CHAPTER - 2

PROGRESSIVE CENSORING UNDER INVERSE SAMPLING
FOR NONPARAMETRIC TWO-SAMPLE PROBLEMS

2.1 INTRODUCTION

A natural characteristic of clinical trials as well as life testing experimentations is that the observations are gathered sequentially in time. Because of practical limitations (on the cost and duration of experimentation), it is not always possible to continue experimentation until all individuals responded and then to draw statistical inference. Usually the experiment is curtailed either after a specified length of time or after a prespecified proportion (or number) of individuals responded. In statistical terminology such procedures are termed respectively the truncated or censored procedures. In the truncated scheme, the duration of the experiment is prefixed, but the number of respondants is a random variable. On the other hand, in the censored case, the number of respondants is prefixed, but the duration of the experiment is a random variable. In statistical literature truncation and censoring are sometimes referred to as type-I and type-II censorings respectively. However, in most practical problems, fixed point truncation or censoring may lead to considerable loss of efficiency relative to cost. This drawback is particularly felt in exploratory studies. A too early termination of the experiment may lead to an adequately small set of observations and thus increase the risk of incorrect decision. Unnecessary prolongation of the experiment may lead to loss of valuable time with practically no extra gain in efficiency. To avoid this Chatterjee and Sen (1973) (see also Sen (1978)) introduced progressive censoring scheme (PCS) which allows one to monitor the experiment from the beginning until a statistically valid decision (with prescribed risk) is made. Further extension of this PCS are given by Sen (1976 a,b,c), Majumdar and Sen (1977, 1978 a,b), Halperin and Ware (1974), Gardiner and Sen (1978), Sinha and Sen (1979 a,b) and Davis (1978) among others. The above PCS techniques are based on of fixed sample sizes drawn from both populations. Wolfe (1977) proposed a partially sequential testing procedure for a two-sample nonparametric problem using one sample sizes of fixed size and other of random size using inverse sampling scheme. The inverse sampling scheme has been discussed in section 1.2 of chapter 1 of the present dissertation. Following Wolfe's inverse sampling scheme, Bandyopadhyay
(1980) proposed a PCS technique for nonparametric two sample problem based on a fixed number of observations from one sample and a random number of observations from the other sample. But this scheme is appropriate only when \(m\) and \(r\) both large and \(r/m\) is close to zero. It is also observed that the procedure discussed by both Wolfe (1977) and Bandyopadhyay (1980) are workable only for one-sided alternatives but not for two-sided alternatives.

Bandyopadhyay and Chattopadhyay (1995) attempts to develop a procedure (progressively censored), by generalising Wolfe’s inverse sampling technique, which is useful for both one and two-sided alternatives and is appropriate when \(m\) and \(r\) are both large but \(r/m\) is any real number lying between \([0, \infty)\). The present chapter contains a detailed discussion of the same. In this connection it would be worthwhile to mention that the test procedure discussed by Bandyopadhyay (1980) is dependent on the number of failures in between the successive successes (i.e. one has to observe the \(Y\)-variables in groups). But, in the proposed procedure, one has to carry out the test at each drawing of \(Y\)-observations. Section 2.2 of the present chapter has been devoted for formulation of the problem. Asymptotic null distribution has been studied in section 2.3. In section 2.4 unbiasedness and consistency properties have been explored. Asymptotic power has been studied in section 2.5. Saving under alternatives has been discussed in section 2.6. Some numerical computation are provided in section 2.7. The chapter ends up with concluding remarks in section 2.8.

### 2.2 FORMULATION OF THE TEST CRITERIA

Let \(F_1\) and \(F_2\) be two univariate continuous distribution functions (df’s). Here we want to test

\[
H: F_1 = F_2
\]  
(2.2.1)

against a composite alternative \(H_a\).

Let \(X_m = (X_1, X_2, \ldots, X_m)\) be a random sample of fixed size \(m\) from the distribution \(F_1\). On the basis of \(X_m\) and by drawing observations from \(F_2\) one-by-one, our object is to develop a suitable nonparametric test for \(H\) against \(H_a\).

**Location Problem:** Here we take \(F_1 = F(x)\) and \(F_2 = F(x-\theta)\), \(-\infty < \theta < \infty\). Let \(F\) be such that, for every \(u \in (0,1)\), there exist \(\xi_u\) for which we have

\[
F(\xi_u) = u \text{ (uniquely)}.
\]  
(2.2.2)
Now, based on $X_m$, we choose a region $A_m$ in the domain of $Y$ in such a way that, for large $m$

$$P_{F, \theta}(Y \in A_m | X_m) > (or <) P_{F}(Y \in A_m | X_m)$$

according as $\theta > (or <) 0$. (2.2.3)

If $\xi_u^{(E)}$ is the $u$-th quantile of the empirical df based on $X_m$, then we choose

$$A_m = \{ Y: Y > \xi_u^{(E)} \}$$

(2.2.4)

Now, writing

$$p_m(F, \theta) = 1 - q_m(F, \theta) = P_{F, \theta}(Y \in A_m | X_m) = 1 - F(\xi_u^{(E)} - \theta)$$

(2.2.5)

and

$$p_m(F) = 1 - q_m(F) = P_{F}(Y \in A_m | X_m) = 1 - F(\xi_u^{(E)})$$

(2.2.6)

we have, as $m \to \infty$,

$$P_m(F, \theta) \overset{p}{\to} p(F, \theta) = 1 - F(\xi_u - \theta)$$

(2.2.7)

and

$$P_m(F) \overset{p}{\to} p(F) = 1 - F(\xi_u) = 1 - u$$

(2.2.8)

Hence, given $u, p(F) = p_0 = 1 - q_0$ is a known quantity and by (2.2.2), $p(F, \theta) > (or <) p_0$ according as $\theta > (or <) 0$. Clearly (2.2.3) is satisfied in the region $A_m$.

After fixing the region $A_m$, $Y$-observations are drawn sequentially and, for each drawing of $Y$-observations, we first define

$$d_m(j) = 1 \text{ or } 0 \text{ according as } Y_j \in A_m \text{ or } \not\in A_m, j \geq 1$$

(2.2.9)

and

$$U_m = \frac{q_0}{p_0} d_m(j) - (1 - d_m(j)) = (d_m(j) - p_0) / p_0$$

(2.2.10)

We now consider the statistics:

$$S_{mk} = \sum_{j=1}^k U_m, k \geq 1, S_{m0} = 0. \text{ (2.2.11)}$$

Thus, on the basis of the random variable $X_m$ and $\{ Y_j, j \geq 1 \}$, we get the random variables:

$$\{S_{mk}, k \geq 1\} \text{ (2.2.12)}$$

Motivated by Sen (1978), we construct, for each $s (\geq 1)$, the following Kolmogorov-Smirnov (K-S) type statistics:

$$T_{ms} = \max_{1 \leq s \leq s} S_{mk} \text{ (one-sided case)} \text{ (2.2.13)}$$

$$= \max_{1 \leq s \leq s} |S_{mk}| \text{ (two-sided case)} \text{ (2.2.14)}$$
where the upper limit of $s$ is a random variable (N) depending on $A_m$ and a pre-fixed integer $r$. It is determined in such a way that, for large $m$,

$$E_{F, \theta}(N|X_m) < E_{F}(N|X_m) \text{ for all } (F, \theta) \in H_k. \quad (2.2.15)$$

