CHAPTER-TWO

Study of some new properties of ijSO sets & Some new notions by ijSO sets

In this chapter we define and study some new properties for bitopological spaces and fuzzy bitopological spaces in connection with ijSO sets. Here we study some relation between ij-semi open sets and ij-fuzzy semi open sets between a bitopological space and the induced fuzzy bitopological space as well as fuzzy bitopological space and the bitopological space or fuzzy bitopological space induced by it. Also we generalize the concepts of semi continuous and semi open type mappings in general topology as ij-S* continuous and ij-S* open mapping between a bitopological space and a topological space as well as we study such functions in fuzzy setting also. Lastly we define some new type of convergence of net, filter and filterbase in bitopological spaces and some relationship between crisp setting and fuzzy setting.

2.1. The Bitopological Space (X, τ₁≤τ₂)

Here we study some properties in a bitopological space (X, τ₁, τ₂) where the topology τ₁ is weaker than τ₂, i.e. τ₁≤τ₂. We denote such bitopological space by (X, τ₁≤τ₂).

Theorem 2.1.1. Let (X, τ₁≤τ₂) be a bitopological space, then 12SO(X) ⊆ SO(X,τ₁)∩SO(X,τ₂) ⊆ SO(X,τ₁)∪SO(X,τ₂) ⊆ 21SO(X).

Proof: Let A∈12SO(X), so there exists a τ₁-open set U such that U⊆A⊆τ₂-cl(U) ⊆ τ₁-cl(U), since τ₁≤τ₂ and also U is also a τ₂-open set, i.e. A∈SO(X,τ₁) ∩ SO(X,τ₂).

Again, let A∈SO(X,τ₁)∪SO(X,τ₂).
If \( A \in \text{SO}(X, \tau_1) \) so their exists a \( \tau_1 \)-open set \( U \) such that \( U \subseteq A \subseteq \tau_1\text{-cl}(U) \), since \( \tau_1 \subseteq \tau_2 \), \( U \) is also \( \tau_2 \)-open set, so \( A \in 21\text{SO}(X) \).

If \( A \in \text{SO}(X, \tau_2) \) so their exists a \( \tau_2 \)-open set \( U \) such that \( U \subseteq A \subseteq \tau_2\text{-cl}(U) \subseteq \tau_1\text{-cl}(U) \), since \( \tau_1 \subseteq \tau_2 \), so \( A \in 21\text{SO}(X) \).

Hence the theorem follows.

**Corollary 2.1.2.** Let \( (X, \tau_1 \leq \tau_2) \) be a bitopological space, then for any \( A \subseteq X \), \( 12\text{sint}(A) \subseteq \tau_1\text{-sint}(A) \subseteq 21\text{sint}(A) \), \( 12\text{sint}(A) \subseteq \tau_2\text{-sint}(A) \subseteq 21\text{sint}(A) \).

**Corollary 2.1.3.** Let \( (X, \tau_1, \tau_2) \) be a bitopological space, then \( 12\text{SC}(X) \subseteq \text{SC}(X, \tau_1) \cup \text{SC}(X, \tau_2) \subseteq 21\text{SC}(X) \).

**Corollary 2.1.4.** Let \( (X, \tau_1 \leq \tau_2) \) be a bitopological space, then for any \( A \subseteq X \), \( 21\text{scl}(A) \subseteq \tau_1\text{scl}(A) \subseteq 12\text{scl}(A) \), \( 21\text{scl}(A) \subseteq \tau_2\text{scl}(A) \subseteq 12\text{scl}(A) \).

**Theorem 2.1.5.** If \( (X, \tau_2) \) is extremely disconnected(ED) space then intersection of any two \( 12\text{SO} \) sets in the bitopological space \( (X, \tau_1 \leq \tau_2) \) is also a \( 12\text{SO} \) set.

**Proof:** Let us recall the definition of extremely disconnected(ED) topological space, which is defined as: A topological space \( (X, \tau) \) is extremely disconnected if \( \tau \)-closure of every \( \tau \)-open set is \( \tau \)-open in \( (X, \tau) \).

Let \( A \) and \( B \) be two \( 12\text{SO} \) sets, so there exist two \( \tau_1 \)-open sets \( U \) and \( V \) such that \( U \subseteq A \subseteq \tau_2\text{-cl}(U) \) and \( V \subseteq B \subseteq \tau_2\text{-cl}(V) \). It is to be noted that \( U \) and \( V \) are also \( \tau_2 \)-open sets.

We have \( \tau_2\text{-cl}(U \cap V) \subseteq \tau_2\text{-cl}(U) \cap \tau_2\text{-cl}(V) \).

Now let \( x \in \tau_2\text{-cl}(U \cap V) \cap \tau_2\text{-cl}(V) \), let \( W \) be an \( \tau_2 \)-open set containing \( x \), let if possible \( U \cap V \cap W = \emptyset \). So \( U \subseteq X - V \cap W \), so \( \tau_2\text{-cl}(U) \subseteq X - V \cap W \), i.e. \( \tau_2\text{-cl}(U) \cap V \cap W = \emptyset \). Since \( \tau_2\text{-cl}(U) \) is also \( \tau_2 \)-open, we have similarly \( \tau_2\text{-cl}(U) \cap \tau_2\text{-cl}(V) \cap W = \emptyset \), which is impossible since \( x \in \tau_2\text{-cl}(U) \cap \tau_2\text{-cl}(V) \cap W \), so \( U \cap V \cap W \neq \emptyset \), i.e. \( x \in \tau_2\text{-cl}(U \cap V) \), i.e. \( \tau_2\text{-cl}(U) \cap \tau_2\text{-cl}(V) \subseteq \tau_2\text{-cl}(U \cap V) \). Hence \( \tau_2\text{-cl}(U) \cap \tau_2\text{-cl}(V) = \tau_2\text{-cl}(U \cap V) \).

