_1, \tau_2)\) where \(X=\{x, y, z\}\), \(\tau_1=\{\emptyset, \{x\}, \{x, y\}, X\}\), \(\tau_2=\{\emptyset, X\}\), all 12SO set are \(\emptyset, \{x\}, \{x, y\}, \{x, z\}\) and \(X\), so the bitopological space is 12S* connected space.

Now \(\{x_5, y_5\}\) is \(\omega(\tau_1)\)-open set with \(\omega(\tau_2)\)-closure \(\{x_5, y_5, z_5\}\), so \(\{x_5, y_5, z_5\}\) is a non zero proper fuzzy set in \(X\) which is simultaneously 12FSO set and 12FSC set in \((X, \omega(\tau_1), \omega(\tau_2))\), so \((X, \omega(\tau_1), \omega(\tau_2))\) is not 12FS* connected.

Theorem 4.3.14. If \((X, \omega(\tau_1), \omega(\tau_2))\) is s-ijFS* connected then \((X, \tau_1, \tau_2)\) is ijS* connected.

Proof: The proof is similar as that of Theorem 4.3.12, since both \(\chi_A\) and \(\chi_B\) are both non zero proper ijFSO sets with \(\chi_A + \chi_B=1\) in \((X, \omega(\tau_1), \omega(\tau_2))\).

Remark 4.3.15. The converse of the above theorem are not true in general. e.g. see Remark 4.3.13 (here if we take 12SO sets \(\alpha=\{x_6, y_6, z_6\}\) and \(\beta=\{x_6, y_6, z_5\}\) then \(\alpha+\beta\geq 1\).