In particular, for testing $H$ against $\theta > 0$, we can choose

$$N = \min \left\{ k : \sum_{j=1}^{k} d_m(j) = r \right\} \quad (2.2.16)$$

and, for testing $H$ against $\theta \neq 0$, we can choose

$$N = \min \left\{ N_1^*, N_2^* \right\} \quad (2.2.17)$$

where $N_1^*$ is given by (2.2.16) and $N_2^*$ is given by

$$N_2^* = \min \left\{ k : \sum_{j=1}^{k} (1 - d_m(j)) = r' \right\} \quad (2.2.18)$$

and $r'$ (a function of $r$) is so determined that, as $m \to \infty$,

$$[E_{H}(N_1^*|X_m)/ E_{H}(N_2^*|X_m)] \to 1 \quad (2.2.19)$$

Now, (2.2.8) and (2.2.19) lead to

$$r' = \lfloor rq_0/p_0 \rfloor \quad (2.2.20)$$

$[x]$ denote the largest integer less than or equal to $x$. Note that (2.2.17) is equivalent to

$$N = \min \left\{ k : \max \left\{ \sum_{j=1}^{k} d_m(j) - r, \sum_{j=1}^{k} (1 - d_m(j)) - r' \right\} = 0 \right\} \quad (2.2.21)$$

In practice one would use (2.2.21) instead of (2.2.17). Thus the upper limit of $s$ is given by (2.2.16) or (2.2.21) according as the problem is one-sided or two-sided.

Observe that $T_{ms}$ is non-decreasing in $s$, so that one can write

$$T_{mN} = \max_{1 \leq k \leq N} T_{mk} \quad (one\text{-sided} \text{ case}) \quad (2.2.22)$$

$$= \max_{1 \leq k \leq N} S_{mk} \quad (two\text{-sided} \text{ case}) \quad (2.2.23)$$

Now, under $H$, if we make probability integral transformation $X_i \to X'_i = F(X_i)$, $i=1,2,\ldots,m$ and $Y_j \to Y'_j = F(Y_j)$, $j=1,2,\ldots$, then $N$ determined by (2.2.16) and (2.2.21) and the random variables given by (2.2.10) will remain unchanged. Thus, under $H$, the joint distribution of $\{ S_{mk}, 1 \leq k \leq N \}$ is completely distribution free and hence the tests provided by (2.2.13) and (2.2.14) are exactly distribution free. Let, for every preassigned level $\alpha$: $0 < \alpha < 1$, $T_{m\alpha}(\alpha)$ be defined by

$$P_H(T_{mN} > T_{m\alpha}(\alpha)) \leq \alpha < P_H(T_{mN} \geq T_{m\alpha}(\alpha)). \quad (2.2.24)$$
Note that, if we continue to draw $Y$-observations sequentially for getting at most $r$ success for one-sided alternatives (either at most $r$ successes or at most $r'$ failures for two-sided problem), at each drawing, it is expected that $T_{ms}$ would be close to zero when $H$ is true, and for at least one drawing, $T_{ms}$ would be positive when $H$ is not true. Hence at the level of significance $\alpha$, we can formulate our test as:

Reject $H$ at the $s$-th ($1 \leq s \leq N$) drawing if

$$T_{ms} \leq T_m(\alpha), j=1,2, \ldots, s-1, T_{ms} > T_m(\alpha)$$

(2.2.25)

Accept $H$ if no $s$ satisfying (2.2.25) holds, i.e., $T_{ms} \leq T_m(\alpha)$.

Here $T_m(\alpha)$ depends on $(m, r, A_m)$. For fixed $m$ and $r$, it is not easy to evaluate this. In the next section, we shall show that, if $m$ and $r$ are sufficiently large so that $r/m$ is finite, $T_m(\alpha)$ can be approximated by a known number which can be easily evaluated.

**Scale Problem:** Here we take $F_1(x) = F(x)$ and $F_2(x) = F(xe^{-\theta})$, $-\infty < \theta < \infty$, with $F(0) = \frac{1}{2}$ (uniquely). For any $0 < u < v < 1$, with $u+v = 1$, we define $\xi_u$ and $\xi_v$ as in (2.2.2).

Then taking $\xi_u(\Theta)$ and $\xi_v(\Theta)$ as in the location problem, we choose

$$A_m = \{ Y: Y < \xi_u(\Theta) \text{ and } Y > \xi_v(\Theta) \}$$

(2.2.26)

Obviously the requirement (2.2.3) is satisfied for this region. Hence we proceed in the same line as in the location problem. Thus, we see that the theory for the location and the scale problems are essentially same. So in the subsequent sections, we give the details of development for the location problem only.

### 2.3 Asymptotic Null Distribution of the Test Criteria

We assume that, for each $m$, there is a positive integer $r = r(m)$, such that, as

$$m \to \infty, r \to \infty,$$

but

$$r/m \to \lambda \in [0, \infty).$$

(2.3.1)

Hereafter, when we make $m \to \infty$, we assume that (2.3.1) holds. Let us also consider the following sequence of positive integers:

$$v = v(m) = [r/p_0], m \geq 1.$$  

(2.3.2)

Let us now introduce the following stochastic processes:

$$Z_m(t) = [(r_0/p_0) (1 + r/mp)]^{1/2} (S_m[t_0] + (vt-[vt])U_m[t](vt)], 0 \leq t \leq 1.$$  

(2.3.3)

The stochastic process $Z_m = \{ Z_m(t), 0 \leq t \leq 1 \}$ belongs to $C[0,1]$, the space of all real continuous functions over the closed intervals $[0,1]$ with $Z_m(0) = 0$ for every $m \geq 1$.

Hence from (2.2.13) – (2.2.14), we get
\[
\left[ \frac{r \sigma^2}{\sigma^2 + \frac{r}{m \theta}} \right]^{-\frac{1}{2}} \sup_{0 \leq t \leq 1} Z_n(t) = \sup_{0 \leq t \leq 1} Z_n(t) \quad \text{(one-sided case)}
\]
\[= \sup_{0 \leq t \leq 1} \left| Z_n(t) \right| \quad \text{(two-sided case)}
\]

Let \( Z = \{ Z(t), 0 \leq t \leq 1 \} \) be a Gaussian process on \( C[0,1] \) having \( E(Z(t)) = 0, E(Z(t)Z(t')) = (1+\lambda/p_0)^{-1/2} \) \( t(1+\lambda t'/p_0), 0 \leq t \leq t' \leq 1 \).

We shall establish, under \( H \), the weak convergence of \( Z_m \) to \( Z \) and utilise the representations (2.3.4) and (2.3.5) to get the desired distributional results. For this we shall first show that the finite dimensional distributions of \( Z_m \) defined by (2.3.3) converges to those of \( Z \).