Now, \( U \cap V \subseteq A \cap B \subseteq \tau_2\text{-cl}(U) \cap \tau_2\text{-cl}(V) = \tau_2\text{-cl}(U \cap V) \), so \( A \cap B \) is also \( 12\text{SO} \) set.
2.2 More on Induced Spaces

If \((X, \delta_1, \delta_2)\) is a fuzzy bitopological space then for each \(t \in [0, 1)\) \((X, \iota(t \delta_1), \iota(t \delta_1))\) is a bitopological space, again if \((X, \tau_1, \tau_2)\) is bitopological space then \((X, \omega(t \tau_1), \omega(t \tau_2))\) is a fuzzy bitopological space for each \(t \in [0, 1)\) and \((X, \omega(\tau_1), \omega(\tau_2))\) is a fuzzy bitopological space.

**Theorem 2.2.1.** If \(A\) is a ijFSO of a fuzzy bitopological space \((X, \omega(t \tau_1), \omega(t \tau_1)))\) then \(A^{-1}(t, 1]\) is ijSO set in the bitopological space \((X, \iota(t \delta_1), \iota(t \delta_1)).\)

**Proof:** Let \(A\) is a ijFSO of a fuzzy bitopological space \((X, \omega(t \delta_1), \omega(t \delta_1)), \exists \) a \(\omega(t \delta_1)\)-fuzzy open set \(U\) such that \(U \subseteq A \subseteq \omega(t \delta_1)-cl(U), \vdots, U^{-1}(t, 1] \subseteq A^{-1}(t, 1]\) \subseteq (\omega(t \delta_1)-cl(U))^{-1}(t, 1].\)

Now, since \(\omega(t \delta_1)\subseteq \omega(t \delta_1)\) and \((\omega(t \delta_1))-cl(U))^{-1}(t, 1]\subseteq t \delta_1)-cl(U^{-1}(t, 1])\) (by Theorem 1.6.5).

So \(\omega(t \delta_1)-cl(U) \subseteq \omega(t \delta_1)-cl(U)\)
and so \((\omega(t \delta_1))-cl(U))^{-1}(t, 1]\subseteq (\omega(t \delta_1))-cl(U))^{-1}(t, 1].\)

Hence \(U^{-1}(t, 1]\subseteq A^{-1}(t, 1]\subseteq t \delta_1)-cl(U^{-1}(t, 1]).\)

Since \(U \in \omega(t \delta_1)\) so \(U^{-1}(t, 1] \in \iota(\omega(t \delta_1))= \iota(t \delta_1)\) ( \(\therefore \iota \omega \iota = \text{identity}\) (by Theorem 1.6.4(i)).

Hence, \(A^{-1}(t, 1]\) is ijSO set in \((X, \iota(t \delta_1), \iota(t \delta_1)).\)

**Theorem 2.2.2.** If \(A\) is a ijFSO of a fuzzy bitopological space \((X, \omega(t \tau_1), \omega(t \tau_2))\) then \(A^{-1}(t, 1]\) is ijSO set in the bitopological space \((X, \tau_1, \tau_2).\)

**Proof:** Let \(A\) is a ijFSO of a fuzzy bitopological space \((X, \omega(t \tau_1), \omega(t \tau_2)), \exists \) a \(\omega(t \tau_1)\)-fuzzy open set \(U\) such that \(U \subseteq A \subseteq \omega(t \tau_1)-cl(U), \vdots, U^{-1}(t, 1]\subseteq A^{-1}(t, 1]\) \subseteq (\omega(t \tau_1)-cl(U))^{-1}(t, 1].\)

Now, since \(\omega(t \tau_1)\subseteq \omega(t \tau_1)\) and \((\omega(t \tau_1))-cl(U))^{-1}(t, 1]\subseteq \tau_2)-cl(U^{-1}(t, 1])\) (by Theorem 1.6.5), so \(\omega(t \tau_1)-cl(U) \subseteq \omega(t \tau_1)-cl(U)\)
and so \((\omega(t \tau_1))-cl(U))^{-1}(t, 1]\subseteq (\omega(t \tau_1))-cl(U))^{-1}(t, 1].\)

Hence \(U^{-1}(t, 1]\subseteq A^{-1}(t, 1]\subseteq \tau_2)-cl(U^{-1}(t, 1]).\)

Since \(U \in \omega(t \tau_1)\) so \(U^{-1}(t, 1] \in \iota(\omega(t \tau_1))= \tau_1\) ( \(\therefore \iota \omega \iota = \text{identity}\) (by Theorem 1.6.4).

Hence, \(A^{-1}(t, 1]\) is ijSO set in \((X, \tau_1, \tau_2).\)
Theorem 2.2.3. If $A$ is a ijFSO of a fuzzy bitopological space $(X, \omega(\tau_1), \omega(\tau_2))$ then $A^{-1}(t,1]$ is ijSO set in the bitopological space $(X, \tau_1, \tau_2)$.

Proof: Let $A$ is a ijFSO of a fuzzy bitopological space $(X, \omega(\tau_1), \omega(\tau_2))$, so $\exists$ a $\omega(\tau_1)$-fuzzy open set $U$ such that $U \leq A \leq \omega(\tau_2)-cl(U)$,

$\vdots U^{-1}(t,1] \subseteq A^{-1}(t,1] \subseteq (\omega(\tau_1)-cl(U))^{-1}(t,1] \subseteq \tau_1-cl(U^{-1}(t,1])$.

Since $U \in \omega(\tau_1)$ so $U^{-1}(t,1] \in \tau_1$.

Hence, $A^{-1}(t,1]$ is ijSO set in $(X, \tau_1, \tau_2)$.