**Theorem 2.3.1:** Under \( H \), for any positive integer \( d \) and for any set of real numbers, \( 0 \leq t_1 < t_2 < \ldots < t_d \leq 1 \), as \( m \to \infty \),

\[
(Z_m(t_1), \ldots, Z_m(t_d)) \overset{D}{\longrightarrow} (Z(t_1), \ldots, Z(t_d))
\]

**Proof:** Note that, for any \( t \),

\[
|Z_m(t) - \left[ (r \sigma^2/p_0^2) \left( 1 + \frac{r}{m \theta} \right) \right]^{1/2} S_m(\nu t) - \left[ (r \sigma^2/p_0^2) \left( 1 + \frac{r}{m \theta} \right) \right]^{1/2} U_{m(\nu t)}| \overset{p}{\longrightarrow} 0, \quad k = 1, 2, \ldots, d.
\]

Thus, to prove (2.3.7), it suffices to show that, as \( m \to \infty \),

\[
[(r \sigma^2/p_0^2) \left( 1 + \frac{r}{m \theta} \right)]^{1/2} \sup_{0 \leq t \leq 1} S_m(\nu t) \overset{D}{\longrightarrow} (Z(t_1), \ldots, Z(t_d))
\]

which will follow, if we can show that, for all nonnull \( \ell = (\ell_1, \ell_2, \ldots, \ell_d)' \)

\[
S_m(\ell) = [(r \sigma^2/p_0^2) \left( 1 + r/m \theta \right)]^{1/2} \sum_{j=1}^{d} \ell_j S_m(\nu t_j) \overset{D}{\longrightarrow} N(0, \Lambda \ell \Lambda \ell')
\]

where

\[
(1+\lambda \theta/p_0) \Lambda = \langle (t_j, 1+\lambda t_j/p_0) \rangle_{j=1}^{d}, \quad j=1 \chi \delta, \chi \delta.
\]

Let us now re-write \( S_m(\ell) \) as

\[
S_m(\ell) = [(r \sigma^2/p_0^2) \left( 1 + r/m \theta \right)]^{1/2} \sum_{j=1}^{d} \ell_j U_{mj}
\]

where

\[
b_j = \ell_1 + \ell_2 + \ldots + \ell_d, 1 \leq j \leq [\nu t_1] \]
\[
= \ell_2 + \ldots + \ell_d, [\nu t_1] + 1 \leq j \leq [\nu t_2]
\]
Writing
\[ a_m(F) = E_H(r^{-1/2}b_j | X_m) = (b/p_0) r^{-1/2} (p_m(F) - p_0), \]
we have
\[ a_m(F) = \sum_{j=1}^{[vt_d]} a_m(F) = (1/p_0) r^{-1/2} (p_m(F) - p_0) \sum_{j=1}^{[vt_d]} b_j, \]

and
\[ v_m^2(F) = V_H(1/p_0 \sum_{j=1}^{[vt_d]} b_j | X_m) = (1/p_0) r^{-1/2} p_m(F) q_m(F) \sum_{j=1}^{[vt_d]} b_j, \]

it can be easily observed that, because of (2.2.8) and (2.3.14), as \( m \to \infty \),

\[ v_m^2(F) \xrightarrow{p} V_0^2 = (q_0/p_0^2) \lim_{m \to \infty} \left( \frac{1}{V_0} \sum_{j=1}^{[vt_d]} b_j^2 \right) \]

where
\[ V_0^2 = (q_0/p_0^2) \lim_{m \to \infty} \left( \frac{1}{V_0} \sum_{j=1}^{[vt_d]} b_j^2 \right) \]

Now, writing \( dP_m = dP (r^{-1/2}b_j | X_m) \), we have for any \( \varepsilon > 0 \)
\[ v_m^{-2}(F) \sum_{j=1}^{[vt_d]} \left[ dP_m \left( x - a_m(F) \right)^2 \right] \]

\[ \leq \varepsilon^{-2} v_m^{-2}(F) \sum_{j=1}^{[vt_d]} \left[ dP_m \left( x - a_m(F) \right)^2 \right] \]

\[ \leq \varepsilon^{-2} v_m^{-2}(F) \mu_{m_4}(F) \sum_{j=1}^{[vt_d]} b_j^4 \]

where
\[ \mu_{m_4}(F) = p_0^{-4} \left[ p_m(F) - 4p_m^2(F) + 6p_m^3(F) - 3p_m^4(F) \right]. \]

Note that, because of (2.2.8) and (2.3.14), the right hand member of (2.3.18) converges in probability to zero. Thus, by using a standard result (see Problem 4, Hajek and Sidak (1967, Ch V, p-194)), it follows that for given \( X_m \) under \( H \), as \( m \to \infty \),

\[ v_m^{-1}(F) \left[ r^{1/2} \sum_{k=1}^{d} \ell_k S_m \left( v_{t_k} - a_m(F) \right) \right] \xrightarrow{d} N(0,1). \]

Since, as \( m \to \infty \),
\[ m^{1/2} \left[ p_m(F) - p_0 \right] \xrightarrow{d} N(0, p_0q_0) \]
it is obvious from (2.3.1) and (2.3.14)-(2.3.15) that, as $m \to \infty$,

$$a_m(F) \xrightarrow{D} N(0, \sigma_0^2)$$  \hspace{1cm} (2.3.22)

where

$$\sigma_0^2 = (\lambda q_0/p_0^3) \lim_{m \to \infty} \left[ \frac{1}{\lambda} \sum_{j=1}^{[\frac{u_j}{\lambda}]} b_j^2 \right]$$

$$= (\lambda q_0/p_0^3) \left[ \sum_{k=1}^{d} \xi_k \xi_k \right].$$  \hspace{1cm} (2.3.23)

Hence, using (2.3.16), (2.3.20), (2.3.22) and a standard result (see Problem 6, Hajek and Sidak (1967, Ch V, p-196)), we have, under H, as $m \to \infty$,

$$t^{-1/2} \sum_{k=1}^{d} \xi_k S_m[ v_k ] \xrightarrow{D} N\left(0, \sigma_0^2 + \nu_0^2\right).$$  \hspace{1cm} (2.3.24)

Note that,

$$\sigma_0^2 + \nu_0^2 = (q_0/p_0^3) \left(1 + \frac{\lambda}{p_0} \right) \xi \wedge \xi.$$

Finally, using (2.3.1), (2.3.11) and (2.3.24), we have the required result.

**Theorem 2.3.2:** Under H, in the uniform topology on $C[0,1]$, as $m \to \infty$,

$$Z_m \xrightarrow{D} Z$$  \hspace{1cm} (2.3.26)

where Z is a Gaussian process given by (2.3.6).

**Proof:** By our previous theorem and Theorem 8.1 of Billingsley (1968, p-54), we need only to show that $\{Z_m\}$ is tight under H. Since $Z_m(0) = 0$, $\{Z_m(0)\}$ is tight. Hence to prove the tightness, we have to prove that, for each $\varepsilon > 0$ and $\eta > 0$, there exists a $\beta \in (0,1)$ such that, for all $\ell$

$$P_H\left( \left[ (q_0/p_0^3)(1+\nu/p_0) \right]^{-1/2} \max_{1 \leq k \leq \ell} |S_{m,k} - S_m| > \varepsilon \right) \leq \eta \beta$$  \hspace{1cm} (2.3.27)

holds for sufficiently large $m$ (see Billingsley (1968, p-59)). Note that, the random variable $U_{mj}$, $j \geq 1$ (although not independently distributed) have a stationary distribution. Hence (2.3.27) reduces to

$$P_H\left( \left[ (q_0/p_0^3)(1+\nu/p_0) \right]^{-1/2} \max_{1 \leq k \leq \ell} |S_{mk}| > \varepsilon \right) \leq \eta \beta$$  \hspace{1cm} (2.3.28)

Introducing

$$S_{mk} = \sum_{j=1}^{k} \frac{d(m(j) - p_m(F))}{p_0},$$

we get,

$$S_{mk} = S_{mk}^0 + \left( k/p_0 \right) (p_m(F) - p_0).$$  \hspace{1cm} (2.3.30)

Hence the left hand member of (2.3.28) is less than or equal to
Under $H$ and given $X_m$, $S_{mk}$ denotes the sum of iid random variables with zero expectation and the fourth order moments $\mu_{m4}(F)$ given by (2.3.19). Then applying Kolmogorov's inequality (using fourth order moments on the conditional probability given $X_m$), the first term of (2.3.31) is

$$\leq (16[\nu/\varepsilon^2](vp_0^2)[(1+r/mpo)]^2 E_H(\mu_{m4}(F))]$$

which, as $p_m(F)$ is a bounded random variable, tends to zero, as $m \to \infty$. Again, writing $V$ as $N(0,1)$ random variable, as $m \to \infty$, the second member of (2.3.31) tends to

$$P(|V|>\varepsilon \lambda_0(1+r/mpo)\lambda_0) \leq (4\lambda^2 \beta^2 / \epsilon^2 \lambda_0(1+r/mpo)) \lambda_1 E(V^2).$$

Hence, using (2.3.31)-(2.3.33), we get

$$\limsup_{m \to \infty} P_H\left( \left( q_0/\rho_0^2 \right) (1+r/mpo) \lambda_1 \ | S_{mk} > \frac{1}{2} \varepsilon \lambda_1 \right) \leq$$

$$\leq (4\lambda^2 \beta^2 / \epsilon^2 \lambda_0(1+r/mpo)) \lambda_1 E(V^2).$$

So, for every $\varepsilon > 0$ and $\eta > 0$, the right hand member of (2.3.34) can be made less than $\eta \beta$ by choosing $\beta > 0$ sufficiently small. Thus we have the required result. Q.E.D.