Theorem 2.2.4. If $A$ is a ijSO of a bitopological space $(X, \tau_1, \tau_2)$ then $\chi_A$ is ijFSO - set in the fuzzy bitopological space $(X, \omega(\tau_1), \omega(\tau_2))$.

Proof: Let $A$ is a ijSO of a bitopological space $(X, \tau_1, \tau_2)$, so $\exists$ a $\tau_1$-open set $U$ such that $U \subseteq A \subseteq \tau_2-cl(U)$.

We will see that $\chi_{\tau_2-cl(U)} = \omega(\tau_1)-cl(\chi_U)$:

Let $\alpha \in \omega(\tau_1)$ and $\chi_U \leq 1 - \alpha$, i.e. $\alpha \leq 1 - \chi_U = \chi_{(X-U)}$.

So, for $0 \leq t < 1$, $\alpha^{-1}(t,1] = (\chi_{(X-U)})^{-1}(t,1] = X - U$, i.e. $U \subseteq X - \alpha^{-1}(t,1]$, also $\alpha^{-1}(t,1] \in \tau_1$ for all $0 \leq t < 1$. So $U \subseteq X - \alpha^{-1}(0,1]$, i.e. $\tau_2-cl(U) \subseteq X - \alpha^{-1}(0,1]$, so $\chi_{\tau_2-cl(U)} \wedge \alpha = 0$ and so $\chi_{\tau_2-cl(U)} \leq 1 - \alpha$,

so every closed set containing $\chi_U$ in $\omega(\tau_1)$ contains $\chi_{\tau_2-cl(U)} \supseteq \chi_U$.

Again $\chi_{\tau_2-cl(U)}$ is a closed set in $\omega(\tau_1)$, since $1 - \chi_{\tau_2-cl(U)} = \chi_{X-\tau_2-cl(U)}$ an open set in $\omega(\tau_1)$ i.e. $\chi_{\tau_2-cl(U)} = \omega(\tau_1)-cl(\chi_U)$.

Now $\chi_U \leq \chi_A \leq \chi_{\tau_2-cl(U)} = \omega(\tau_1)-cl(\chi_U)$, since $\chi_U \in \omega(\tau_1)$,

so $\chi_A$ is a ijFSO in $(X, \omega(\tau_1), \omega(\tau_2))$. 
2.3 Some New types of Mappings

**Definition 2.3.1.** Let \((X, \tau_1, \tau_2)\) be a bitopological space and \((Y, \sigma)\) be a topological space then a mapping \(f : X \rightarrow Y\) is said to be \(ijS^*\) continuous iff for any \(V \in \sigma\), \(f^{-1}(V)\) is a \(ijSO\) set in \((X, \tau_1, \tau_2)\).

**Theorem 2.3.2.** Let \((X, \tau_1, \tau_2)\) be a bitopological space and \((Y, \sigma)\) be a topological space and \(f : X \rightarrow Y\) then the following statements are equivalent:

(i) \(f\) is \(ijS^*\) continuous function.

(ii) for any closed set \(F\) in \((Y, \sigma)\), \(f^{-1}(F)\) is \(ijSC\) set.

(iii) for any \(A \subseteq X\) \(f(ijscl(A)) \subseteq \sigma-cl(f(A))\).

(iv) for any \(B \subseteq Y\), \(ijsccl(f^{-1}(B)) \subseteq f^{-1}(\sigma-cl(B))\).

(v) for any \(B \subseteq Y\), \(f^{-1}(\sigma-int(B)) \subseteq ijscint(f^{-1}(B))\).

**Proof:** (i)\(\Rightarrow\)(ii): Let \(F\) be a closed set in \(Y\), so \(Y - F\) is open in \(Y\). So, \(f^{-1}(Y - F) = X - f^{-1}(F)\) is \(ijSO\) in \(X\), i.e. \(f^{-1}(F)\) is \(ijSC\) in \(X\).

(ii)\(\Rightarrow\)(i): The proof is similar to that of above.

(iii)\(\Rightarrow\)(iv): For any \(A \subseteq X, f(A) \subseteq \sigma-cl(f(A))\). \(A = f^{-1}(f(A)) \subseteq f^{-1}(\sigma-cl(f(A)))\),

\[\therefore ijscl(A) \subseteq f^{-1}(\sigma-cl(f(A)))\]

\(\therefore f(ijscl(A)) \subseteq f(f^{-1}(\sigma-cl(f(A)))) \subseteq \sigma-cl(f(A))\).

(iv)\(\Rightarrow\)(iii): Let \(B\) be a closed set in \(Y\), \(\therefore ijscl(f^{-1}(B)) \subseteq f^{-1}(\sigma-cl(f^{-1}(B)))\).

But \(f^{-1}(B) \subseteq ijscl(f^{-1}(B))\), so \(f^{-1}(B) = ijscl(f^{-1}(B))\), i.e. \(f^{-1}(B)\) is \(ijSC\), i.e. \(f\) is \(ijS^*\) continuous.

(i)\(\Rightarrow\)(v): Since \(f^{-1}(\sigma-int(B))\) is a \(ijSO\) and \(f^{-1}(\sigma-int(B)) \subseteq f^{-1}(B)\), so \(f^{-1}(\sigma-int(B)) \subseteq ijscint(f^{-1}(B))\).

(v)\(\Rightarrow\)(i): Let \(B\) be any open set in \(Y\),
also \( f^{-1}(B) = f^{-1}(\sigma \text{-int}(B)) \subseteq ijSint f^{-1}(B) \) but \( ijSint(f^{-1}(B)) \subseteq f^{-1}(B) \).

i.e. \( f^{-1}(B)=ijSint(f^{-1}(B)) \) i.e. \( f^{-1}(B) \) is an ijSO set, so \( f \) is ijS* continuous function.

\[
\begin{align*}
(v) & \quad \iff \quad (i) \quad \iff \quad (ii) \\
(iv) & \quad \iff \quad (iii)
\end{align*}
\]

**Theorem 2.3.3.** Let \((X, \tau_1, \tau_2)\) be a bitopological space and \((Y, \sigma)\) be a topological space then \( f : X \to Y \) is ijS* continuous function iff for any \( x \in X \) and any open set \( V \) in \( Y \) containing \( f(x) \) \( \exists \) an ijSO set \( U \) in \( X \) containing \( x \) such that \( f(U) \subseteq V \).