Now, from the covariance structure (2.3.6), we observe that the process $Z(t)$ is equivalent to the process

$$Z(t) = \sup_{0 \leq t \leq 1} \left( 1+r/mpo \right) \left[ W(t) + (\lambda_0/\rho_0) V(t) \right],$$

where $W(t)$ is a Wiener process on $C[0,1]$ and $V$ is a $N(0,1)$ random variable distributed independently of $W(t)$. We shall now use the following important result to determine the cutoff point of the proposed test statistics:

**Result 2.3.1**: Let $W(t)$ be a Wiener process on $C[0,1]$ and $V$ be a $N(0,1)$ variable distributed independently of $W(t)$. Further suppose $c$ and $\sigma \geq 0$ are any two constants independent of $t$ and $T \in [0,1]$. Then for any $c \geq 0$

$$P \left\{ \sup_{0 \leq t \leq 1} \left( 1+r/\rho_0^2 \right) \left( W(t) + V(t) \right) + bt > c \right\} = g_1(c, b, T),$$

$$P \left\{ \sup_{0 \leq t \leq 1} \left( 1+r/\rho_0^2 \right) \left( W(t) + V(t) \right) + bt > c \right\} = g_2(c, b, T),$$

where

$$g_1(c, b, T) = \Phi\left( [T(1+\sigma^2 T)]^{1/2}(1+\sigma^2)\beta(bT-c) \right) \exp(2(1+\sigma^2)bc + 2\sigma^2(1+\sigma^2)c^2).$$

$$\Phi\left( [T(1+\sigma^2 T)]^{1/2}(1+\sigma^2)\beta(bT+c) \right).$$

(2.3.36)
\[ g_2(c,b,T) = \Phi\left( [T(1+\sigma^2 T)]^{1/2}(1+\sigma^2)^{1/2} (bT-c) \right) \]
\[ + \Phi\left( [T(1+\sigma^2 T)]^{1/2}(1+\sigma^2)^{1/2} (bT+c) \right) \]
\[ + \sum_{j=1}^{4} \left\{ \sum_{i=1}^{i} \exp \left( ((1+\sigma^2)cb_{ij} \right) \right. \]
\[ + \sigma^2 \left( 1+\sigma^2 \right) d_{ij}^2 c^2/2 \right) \Phi\left( [T(1+\sigma^2 T)]^{1/2}(1+\sigma^2)^{1/2} (cd_{ij}^* - \sigma^2 c_{ij}T-bT) \right) \]
\[ + \sum_{j=1}^{4} \exp \left( - (1+\sigma^2)cb_{ij} \right) + \sigma^2 \left( 1+\sigma^2 \right)d_{ij}^2 c^2/2 \right) \]
\[ \Phi\left( [T(1+\sigma^2 T)]^{1/2}(1+\sigma^2)^{1/2} (cd_{ij}^* - \sigma^2 c_{ij}T-bT) \right) \}
\[ (2.3.39) \]

and
\[ d_{ij} = d_{ij} = 2(2j-1), d_{ij} = d_{ij} = 4j, d_{ij}^* = d_{ij}^* = -(4j-3) \]
\[ (2.3.40) \]

**Proof:** Observe that the left hand member of (2.3.36) can be re-written as
\[ \int_{0}^{\infty} P\{ W(t) > c(1+\sigma^2)^{1/2} - (\sigma v+b(1+\sigma^2)^{1/2})t \} \text{d}\Phi(v), \] (2.3.41)
and from Anderson(1960), we have the following form of (2.3.41)
\[ \int_{0}^{\infty} \{ \Phi\left( T^{1/2} [(\sigma v+b(1+\sigma^2)^{1/2})T - c(1+\sigma^2)^{1/2}] \right) \]
\[ + \exp(2(1+\sigma^2)^{1/2} c([\sigma v+b(1+\sigma^2)^{1/2}]T)
\[ +c(1+\sigma^2)^{1/2})] \} \text{d}\Phi(v) \]
\[ \Phi\left( [T(1+\sigma^2 T)]^{1/2}(1+\sigma^2)^{1/2} (bT-c) \right) \exp(2(1+\sigma^2)^{1/2} (c+bT +2c\sigma^2 T)). \]
\[ = g_3(c,b,T) \]
\[ (2.3.42) \]
Also, the left hand member of (2.3.37) can be re-written as
\[ \int_{0}^{\infty} P\{ W(t) > c(1+\sigma^2)^{1/2} - (\sigma v+b(1+\sigma^2)^{1/2})t \}
\[ \text{or} W(t) < -c(1+\sigma^2)^{1/2} - (\sigma v+b(1+\sigma^2)^{1/2})t \text{for some t} \text{e} [0,T] \} \text{d}\Phi(v). \]
\[ (2.3.43) \]
Further from Anderson(1960), we have the following form of (2.3.43):
\[ \int_{0}^{\infty} \left[ P_1(v,T) + P_2(v,T) \right] \text{d}\Phi(v). \]
\[ (2.3.44) \]
where
\[ P_1(v,T) = \Phi\left( T^{1/2} [(\sigma v+b(1+\sigma^2)^{1/2})T - c(1+\sigma^2)^{1/2}] \right) \]
\[ + \sum_{j=1}^{2} \left\{ \sum_{i=1}^{i} (-1)^{i-1} \exp[|d_{ij}|(1+\sigma^2)^{1/2} c([\sigma v+b(1+\sigma^2)^{1/2}]T] \right. \]
\[ \Phi\left( T^{1/2} [d_{ij}^* c (1+\sigma^2)^{1/2} - \sigma v T-b(1+\sigma^2)^{1/2} T]) - \sum_{j=1}^{4} (-1)^{i-1} \exp[-d_{ij}(1+\sigma^2)^{1/2} c([\sigma v+b(1+\sigma^2)^{1/2}]T] \right) \]
\[ \Phi\left( T^{1/2} [d_{ij}^* c (1+\sigma^2)^{1/2} + \sigma v T+b(1+\sigma^2)^{1/2} T]) \right), \]
\[ (2.3.45) \]
\[
P_2(v,T) = \Phi(-T^{1/2}[\sigma_v T + b(1 + \sigma^2)^{1/2} T + c(1 + \sigma^2)^{1/2}]) + \sum_{j=1}^{3} \sum_{i=1}^{3} (-1)^{i-1} \exp[-d_{ij}(1 + \sigma^2)^{1/2}/c(\sigma_v + b(1 + \sigma^2)^{1/2})].
\]
\[
\Phi(T^{1/2}[d_{ij}^* c (1 + \sigma^2)^{1/2} + \sigma_v T + b(1 + \sigma^2)^{1/2} T]) - \sum_{j=3}^{4} (-1)^{i-1} \exp[d_{ij}(1 + \sigma^2)^{1/2}/c(\sigma_v + b(1 + \sigma^2)^{1/2})].
\]
\[
\Phi(T^{1/2}[d_{ij}^* c (1 + \sigma^2)^{1/2} - \sigma_v T - b(1 + \sigma^2)^{1/2} T])\}
\]

Now note that for any constants \(w_1, w_2\) and \(w_3\), it can be easily proved that
\[
\int_{\infty}^{\infty} \Phi(w_1 + w_2 x) \, d\Phi(x) = \Phi(w_1/(1+w_2^{1/2})
\]
and
\[
\int_{\infty}^{\infty} \exp(w_3 x) \Phi(w_1 + w_2 x) \, d\Phi(x) = \exp(w_3^{1/2}) \Phi(w_1 + w_2 w_3)
\]
and so
\[
\int_{\infty}^{\infty} \Phi(-T^{1/2}[\sigma_v T + b(1 + \sigma^2)^{1/2} T + c(1 + \sigma^2)^{1/2}]) \, d\Phi(v)
\]
\[
= \Phi[T(1+\sigma^2 T)]^{1/2}(-c(1+\sigma^2)^{1/2}-(1+\sigma^2)^{1/2} bT)
\]
and
\[
\int_{\infty}^{\infty} \Phi(T^{1/2}[\sigma_v T + b(1 + \sigma^2)^{1/2} T - c(1 + \sigma^2)^{1/2}]) \, d\Phi(v)
\]
\[
= \Phi[T(1+\sigma^2 T)]^{1/2}(-c(1+\sigma^2)^{1/2}+(1+\sigma^2)^{1/2} bT).
\]

Also
\[
\int_{\infty}^{\infty} \exp[d_{ij}(1 + \sigma^2)^{1/2} c (\sigma_v + b(1 + \sigma^2)^{1/2})].
\]
\[
\Phi(T^{1/2}[d_{ij}^* c (1 + \sigma^2)^{1/2} - \sigma_v T - b(1 + \sigma^2)^{1/2} T)] \, d\Phi(v)
\]
\[
= \exp[(1+\sigma^2)^{1/2} d_{ij} cb + \sigma^2(1+\sigma^2) d_{ij}^2 c^2/2] \Phi[T(1+\sigma^2 T)]^{1/2}
\]
\[
[d_{ij}^c c (1+\sigma^2)^{1/2} -(1+\sigma^2)^{1/2} bT - \sigma^2 (1+\sigma^2)^{1/2} d_{ij}^T]
\]
and
\[
\int_{\infty}^{\infty} \exp[-d_{ij}(1 + \sigma^2)^{1/2} (\sigma_v + b(1 + \sigma^2)^{1/2})] \Phi(T^{1/2}[d_{ij}^* c (1 + \sigma^2)^{1/2} + \sigma_v T
\]
\[
+ b(1+\sigma^2)^{1/2} T]) \, d\Phi(v)
\]
\[
= \exp[-(1+\sigma^2)^{1/2} d_{ij} cb + \sigma^2(1+\sigma^2) d_{ij}^2 c^2/2] \Phi[T(1+\sigma^2 T)]^{1/2}
\]
\[
[d_{ij}^c c (1+\sigma^2)^{1/2} +(1+\sigma^2)^{1/2} bT + \sigma^2 (1+\sigma^2)^{1/2} d_{ij}^T].
\]

Hence, from (2.3.44)-(2.3.48), we have the required result (2.3.36) and (2.3.37).

Now, using (2.3.36) and (2.3.37), we get (taking \(\sigma = (\lambda/p_0)^{1/2}\))
\[
P(\sup_{0 \leq t \leq 1} Z(t) > c) = g_1(c, 0, 1)
\]
and
\[
P(\sup_{0 \leq t \leq 1} |Z(t)| > c) = g_2(c, 0, 1)
\]
**Theorem 2.3.3:** For every \( c \geq 0 \)

\[
\lim_{m \to \infty} P_H\left( \left( r q_0 / p_0^2 \right) \left( 1 + r / m p_0 \right) \right)^{1/2} T_{mN} \leq c \) = 1 - \( g_1(c, 0, 1) \) (one-sided case), \( (2.3.51) \)

\[
= 1 - g_2(c, 0, 1) \) (two-sided case). \( (2.3.52) \)

The proof of the theorem depends on the following lemmas:

**Lemma 2.3.1:** If \( 0 < p_0 < 1 \), then under \( H \), as \( m \to \infty \),

\[
N / r \overset{p}{\longrightarrow} 1 / p_0 \] \( (2.3.53) \)

**Proof:** If \( N \) is defined by (2.2.16) then the result is immediate from (2.2.8) and the fact that, given \( X_m \) and under \( H \), \( N \) has the negative binomial \((r, p_m(F))\) distribution. If \( N \) is defined by (2.2.21), which is same as the representation (2.2.17), then also the result is immediate from (2.2.8) and using the following representation

\[
N = \min \left( N_1^*, N_2^* \right) = \frac{1}{2} \left( N_1^* + N_2^* - |N_1^*, N_2^*| \right) \] \( (2.3.54) \)

**Lemma 2.3.2:** Under \( H \), as \( m \to \infty \),

\[
n^{-1/2} \left[ T_{mN} - T_{mv} \right] \overset{p}{\longrightarrow} 0 \] \( (2.3.55) \)

**Proof:** Note that, for any fixed \( \epsilon (\geq 0) \),

\[
P_H\left( | T_{mN} - T_{mv} | > \epsilon n^{1/2} \right) \leq P_H\left( |N - \nu| > \epsilon \nu n \right) + P_H\left( \max_{|s| = \nu} | T_{mn} - T_{mv} | > \epsilon \nu^{1/2} \right). \] \( (2.3.56) \)

As \( S_{mn} \leq T_{mv} \), we have, for \( n \geq \nu \)

\[
| T_{mn} - T_{mv} | \leq \max_{v \leq \nu} | S_{mk} - S_{mv} | \] \( (2.3.57) \)

and, for \( n \leq \nu \),

\[
| T_{mn} - T_{mv} | \leq \max_{n \leq \nu} | S_{mk} - S_{mn} | \leq \max_{n \leq \nu} | S_{mk} - S_{mv} | + | S_{mn} - S_{mv} | \] \( (2.3.58) \)

Hence we get

\[
v^{-1/2} \max_{v \nu \leq s \leq \nu \nu} | T_{mn} - T_{mv} | \leq v^{-1/2} \max_{v \nu \leq s \leq \nu \nu} | S_{mk} - S_{mv} | \]

\[
\leq v^{1/2} \max_{v \nu \leq s \leq \nu \nu} | S_{mk} - S_{mv} | + \left( \epsilon / p_0 \right) v^{1/2} (p_m(F) - p_0). \] \( (2.3.59) \)

Similarly

\[
v^{-1/2} \max_{v \nu \leq s \leq \nu \nu} | T_{mn} - T_{mv} | \leq v^{-1/2} \max_{v \nu \leq s \leq \nu \nu} | S_{mk} - S_{mv} | + \left( \epsilon / p_0 \right) v^{1/2} (p_m(F) - p_0) \]

\( (2.3.60) \)

where \( S_{mk} \)'s are as defined in (2.3.29). Now, applying the same technique as in the proof of (2.3.34), we have
\[ \limsup_{m \to \infty} \mathbb{P}_H\left( \max_{|n| \leq \varepsilon \sqrt{n}} |T_{mn} - T_{mv}| > \varepsilon \sqrt{\lambda q_0/p_0} \right) \leq (5\lambda q_0/p_0^2) \varepsilon^4. \]  