**Proof:** Let \( f \) is ijS* continuous function, let \( x \in X \) and \( V \) a open set in \( Y \) containing \( f(x) \), so \( U=f^{-1}(V) \) is a ijSO set in \( X \) containing \( x \) such that \( f(U) \subseteq V \).

Conversely Let for any \( x \in X \) and any open set \( V \) in \( Y \) containing \( f(x) \) \( \exists \) an ijSO set \( U \) in \( X \) containing \( x \) such that \( f(U) \subseteq V \). Let \( W \) be an open set in \( Y \).

Let \( x \in f^{-1}(W) \), i.e. \( f(x) \in W \), so \( \exists \) an ijSO set \( U_x \) in \( X \) containing \( x \) such that \( f(U_x) \subseteq W \). \( \forall \ x \in W \).

\[ \therefore f^{-1}(W) = \bigcup \{ x : x \in f^{-1}(W) \} \subseteq \bigcup \{ U_x : x \in f^{-1}(W) \} \subseteq f^{-1}(W), \]

i.e. \( f^{-1}(W) = \bigcup \{ U_x : x \in f^{-1}(W) \} \), i.e. \( f^{-1}(W) \) is an ijSO set, i.e. \( f \) is ijS* continuous mapping.

**Theorem 2.3.4.** Let \((X, \tau_1, \tau_2)\) be a bitopological space and \((Y, \sigma)\) be a topological space then \( f : X \to Y \) is ijS* continuous function iff for any base \( \mathcal{B} \) of \((Y, \sigma)\), \( V \in \mathcal{B} \), \( f^{-1}(V) \) is a ijSO set.

**Proof:** The proof follows from the fact that arbitrary union of ijSO set is a ijSO set.

**Definition 2.3.5.** Let \((X, \tau_1, \tau_2)\) be a bitopological space and \((Y, \sigma)\) be a topological space then a mapping \( f : Y \to X \) is said to be ijS* open mapping if for any \( V \in \sigma \), \( f(V) \) is ijSO set in \((X, \tau_1, \tau_2)\).
Theorem 2.3.6. Let \((X, \tau_1, \tau_2)\) be bitopological space, \((Y, \sigma)\) and \((Z, \eta)\) are topological spaces, \(f: Y \to X\) is a \(ijS^*\) open mapping and also onto, \(g: X \to Z\) be any mapping, if the composition function \(gof: Y \to Z\) is continuous mapping, then \(g\) is a \(ijS^*\) continuous mapping.

**Proof:** Let \(gof\) is continuous, let \(V\) is a open set in the topological space \((Z, \eta)\), so \((gof)^{-1}(V) = f^{-1}(g^{-1}(V))\) is a open set in the topological space \((Y, \sigma)\), again since \(f\) is \(ijS^*\) open mapping so \(f(f^{-1}(g^{-1}(V)))\) is a \(ijSO\) set in the bitopological space \((X, \tau_1, \tau_2)\), since \(f\) is onto mapping so \(f(f^{-1}(g^{-1}(V))) = g^{-1}(V)\), which shows that \(g\) is a \(ijS^*\) continuous mapping.

Remark 2.3.7. The converse of the above theorem is not true in general, e.g. Let us consider the bitopological space \((X, \tau_1, \tau_2)\), where \(X = \{a, b, c\}\), \(\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}\), \(\tau_2 = \{\emptyset, X\}\).

The 12SO sets in \(X\) are: \(\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\).

Again we consider the topological spaces \((Y, \sigma)\) and \((Z, \eta)\) where \(Y = \{x, y, z\}\), \(\sigma = \{\emptyset, \{x\}, \{x, y\}, Y\}\), \(Z = \{w, t\}\) and \(\eta = \{\emptyset, \{t\}, Z\}\).

Now we consider the mapping \(f: (Y, \sigma) \to (X, \tau_1, \tau_2)\) defined by \(f(x) = a, f(y) = b, f(z) = c\) which is surjective and \(12S^*\) open mapping, since image of open sets in \(Y\) under \(f\) are: \(\emptyset, \{a\}, \{a, b\}, X\) which are 12SO sets in \(X\).

We consider the mapping \(g: (X, \tau_1, \tau_2) \to (Z, \eta)\) defined by \(g(a) = w, g(b) = t, g(c) = w\) is \(12S^*\) continuous mapping, since the inverse image of open sets in \(Z\) under the mapping \(g\) are: \(\emptyset, \{b\}, X\) which are 12SO in \(X\).

Now the mapping \(gof: (Y, \sigma) \to (Z, \eta)\) is defined as \(gof(x) = w, gof(y) = t, gof(z) = w\) is not a continuous mapping, since the inverse image of the open set \(\{t\}\) in \(Z\) under \(gof\) is \(\{y\}\) which is not open in \(Y\).

Definition 2.3.8. Let \((X, \delta_1, \delta_2)\) be a fuzzy bitopological space and \((Y, \eta)\) be a fuzzy topological space, then a mapping \(f: (X, \delta_1, \delta_2) \to (Y, \sigma)\) is said to be \(ijFS^*\) continuous if for any \(\alpha \in \eta\), \(f^{-1}(\alpha)\) is a \(ijFSO\) set.