(2.3.61)

Since \( \varepsilon \) is arbitrary the right hand member of (2.3.56) converges to zero as \( m \to \infty \). Hence, using Lemma 2.3.1, our required result follows. \textbf{Q.E.D.}

\textbf{Proof of Theorem 2.3.3:} By Lemma 2.3.2 and (2.3.1)-(2.3.2), we get

\[ \left[ (\mu_0/p_0^2)(1+r/mp_0) \right]^{1/2} (T_{mN} - T_{mv}) \xrightarrow{p} 0, \text{ as } m \to \infty. \]  

(2.3.62)

Hence, using (2.3.4), (2.3.5), (2.3.49), (2.3.50) and Theorem 2.3.2 we get (2.3.51) and (2.3.52). \textbf{Q.E.D.}

Let \( c_+^a \) and \( c_a \) be the values of \( c \) for which the right hand members of (2.3.49) and (2.3.50) are equal to \( \alpha \in (0,1) \). Hence, by virtue of (2.3.51) and (2.3.52) we have

\[ T_m(\alpha) \approx \left[ (\mu_0/p_0^2)(1+r/mp_0) \right]^{1/2} c_+^a \text{ (one-sided case)}, \]  

(2.3.63)

\[ \approx \left[ (\mu_0/p_0^2)(1+r/mp_0) \right]^{1/2} c_a \text{ (two-sided case)}. \]  

(2.3.64)

\textbf{Remark:} Note that Theorem 2.3.3 may be proved by using Theorem 17.1 of Billingsley (1968) and using the space \( D[0,1] \).

\textbf{2.4 UNBIASEDNESS AND CONSISTENCY OF THE PROPOSED TESTS}

\textbf{Unbiasedness:} For the one-sided case, instead of observing \( S_{mk} \)'s at each drawing of \( Y \)-observations, it is sufficient to observe \( S_{mk} \)'s only at the occurrence of the successes, and hence, we can re-write \( T_{mN} \) as

\[ T_{mN} = \max_{1 \leq s \leq r} \left( k/p_0 - N_k \right) \]  

(2.4.1)

where \( N_k \) is given by (2.2.16) with \( r \) replaced by \( k \left( 1 \leq k \leq r \right) \). Now, defining \( d'_m(j) \) as in (2.2.9) with \( Y_j \) replaced by \( Y_j' \), it is obvious that, if \( Y_j' \leq Y_j \),

\[ d'_m(j) \leq d_m(j) \text{ for all } j. \]  

(2.4.2)

Thus, writing

\[ N_k' = \min \{ n: \sum_{j=1}^n d'_m(j) = k \} \]  

(2.4.3)

we have

\[ N_k \leq N_k' \text{ for all } k. \]  

(2.4.4)

Finally, defining \( T'_{mN'} \) as in (2.4.1) with \( N_k \) replaced by \( N_k' \), we get

\[ T'_{mN'} \leq T_{mN} \]  

(2.4.5)
Now, using the same technique as in the proof of Lemma 3 of Lehmann (1986, p 234), we conclude that the test for the one-sided problem is unbiased.

**Remark:** Here it is not possible to show that the test for the two-sided problem is unbiased. In the next section we shall see that the test is asymptotically unbiased.

**Consistency:** Consider a fixed alternative given by

\[ H_a: F_1 = F(x), F_2 = F(x-\theta), \theta \neq 0. \]  

Then, it can be easily shown that, under any \((F,\theta)\), as \(m \to \infty\),

\[ S_{mn}/r \xrightarrow{p} \mu(F,\theta) = (p(F,\theta) - p(F))/p_0^2 \]  

which by (2.2.3), is positive if \(\theta > 0\) and non-zero if \(\theta \neq 0\).

Now, as

\[ \max_{1 \leq k \leq r} S_{mk} \geq S_{mn}, \max_{1 \leq k \leq r} |S_{mk}| \geq |S_{mn}| \]

our proposed tests are consistent against the respective alternatives.

### 2.5 ASYMPTOTIC POWER OF THE PROPOSED TESTS

As the proposed tests are consistent against fixed alternatives, we, for the study of asymptotic power properties, shall confine ourselves to some local alternatives for which asymptotic power function exists and is different from 0 and 1.

Let \(\{H_{am}\}\) be a sequence of local alternative hypotheses:

\[ H_{am}: F_1 = F(x), F_2 = F(x-\theta_m), \]  

where, as \(m \to \infty\),

\[ [m(r/p_0)/(m+r/p_0)]^{1/2} \theta_m \to \theta \neq 0. \]  

Now, letting \(\xi_u\) as in (2.2.2) and assuming that \(F'(x) = f(x)\) exists at \(x = \xi_u\) with \(f(\xi_u) > 0\), we prove the following theorem:

**Theorem 2.5.1:** Under \(\{H_{am}\}\), as \(m \to \infty\),

\[ Z_m \xrightarrow{d} Z + \mu, \]  

where \(Z\) is a Gaussian process given by (2.3.6) and the drift function \(\mu = \{\mu(t), 0 \leq t \leq 1\}\) is given by

\[ \mu = \Theta f(\xi_u)/(p_0q_0)^{1/2} t, 0 \leq t \leq 1. \]
Proof: To prove the theorem we first show that the finite dimensional distributions of 
\{Z_m\} converges to those of \(Z + \mu\). For this, as in the proof of theorem 2.3.1, it is 
enough to show that, under \(\{\text{H}_m\}\), as \(m \to \infty\),

\[ S_m(t) \xrightarrow{d} N(\sum_{k=1}^{d} \mu_k(t_k), \Gamma^m). \]  

(2.5.5)

Now, defining \(a_m(F, \theta_m), a_m(F, \theta_m)\) and \(v_m^2(F, \theta_m)\) as in (2.3.15) replacing \(p_m(F)\) by
\(p_m(F, \theta_m)\) and noting the fact that, as \(m \to \infty\),

\[ m^{1/2}(p_m(F, \theta_m)-p_0) \xrightarrow{d} N(\theta[(1+\lambda/p_0)/(\lambda/p_0)]^{1/2} f(\xi), p_0 q_0), \]  

(2.5.6)

we observe

\[ v_m^2(F, \theta_m) \xrightarrow{d} v_0^2, \]  

(2.5.7)

\[ a_m(F, \theta_m) \xrightarrow{d} N(\theta[(1/p_0)(1+\lambda/p_0)]^{1/2} \sum_{k=1}^{d} \mu_k(t_k), \sigma_0^2), \]  

(2.5.8)

where \(v_0^2\) and \(\sigma_0^2\) are as defined in (2.3.17) and (2.3.23) respectively. Now, applying
the same technique as in the proof of (2.3.11), (2.5.5) follows. Again, as in (2.3.34),
we have

\[
\lim \sup_{m \to \infty} P(\frac{(v_0/p_0^2)(1+\lambda/p_0)}{\varepsilon^2}(\max_{1 \leq k \leq d} |S_{mk}| > 2^{1/2} \varepsilon^{1/2}|H_m)) \\
\leq (4\lambda\beta^2 / \varepsilon^2 p_0)(1+\lambda/p_0)^{-1} E(V^2) \]  

(2.5.9)

where \(V^*\) has \(N(\theta(p_0 q_0)^{-1/2}(1+\lambda/p_0)/(\lambda/p_0))^{1/2}, 1)\) distribution. By the same argument as
in Theorem 2.3.2, it follows that \(\{Z_m\}\) is also tight under \(\{\text{H}_m\}\). Thus we have the 
required result.