Remark 2.3.9. A \(ijFS^*\) continuous function may not be a \(jiFS^*\) continuous function.
e.g., we consider the fuzzy bitopological space \((X, \delta_1, \delta_2)\) where \(X=\{x, y\}, \delta_1=\{0, 1, \{x, y\}\}, \delta_2=\{0, 1, \{x, y\}\}\) and we consider the fuzzy topological space \((Y, \eta)\) where \(Y=\{a, b, c\}, \eta=\{0, 1, \{a, b, c\}\}\). We consider the mapping \(f : X \to Y\) defined as \(f(x)=a, f(y)=b\) then the mapping is 12FS* continuous but not 21FS* continuous.

**Theorem 2.3.10.** Let \((X, \delta_1, \delta_2)\) be a fuzzy bitopological space and \((Y, \eta)\) be a fuzzy topological space, \(f : X \to Y\) be a mapping then the following conditions are equivalent:

(i) \(f\) is \(ijFS^*\) continuous function.

(ii) for any fuzzy closed set \(\beta\) in \((Y, \eta)\) \(f^{-1}(\beta)\) is \(ijFSC\)

(iii) for any fuzzy set \(\beta\) of \(Y\) \(ijfscl(f^{-1}(\beta)) \subseteq f^{-1}(\eta-cl(\beta))\)

(iv) for any fuzzy set \(\alpha\) of \(X\), \(f(ijfscl(\alpha)) \subseteq \eta-cl(f(\alpha))\).

(v) for any fuzzy set \(\beta\) of \(Y\) \(f^{-1}(\eta-fint(\beta)) \subseteq ijfsint(f^{-1}(\beta))\)

**Proof:** Similar as Theorem 2.3.2.

**Theorem 2.3.11.** Let \((X, \tau_1, \tau_2)\) bitopological space and \((Y, \sigma)\) be a topological space, let \(f : X \to Y\) be a mapping if \(f : (X, \omega(\tau_1), \omega(\tau_2)) \to (Y, \omega(\sigma))\) is \(ijFS^*\) continuous mapping then \(f : (X, \tau_1, \tau_2) \to (Y, \sigma)\) is \(ijS^*\) continuous mapping.

**Proof:** Let \(f : (X, \omega(\tau_1), \omega(\tau_2)) \to (Y, \omega(\sigma))\) is \(ijS^*\) continuous mapping and \(U \in \sigma\) and \(x \in f^{-1}(U)\), \(\therefore f(x) \in U\), so \(\chi_U\) is a fuzzy open set in \((Y, \omega(\sigma))\) with \(x_1 \in f^{-1}(\chi_U)\), since \(f^{-1}(\chi_U)\) is a \(ijFSO\) set in \((X, \omega(\tau_1), \omega(\tau_2))\), so there exists a \(ijFSO\) \(\alpha\) in \((X, \omega(\tau_1), \omega(\tau_2))\) such that \(x_1 \in \alpha \subseteq f^{-1}(\chi_U)\), so for any \(r \in [0, 1]\)

\[x \in \alpha^{-1}(r, 1] \subseteq (f^{-1}(\chi_U))^{-1}(r, 1] = (\chi_U \circ f)^{-1}(r, 1] = f^{-1}(\chi_U^{-1}(r, 1]) = f^{-1}(U),\]

since \(\alpha^{-1}(r, 1]\) is \(ijSO\) set in \((X, \tau_1, \tau_2)\), so \(f^{-1}(U)\) is a \(ijSO\) set in \((X, \tau_1, \tau_2)\). Hence the theorem follows.

**Definition 2.3.12.** Let \((X, \tau_1, \tau_2)\) be a bitopological space and we consider \(R\) with the usual topology \(u\), then a mapping \(f : X \to R\) is said to be \(ijS^*\) lower semi continuous (\(ijS^*\) upper semi continuous) if for any \(r \in R\), \(f^{-1}(r, \infty)\) (resp. \(f^{-1}(\infty, r)\)) is \(ijSO\) set in \((X, \tau_1, \tau_2)\) or equivalently \(f^{-1}(\infty, r]\) (resp. \(f^{-1}[r, \infty)\)) is \(ijSC\) set in \((X, \tau_1, \tau_2)\).
Corollary 2.3.13. If \( f : X \to \mathbb{R} \) is \( ijS^* \) continuous then \( f \) is both \( ijS^* \) upper semi continuous and \( ijS^* \) lower semi continuous. But the converse may not be true which can be seen from the following example:

Consider the bitopological space \((X, \tau_1, \tau_2)\), where \( X = \{x, y, z\}, \tau_1 = \{\emptyset, \{x\}, \{z\}, \{x, z\}, X\}, \tau_2 = \{\emptyset, \{x\}, \{x, z\}\}.\)

All 12SO sets are: \( \emptyset, \{x\}, \{z\}, \{x, z\}, \{y, z\} \) and \( X \)

mapping \( f : (X, \tau_1, \tau_2) \to \mathbb{R}, \) defined as: \( f(x) = 2, f(y) = 1, f(z) = 0. \)

Now, for \( r \in \mathbb{R}, \)

\[
\begin{align*}
f^{-1}(r, \infty) &= \begin{cases} 
X, r < 0 \\
\{x, y\}, 0 \leq r < 1 \\
\{x\}, 1 \leq r < 2 \\
\emptyset, r \geq 2 
\end{cases} \\
\end{align*}
\]

and,

\[
\begin{align*}
f^{-1}(-\infty, r) &= \begin{cases} 
\emptyset, r \leq 0 \\
\{z\}, 0 < r \leq 1 \\
\{y, z\}, 1 < r \leq 2 \\
X, r > 2 
\end{cases} \\
\end{align*}
\]

This shows that \( f \) is both \( 12S^* \) lower and upper semi continuous, but \( f \) is not \( 12S^* \) continuous, since \( f^{-1}(0.5, 1.5) = \{y\} \) which is not 12SO set.

But in a certain bitopological spaces the converse of the above corollary is also true. It can be seen from the following theorem:

Theorem 2.3.14. If in the bitopological space \((X, \tau_1 \leq \tau_2)\), \((X, \tau_2)\) is extremely disconnected (ED), then a function \( f : (X, \tau_1, \tau_2) \to \mathbb{R} \) which is both \( 12S^* \) upper and lower semi continuous is \( 12S^* \) continuous.

Proof: Since in the bitopological space \((X, \tau_1 \leq \tau_2)\), \((X, \tau_2)\) is ED, so intersection of any two 12SO sets is 12SO set.

Now for \( a < b, f^{-1}(a, b) = f^{-1}\{(-\infty, b) \cap (a, \infty)\} = f^{-1}(-\infty, b) \cap f^{-1}(a, \infty), \) now both \( f^{-1}(-\infty, b) \) and \( f^{-1}(a, \infty) \) are 12SO set and so \( f^{-1}(-\infty, b) \cap f^{-1}(a, \infty) \) and hence \( f^{-1}(a, b) \) is a 12SO set. Since the collection \( \{(a, b): a < b\} \) forms a base for the usual topology on \( \mathbb{R} \), so \( f \) is a \( 12S^* \) continuous mapping.
2.4. ijS* Induced Fuzzy Supra Topological space

Definition 2.4.1 Let \((X, \tau_1, \tau_2)\) be a bitopological space and we consider the unit interval \(I=[0, 1]\) with the usual topology, then a mapping \(f: X \rightarrow I\) is called ijS* lower semi continuous if for any \(r \in [0, 1]\), \(f^{-1}(r, 1]\) is an ijSO set in \((X, \tau_1, \tau_2)\) or equivalently \(f^{-1}[0, r]\) is an ijSC set in \((X, \tau_1, \tau_2)\).

For any bitopological space \((X, \tau_1, \tau_2)\), let \(\omega_{ij}(X)\) be the set of all ijS* lower semi continuous functions from \((X, \tau_1, \tau_2)\) to \(I\). Now we have the following theorem:

Theorem 2.4.2. \((X, \omega_{ij}(X))\) is a fuzzy supra topological space.

Proof: It is very easy to see that \(0: (X, \tau_1, \tau_2) \rightarrow I\), defined by \(0(x)=0\), for all \(x \in X\), is a member of \(\omega_{ij}(X)\) and \(1: (X, \tau_1, \tau_2) \rightarrow I\), defined by \(1(x)=1\), for all \(x \in X\), is a member of \(\omega_{ij}(X)\).

Now let \(\{f_\alpha: \alpha \in \Delta\}\) be a family of members in \(\omega_{ij}(X)\), let \(h=\sup \{f_\alpha: \alpha \in \Delta\}\), so for \(r \in [0, 1]\), \(h^{-1}(r, 1]=\{x: \sup \{f_\alpha(x): \alpha \in \Delta\}>r\} = \bigcup_{\alpha \in \Delta} \{x: f_\alpha(x)>r\} = \bigcup_{\alpha \in \Delta} f_\alpha^{-1}(r, 1]\) which is an ijSO set, so \(\sup \{f_\alpha: \alpha \in \Delta\} \in \omega_{ij}(X)\).

So, \((X, \omega_{ij}(X))\) is a fuzzy supra topological space.

Theorem 2.4.3. For the bitopological space \((X, \tau_1 \leq \tau_2)\) where \(\tau_2\) is extremely disconnected, then \((X, \omega_{ij2}(X))\) is a fuzzy topological space.

Proof: This is due to the fact that in such bitopological spaces intersection of any two 12SO sets is a 12SO set.

Theorem 2.4.4. For any subset \(U\) of a bitopological space \((X, \tau_1, \tau_2)\) with the induced fuzzy supra topological space \((X, \omega_{ij}(X))\):

\[(i) \quad \omega_{ij}\text{-}\text{cl}(\chi_U) = \chi_{ij\text{scl}(U)}\]
\[(ii) \quad \omega_{ij}\text{-}\text{int}(\chi_U) = \chi_{ij\text{sint}(U)}\]

Proof: (i) Let \(\alpha \in \omega_{ij}(X)\) and \(\chi_U \leq 1 - \alpha\), i.e. \(\alpha \leq 1 - \chi_U = \chi_{X-U}\).

So, \(\alpha^{-1}(0,1] \subseteq (\chi_{X-U})^{-1}(0,1] = X - U\), i.e. \(U \subseteq X - \alpha^{-1}(0,1]\), also \(\alpha^{-1}(0,1]\) is a ijSO set, so \(ij\text{scl}(U) \subseteq X - \alpha^{-1}(0,1]\),
so \( \chi_{ij\text{-}cls}(U) \wedge \alpha = 0 \) and so \( \chi_{ij\text{-}cls}(U) \leq 1 - \alpha \),

hence every fuzzy closed set containing \( \chi_U \) in \( (X, \omega_{ij}(X)) \) contains \( \chi_{ij\text{-}cls}(U) \geq \chi_U 

Again \( \chi_{ij\text{-}cls}(U) \) is a fuzzy closed set in \( (X, \omega_{ij}(X)) \), since \( 1 - \chi_{ij\text{-}cls}(U) = \chi_{X-ij\text{-}cls}(U) \) a fuzzy open set in \( (X, \omega_{ij}(X)) \), i.e. \( \chi_{ij\text{-}cls}(U) = \omega_{ij}\text{-}cl(\chi_U) \).