Q.E.D.

Here it can be noted that Lemmas 2.3.1 and 2.3.2 are also true under \(\{\text{H}_m\}\).

Hence by the above theorem and using (2.3.4), (2.3.5), (2.3.50) and (2.3.51) the AP of
the proposed test is given by

\[ P(\theta) = \lim_{m \to \infty} \text{P}(T_{mn} > T_m(\alpha) \mid H_m) \]  

\[ = P \left\{ \sup_{0 \leq t \leq 1} [Z(t) + \mu(t)] > c_\alpha^+ \right\} \text{ (one-sided case)}, \]  

(2.5.10)

\[ = P \left\{ \sup_{0 \leq t \leq 1} [Z(t) + \mu(t)] > c_\alpha \right\} \text{ (two-sided case)} \]  

(2.5.11)

where \(Z(t)\) and \(\mu(t)\) are respectively given by (2.3.6) and (2.5.4).

Letting

\[ b = \theta f(\xi)/p_0 q_0, \quad \sigma = (\lambda/p_0)^{1/2} \]  

(2.5.12)

and using (2.3.36) and (2.3.37), we have
\[ P(\theta) = g_1(c_\alpha^+, b, 1) \quad \text{(one-sided case)} \]
\[ = g_2(c_\alpha, b, 1) \quad \text{(two-sided case)} \]

where \( c_\alpha^+ \), \( c_\alpha \) are respectively given by (2.3.63) and (2.3.64).

If the test procedure is based on the terminal sample size \( N \), then the AP of such a test under \( \{ H_{\alpha m} \} \) would be given by

\[ P^*(\theta) = 1 - \Phi(\tau_{\alpha-b}) \quad \text{(one-sided case)} \]
\[ = 1 - \Phi(\tau_{\alpha/2-b}) - \Phi(-\tau_{\alpha/2-b}) \quad \text{(two-sided case)} \]

where \( \tau_u \) (\( u=\alpha, \alpha/2 \)) is the critical value of \( N(0, 1) \) distribution and \( b \) is given by (2.5.12).

**Note:** From the Corollary 4 of Anderson (1955), it is not difficult to show that, for all \( b \neq 0 \), \( g_2(c_\alpha, b, 1) \geq g_2(c_\alpha, 0, 1) = \alpha \), which implies that the test provided by the statistic (2.2.23) is asymptotically unbiased against two-sided alternatives.

Since the terminal tests use more information, we would expect these to be more powerful than the corresponding progressively censored tests. Now, to investigate the extent of shortfall in power we, for some selected \( F \), compare numerically \( P(\theta) \) with \( P^*(\theta) \) for different choices of \( \theta \) and \( \lambda/p_0 \). These are given in Tables 2.1 and 2.2. In the next section we shall show that the shortfall can be compensated by the saving in the experimental cost which can be measured by the expected number of \( Y \)-observations needed for the termination of the experiment.

It can be easily seen that the power expressions given by (2.5.13) –(2.5.16) will remain unchanged if the random sample size \( N \) is replaced by the fixed sample size \( v = [r/p_0] \). Thus the efficiency of the proposed test is same as the corresponding test based on fixed sample size. The main purpose served by using inverse sampling technique is that the cost of the experimentation, as represented by the number of second sample observations, is reduced in this way when \( H \) is not true. This is discussed in the next section.

### 2.6 SAVING UNDER ALTERNATIVES

We start this section with the following theorem:

**Theorem 2.6.1:** Let, corresponding to the proposed sampling scheme, \( M_m \) be the number of \( Y \)-observations required at the termination of the experiment. Then

\[ \lim_{m \to \infty} [1 - \mathbb{E}(M_m/N | H_{\alpha m})] = S(\theta) = \int g_1(c_\alpha^+, b, T) dT \quad \text{(one-sided case)} \]
\[ = \int g_2 (\alpha, b, T) dT \quad \text{(two-sided case)} \quad (2.6.2) \]

where \( g_1 (\alpha, b, T) \) and \( g_2 (\alpha, b, T) \) are respectively given by (2.3.36) and (2.3.37).

**Proof:** Define, as in Sen (1978), the following random variable:

\[ M_{m*} = n \text{ if } T_{m,n-1} \leq T_m(\alpha), \quad T_m > T_m(\alpha), \quad 1 \leq n \leq v, \]
\[ = v \text{ if } T_{m,v} \leq T_m(\alpha), \quad (2.6.3) \]

where \( v \) is defined by (2.3.2). Then, under \( H_{am} \) and for every \( \epsilon > 0 \), we have

\[ P \{|M_m - M_{m*}| > \epsilon v\} \leq P \{|N-v|>\epsilon v/2\} + P \{|N-v| \leq \epsilon v/2, |M_m - M_{m*}| > \epsilon v\}. \quad (2.6.4) \]

Now,

\[ |N-v| \leq \epsilon v/2 \Rightarrow |M_m - M_{m*}| = 0 \text{ or } \leq \epsilon v/2 \quad (2.6.5) \]

which implies that the left-hand member of (2.6.4) is

\[ \leq P \{|N-v| > \epsilon v/2\} \to 0, \text{ as } m \to \infty. \quad (2.6.6) \]

Hence

\[ v^{-1} |M_m - M_{m*}| \overset{p}{\longrightarrow} 0, \text{ as } m \to \infty, \quad (2.6.7) \]

which, using Sen (1978, p. 249) and the fact that \( N/v \to 1 \) in probability as \( m \to \infty \), provides the required result. \textbf{Q.E.D.}

The above theorem gives us the limiting value of the expected proportion of saving under \( \{H_{am}\} \), resulting from the adoption of the progressively censored scheme corresponding to the random sample size \( N \). It also follows that the saving is same as that of the progressively censored scheme corresponding to the fixed sample size \( v \). Now, to get some idea about the extent of saving in the progressively censored scheme, we compute (2.6.1) and (2.6.2) for different values of \( \theta \) and \( \lambda/p_0 \). The values are given in Table 2.1 and 2.2. It is seen that the saving increases as \( \theta \) (or \( |\theta| \) ) increases. Thus it can be conjectured that the saving will be more significant if fixed alternative is considered.

**Theorem 2.6.2:** Under any \( (F, \theta) \), as \( m \to \infty \),

\[ N/v \overset{p}{\longrightarrow} \beta(\theta), \quad (2.6.8) \]

where

\[ \beta(\theta) = p_0/p((F, \theta)) \quad \text{(one-sided case)}, \quad (2.6.9) \]
\[ = \min \{ p_0/p((F, \theta), q_0/q((F, \theta)) \} \quad \text{(two-sided case)}, \quad (2.6.10) \]

**Proof:** Proof is immediate from the definition of \( N \) given in (2.2.16) and (2.2.21). \textbf{Q.E.D.}
Since $\beta(\theta) < 1$ under respective alternatives, we have, for large $m$, $M_m$ less than or equal to $M_m^*$ with probability 1. Hence, under fixed alternative, there is saving in the proposed inverse sampling scheme and the limiting proportion of saving is $(1-\beta(\theta))$ for the terminal test.