(ii) Let \( \alpha \in \omega_{ij}(X) \) and \( \alpha \leq \chi_U \), so \( \alpha^{-1}(0,1] \subseteq U \), since \( \alpha^{-1}(0,1] \) is a ijSO set in \( (X, \tau_1, \tau_2) \), so \( \alpha^{-1}(0,1] \subseteq j\text{-}int(\chi_U) \), hence \( \alpha \leq j\text{-}int(\chi_U) \), so \( \omega_{ij}\text{-}int(\chi_U) \leq \chi_{j\text{-}int}(U) \). Again \( \chi_{j\text{-}int}(U) \) is an open set in \( \omega_{ij}(X) \) contained in \( \chi_U \), so \( \chi_{j\text{-}int}(U) \leq \omega_{ij}\text{-}int(\chi_U) \), i.e \( \omega_{ij}\text{-}int(\chi_U) = \chi_{j\text{-}int}(U) \).

**Theorem 2.4.5.** For any bitopological space \( (X, \tau_1, \tau_2) \), for any fuzzy subset \( \lambda \) of \( X \) and \( r \in [0,1) \),

(i) \( (\omega_{ij}\text{-}cl(\lambda))^{-1}(r,1] \subseteq ij\text{-}cls(\lambda^{-1}(r,1]) \).

(ii) \( (\omega_{ij}\text{-}int(\lambda))^{-1}(r,1] \subseteq ij\text{-}int(\lambda^{-1}(r,1]) \)

**Proof:** (i) Let \( C \) be a ijSC set in \( (X, \tau_1, \tau_2) \) with \( \lambda^{-1}(r,1] \subseteq C \), let us define the mapping \( \varphi_C: X \to [0,1] \) as

\[
\varphi_C(x) = \begin{cases} 
1, & \text{if } x \in C \\
r, & \text{if } x \notin C
\end{cases}
\]

It is easy to see that \( \varphi_C \) is a fuzzy closed set in \( (X, \omega_{ij}(X)) \) with \( \lambda \leq \varphi_C \), so \( \omega_{ij}\text{-}cl(\lambda) \leq \varphi_C \). Now \( \omega_{ij}\text{-}cl(\lambda)(x) > r \) implies \( \varphi_C(x) > r \), i.e. \( x \in C \), so \( (\omega_{ij}\text{-}cl(\lambda))^{-1}(r,1] \subseteq C \), hence \( (\omega_{ij}\text{-}cl(\lambda))^{-1}(r,1] \subseteq ij\text{-}cls(\lambda^{-1}(r,1]) \).

(ii) \( \omega_{ij}\text{-}int(\lambda) \leq \lambda \) and so \( (\omega_{ij}\text{-}int(\lambda))^{-1}(r,1] \subseteq \lambda^{-1}(r,1] \). Since \( (\omega_{ij}\text{-}int(\lambda))^{-1}(r,1] \) is an ijSO set in \( (X, \tau_1, \tau_2) \), so \( (\omega_{ij}\text{-}int(\lambda))^{-1}(r,1] \subseteq ij\text{-}int(\lambda^{-1}(r,1]) \).
2.5. Some Definitions on Convergency

Definition 2.5.1. A subset $A$ in a bitopological space $(X, \tau_1, \tau_2)$ is called $i*j$-sclopen if $A$ is simultaneously $ij$SO set and $ji$SC set.

Lemma 2.5.2. The complement of a $i*j$-sclopen set is $j*i$-sclopen set.

Proof: The proof follows from the fact that the complement of a $ij$SO set is $ij$SC set and the complement of $ji$SC set is $ji$SO set.

Theorem 2.5.3. In a bitopological space $(X, \tau_1, \tau_2)$ $ji$-semi closure of a $ij$SO set is $i*j$-sclopen.

Proof: Let $A$ be a $ij$SO set, so $\exists$ a $\tau_i$-open set $U$ such that $U \subseteq A \subseteq \tau_i$-cl($U$), $\therefore$, $jiscl(U) \subseteq jiscl(A) \subseteq \tau_i$-cl($U$), so $U \subseteq jiscl(A) \subseteq \tau_i$-cl($U$), so $jiscl(A)$ is a $ij$SO set and also a $ji$SC set i.e., $i*j$-sclopen.

Theorem 2.5.4. In a bitopological space $(X, \tau_1, \tau_2)$ $ji$-semi interior of a $ij$SC set is $j*i$-sclopen.

Proof: Let $A$ be a $ij$SC set, so $\exists$ a $\tau_i$-closed set $F$ such that $\tau_i$-int($F$) $\subseteq A \subseteq F$, $\therefore$, $\tau_i$-int($F$) $\subseteq jisint(A) \subseteq F$, so $jisint(A)$ is a $ij$SC set and also a $ji$SO set., i.e. $j*i$-sclopen.

Theorem 2.5.5. Let $A$ be a $ij$SO set in a bitopological space $(X, \tau_1, \tau_2)$, then $x \in jiscl(A)$ iff $W \cap A \neq \emptyset$ for every $j*i$-sclopen set $W$ containing $x$.

Proof: The first part of this theorem follows directly, since every $j*i$-sclopen set is $ji$SO set.

Conversely suppose that $W \cap A \neq \emptyset$ for every $j*i$-sclopen set $W$ containing $x$. Let there be a $ji$SO set $V$ containing $x$ and $V \cap A = \emptyset$, so $jiscl(V) \cap A = \emptyset$, since $V \subseteq X-A$ and $X-A$ is $ji$SC set so $jiscl(V) \subseteq X-A$. But by Theorem 2.5.3 $jiscl(V)$ is a $j*i$-sclopen set containing $x$ with $jiscl(V) \cap A = \emptyset$, which is a contradiction, hence $x \in jiscl(A)$.
Here we define some accumulation of net, filters etc. in term of ijSO set in a bitopological spaces.

**Definition 2.5.6.** Let \((X, \tau_1, \tau_2)\) be a bitopological space, a filter (filterbase) \(\mathcal{F}\) on \(X\) is said to ij-S accumulate (i*j-sclo-accumulate) at a point \(x \in X\), if for any ijSO (i*j-sclopen) set \(U\) containing \(x\) and for any \(F \in \mathcal{F}\), \(F \cap U \neq \emptyset\).

**Theorem 2.5.7.** Let \(\mathcal{F}\) be a filter (filterbase) on a bitopological space \((X, \tau_1, \tau_2)\), then \(x \in X\) is a ij-S accumulation point of \(\mathcal{F}\) iff \(x \in \cap \{\text{ijsclo}(F) : F \in \mathcal{F}\}\).

**Proof:** The proof is a easy consequence of the definition 2.5.6.

**Definition 2.5.8.** Let \((X, \tau_1, \tau_2)\) be a bitopological space and \(\{x_n : n \in D\}\) be a net in \(X\), where \((D, \succeq)\). Then \(x_0 \in X\) is said to be an ij-S-accumulation point of the net \(\{x_n : n \in D\}\) iff for every ijSO set \(U\) containing \(x_0\), for each \(n \in D\), \(\exists \ m \in D\) with \(m \succeq n\) with \(x_m \in U\).

\(x_0 \in X\) is said to be an ij-S-limit point of the net \(\{x_n : n \in D\}\) iff for every ijSO set \(U\) containing \(x_0\), \(\exists \ n \in D\), such that for all \(m \in D\) with \(m \succeq n\), \(x_m \in U\).

**Theorem 2.5.9.** Let \((X, \tau_1, \tau_2)\) be a bitopological space. If a fuzzy net \(\{x^n_{\alpha_n} : n \in D\}\) on \(X\) is a \(\alpha\)-net and if the net \(\{x^n : n \in D\}\) has ij-S-accumulation point \(x\) in the bitopological space \((X, \tau_1, \tau_2)\) then for any open set \(\lambda\) in \((X, \omega_{\alpha}(X))\) with \(x_{\alpha} \in \lambda\) implies for each \(n \in D\), there exists \(m \in D\) with \(m \succeq n\) and \(x^n_{\alpha} \in \lambda\).

Conversely, if the given condition holds for a fuzzy \(\alpha\)-net \(\{x^n_{\alpha} : n \in D\}\), \((\alpha \neq 0)\) then \(\{x^n : n \in D\}\) has ij-S-accumulation point \(x\) in the bitopological space \((X, \tau_1, \tau_2)\).

**Proof:** Since \(x_{\alpha} \in \lambda\), so \(\alpha^+ \lambda(x) > 1\), i.e. \(\lambda(x) > 1 - \alpha\). Let \(1 - \alpha < r < \lambda(x)\) and hence \(1 - r < \alpha\), so there exists \(n_0 \in D\), such that \(\alpha_m > 1 - r\) for all \(m \succeq n_0\).
Now \( x \in \lambda^{-1}(r, 1] \) and \( \lambda^{-1}(r, 1] \) is a \( ij\)-SO set in \((X, \tau_1, \tau_2)\), so for each \( n \geq n_0 \) we can find \( m \geq n \) such that \( x_m \in \lambda^{-1}(r, 1] \), i.e. \( \lambda(x_m) > t \), i.e. \( \alpha_m > 1 - r > 1 - \lambda(x_m) \) and hence \( \alpha_m + \lambda(x_m) > 1 \), so \( x_m^\alpha q \lambda \).

Conversely, let \( \{ x_n^\alpha : n \in D \} \) is a fuzzy \( \alpha \)-net on \( X \) (\( \alpha \neq 0 \)), having the given condition. Let \( U \) be a \( ij\)-SO set in \((X, \tau_1, \tau_2)\) with \( x \in U \), so \( x_\alpha q x_U \) and \( x_U \in \omega_\alpha(X) \). So for for each \( n \in D \), there exists \( m \in D \) with \( m \geq n \), \( x_m^\alpha q x_U \) which is only possible iff \( x_m \in U \), i.e. \( x \) is a \( ij\)-S-accumulation point of the net \( \{ x_n^\alpha : n \in D \} \) in the bitopological space \((X, \tau_1, \tau_2)\).

**Theorem 2.5.10.** Let \((X, \tau_1, \tau_2)\) be a bitopological space. If a fuzzy net \( \{ x_n^\alpha : n \in D \} \) on \( X \) is a \( \alpha \)-net and if the net \( \{ x_n : n \in D \} \) has \( ij\)-S-limit point \( x \) in the bitopological space \((X, \tau_1, \tau_2)\) then for any open set \( \lambda \) in \((X, \alpha_\lambda(X))\) with \( x_\lambda q x_U \), there exists \( n \in D \), such that for all \( m \geq n \), \( x_m^\alpha q x_U \).

**Proof:** Similar to that of the Theorem 2.5.9.

The general definition of a net can be found in Dugundji and also mentioned in the first chapter. For our purpose we redefine the definition of net of subsets of a set for very restricted case.

**Definition 2.5.11.** A net of subsets of a non-empty sets \( X \) is a mapping \( \varphi: D \to \wp(X) \), where \((D, \geq)\) is a directed set and \( \wp(X) \) is the power set of \( X \) and \( \varphi(\alpha) \neq \emptyset \) for all \( \alpha \in D \). Let us write \( \varphi(\alpha) \) as \( A_\alpha \), i.e. under the map \( \varphi \), \( \alpha \in D \) mapped to \( A_\alpha \subseteq X \), we will denote this net as \( \{A_\alpha\}_{\alpha \in D} \).

**Definition 2.5.12.** Let \((X, \tau_1, \tau_2)\) be a bitopological space and \( \{A_\alpha\}_{\alpha \in D} \) be a net of subsets of \( X \). Then \( x_0 \in X \) is said to be a \( \tau_i \)-accumulation point (or \( ij\)-S-accumulation or \( i^*j\)-sclo-accumulation) point of the net \( \{A_\alpha\}_{\alpha \in D} \) iff for every \( \tau_i \)-