2.7 NUMERICAL COMPUTATIONS

For different choices of $F$ (e.g. standard normal, Cauchy, double exponential), we compute, for some selected values of $\theta$ and $\lambda/p_0$, $P(\theta)$, $P^*(\theta)$ and $S(\theta)$ at different values of $p_0$ taking $\alpha = 0.05$. While computing $c_{\alpha}^+$, $c_{\alpha}$ and $P(\theta)$, the following approximation due to Schucany and Gray (1968) (see Johnson and Kotz (1970), p-56)

$$1-\Phi(z) = [(z^2+2)(2\pi)^{1/2}]^{-1} z \exp(-z^2/2) \left[ (z^6+6z^4+14z^2-28)/(z^6+5z^4-20z^2-4) \right]$$

is used for $z > 3$. The values of $c_{\alpha}^+$, $c_{\alpha}$ and $P(\theta)$, $P^*(\theta)$ and $S(\theta)$ are tabulated in Table 2.1 and Table 2.2. Also for some selected value of $\theta$, $(1-\beta(\theta))$ is computed for the above choices of $F$. These are given in Table 2.3. For Table 2.1 and 2.2 it is observed that there is negligible differences between the powers with high proportion of saving in most of the cases. From Table 2.3, it is also seen that the saving increases with the increase in $\theta$ (or $|\theta|$).

Table 2.1: Asymptotic powers of the proposed test for one-sided alternative ($\alpha = .5$)

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$p_0$</th>
<th>Normal</th>
<th>Cauchy</th>
<th>Double Exponential</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta = 3.5$, $\alpha = 0.05$</td>
<td>.1</td>
<td>.639</td>
<td>.659</td>
<td>.177</td>
</tr>
<tr>
<td></td>
<td>.3</td>
<td>.827</td>
<td>.843</td>
<td>.283</td>
</tr>
<tr>
<td></td>
<td>.5</td>
<td>.861</td>
<td>.875</td>
<td>.309</td>
</tr>
<tr>
<td></td>
<td>.7</td>
<td>.827</td>
<td>.843</td>
<td>.283</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>.639</td>
<td>.659</td>
<td>.177</td>
</tr>
</tbody>
</table>

| $\theta = 1.87$, $\alpha = 0.05$ | .1 | .541 | .547 | .149 | .089 | .090 | .015 | .257 | .261 | .055 |
| | .3 | .730 | .735 | .238 | .386 | .391 | .093 | .621 | .627 | .183 |
| | .5 | 769 | .774 | .262 | .600 | .606 | .174 | .910 | .913 | .377 |
| | .7 | .730 | .735 | .238 | .386 | .391 | .093 | .621 | .627 | .183 |
| | 9 | .541 | .547 | .149 | .089 | .090 | .015 | .257 | .261 | .055 |
Table 2.2: Asymptotic powers of the proposed test for two-sided alternative ($\alpha = .5$)

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$p_0$</th>
<th>Normal</th>
<th>Cauchy</th>
<th>Double Exponential</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P(\theta)$</td>
<td>$P'(\theta)$</td>
<td>$S(\theta)$</td>
<td>$P(\theta)$</td>
</tr>
<tr>
<td>$\theta = 1.4$</td>
<td>$\xi_{p_0} = 9$, $\xi_{p_0} = 2.105$</td>
<td>$\xi_{p_0} = 9$, $\xi_{p_0} = 2.225$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>.128</td>
<td>.129</td>
<td>.040</td>
<td>.052</td>
</tr>
<tr>
<td>2</td>
<td>.182</td>
<td>.185</td>
<td>.062</td>
<td>.096</td>
</tr>
<tr>
<td>3</td>
<td>.197</td>
<td>.200</td>
<td>.068</td>
<td>.142</td>
</tr>
<tr>
<td>4</td>
<td>.182</td>
<td>.185</td>
<td>.062</td>
<td>.096</td>
</tr>
<tr>
<td>5</td>
<td>.128</td>
<td>.129</td>
<td>.040</td>
<td>.052</td>
</tr>
<tr>
<td>$\theta = 2$</td>
<td>$\xi_{p_0} = 9$, $\xi_{p_0} = 2.225$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td>.201</td>
<td>.216</td>
<td>.086</td>
<td>.052</td>
</tr>
<tr>
<td>2</td>
<td>.305</td>
<td>.327</td>
<td>.110</td>
<td>.138</td>
</tr>
<tr>
<td>3</td>
<td>.334</td>
<td>.358</td>
<td>.124</td>
<td>.229</td>
</tr>
<tr>
<td>4</td>
<td>.305</td>
<td>.327</td>
<td>.110</td>
<td>.138</td>
</tr>
<tr>
<td>5</td>
<td>.201</td>
<td>.216</td>
<td>.086</td>
<td>.052</td>
</tr>
</tbody>
</table>

Table 2.3: Limiting proportions of saving for terminal tests with respective alternatives

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$p_0$</th>
<th>Normal</th>
<th>Cauchy</th>
<th>Double Exponential</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi_{p_0} = 9$, $\xi_{p_0} = 2.105$</td>
<td>$\xi_{p_0} = 9$, $\xi_{p_0} = 2.225$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>.1</td>
<td>.743</td>
<td>.299</td>
<td>.632</td>
</tr>
<tr>
<td>2</td>
<td>.3</td>
<td>.559</td>
<td>.487</td>
<td>.567</td>
</tr>
<tr>
<td>3</td>
<td>.5</td>
<td>.405</td>
<td>.333</td>
<td>.387</td>
</tr>
<tr>
<td>4</td>
<td>.7</td>
<td>.252</td>
<td>.159</td>
<td>.213</td>
</tr>
<tr>
<td>5</td>
<td>.9</td>
<td>.089</td>
<td>.025</td>
<td>.065</td>
</tr>
<tr>
<td>$\xi_{p_0} = 2.225$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>.1</td>
<td>.869</td>
<td>.580</td>
<td>.848</td>
</tr>
<tr>
<td>2</td>
<td>.3</td>
<td>.677</td>
<td>.619</td>
<td>.661</td>
</tr>
<tr>
<td>3</td>
<td>.5</td>
<td>.488</td>
<td>.413</td>
<td>.463</td>
</tr>
<tr>
<td>4</td>
<td>.7</td>
<td>.295</td>
<td>.211</td>
<td>.270</td>
</tr>
<tr>
<td>5</td>
<td>.9</td>
<td>.099</td>
<td>.040</td>
<td>.087</td>
</tr>
<tr>
<td>$\xi_{p_0} = 3.3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>.1</td>
<td>.099</td>
<td>.050</td>
<td>.095</td>
</tr>
<tr>
<td>2</td>
<td>.3</td>
<td>.299</td>
<td>.236</td>
<td>.289</td>
</tr>
<tr>
<td>3</td>
<td>.5</td>
<td>.499</td>
<td>.442</td>
<td>.487</td>
</tr>
<tr>
<td>4</td>
<td>.7</td>
<td>.697</td>
<td>.654</td>
<td>.687</td>
</tr>
<tr>
<td>5</td>
<td>.9</td>
<td>.895</td>
<td>.789</td>
<td>.885</td>
</tr>
<tr>
<td>$\xi_{p_0} = 5.5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>.1</td>
<td>.100</td>
<td>.063</td>
<td>.099</td>
</tr>
<tr>
<td>2</td>
<td>.3</td>
<td>.300</td>
<td>.259</td>
<td>.298</td>
</tr>
<tr>
<td>3</td>
<td>.5</td>
<td>.500</td>
<td>.466</td>
<td>.498</td>
</tr>
<tr>
<td>4</td>
<td>.7</td>
<td>.699</td>
<td>.676</td>
<td>.698</td>
</tr>
<tr>
<td>5</td>
<td>.9</td>
<td>.899</td>
<td>.881</td>
<td>.898</td>
</tr>
</tbody>
</table>

2.8 CONCLUDING REMARKS

The method discussed in this chapter may be extended to the case of general score function as well as for data other than continuous data. Also attempt may be taken to incorporate clinical covariates which normally influence treatment responses.