CHAPTER – III

PROBLEMS ON HYDROMAGNETIC FLOW OF AN OLDROYD FLUID
CHAPTER – III

Introduction:

This chapter concerns with two problems on unsteady MHD flows of an incompressible conducting Oldroyd fluid through porous medium. The first paper deals with the motion between two non-conducting impermeable infinite parallel plates under the action of a uniform body force. The second paper deals with the flow through a rectilinear equilateral tube with impermeable boundary. The flow is generated by an axial pressure gradient which has two parts:

(a) One part is a constant and (b) the other part is an arbitrary function of time.

Basic Equations:

The constitutive equation for Oldroyd two constant model [62], [63] is

\[ \tau_{ij} = -p\delta_{ij} + \tau'_{ij} \]

(1)

\[ \left( 1 + \lambda_1 \frac{D}{Dt} \right) \tau_{ij} = 2\mu \left( 1 + \lambda_2 \frac{D}{Dt} \right) e_{ij} \]

(II)

\[ e_{ij} = (u_{i,j} + u_{j,i})/2 \]

(III)

where \( \frac{D}{Dt} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i} \).

The equation of continuity is

\[ u_{i,i} = 0 \]

(IV)

The equation of motion for unsteady MHD flow through porous medium is

\[ \rho \left( \frac{\partial u_i}{\partial t} + u_j u_{i,j} \right) = p F_i - p_i + \tau'_{ij,j} + \left( j \times B \right)_i - \frac{\mu}{K} u_i \]

(V)

where \( j = \sigma_1 (E + \mathbf{u} \times \mathbf{B}) \) is the electric current density and \( \mathbf{B} = \mu_0 H_c \mathbf{H}_c \) is the magnetic induction vector.

If the liquid is assumed to be of low conductivity i.e. the magnetic Reynolds number be very small, the induced magnetic field produced by the motion of an electrically conducting fluid is neglected. Then the Lorentz force in the absence of electric field will be \( \sigma_1 (\mathbf{u} \times \mathbf{B}) \times \mathbf{B} \).
**NOMENCLATURE**

\[
\begin{align*}
\tau_{ij} &= \text{total stress tensor} \\
\tau'_{ij} &= \text{deviatoric stress tensor} \\
e_{ij} &= \text{rate of strain tensor} \\
p &= \text{pressure} \\
\mu &= \text{co-efficient of viscosity} \\
\lambda_1 &= \text{stress relaxation time} \\
\lambda_2 &= \text{strain retardation time} (\lambda_1 > \lambda_2 > 0) \\
u_i &= \text{velocity component of the fluid} \\
\sigma_1 &= \text{electrical conductivity} \\
\rho &= \text{fluid density} \\
\bar{u} &= \text{velocity of the conducting medium} \\
K &= \text{permeability of the porous medium} \\
\mu_c &= \text{magnetic permeability} \\
\tilde{H}_c &= \text{magnetic field intensity} \\
\tilde{B} &= \text{magnetic induction vector} \\
\tilde{E} &= \text{electric field intensity} \\
\bar{J} &= \text{density of electric current} \\
F_i &= \text{component of external body force per unit mass}.
\end{align*}
\]

The physical significance of $\lambda_1$ is that if the motion is stopped suddenly the stress will decay as $\exp\left(-\frac{t}{\lambda_1}\right)$ and the physical significance of $\lambda_2$ is that if the stress are removed the motion will decay as $\exp\left(-\frac{t}{\lambda_2}\right)$.
3.1 : THE FLOW OF AN ELASTICO-VISCOUS CONDUCTING INCOMPRESSIBLE FLUID BETWEEN TWO INFINITE PARALLEL PLATES THROUGH POROUS MEDIUM UNDER TRANSVERSE UNIFORM MAGNETIC FIELD AND A UNIFORM BODY FORCE

Introduction:

The study of hydromagnetic flow of a conducting liquid through the ducts of rectangular and circular cross-section was investigated by Hartmann and Lazarus [41]. Chang and Lundgren [18], and Shercliff [79], discussed the solution of steady magnetohydrodynamic flow in a straight rectangular duct under a uniform transverse magnetic field. The problem of laminar flow of a viscous incompressible conducting fluid between two parallel plates under the action of transverse magnetic field was studied by many workers. Sengupta and Moitra [76] presented the motion of a conducting visco-elastic fluid between parallel plates which is set in motion by a uniform body force.

The aim of the present paper is to investigate the motion of visco-elastic Oldroyd fluid through porous medium between two parallel plates in presence of transverse magnetic field and a uniform body force. The fluid starts to move under the influence of uniform body force applied at time \( t = 0 \). Explicit expression for the velocity of the liquid has been obtained by using the technique of Laplace transform and the expression for shearing stress at the plates has been shown. The effects of magnetic field, elastic parameter, porous medium and time on velocity and shear stress have been shown graphically.

Formulation of the Problem:

We now consider the flow of an elastico-viscous Oldroyd type liquid through a porous medium between two parallel plates. The fluid initially at rest is set into motion by the action of uniform body force in presence of uniform transverse magnetic field.

* Published in Indian Journal of Theoretical Physics, Vol. 46, No. 3, p. 253 (1998)
Assuming the plates to be horizontal, so that y-axis is measured in a vertical direction and z-axis is measured in a horizontal direction normal to the x-axis. The spacing between the plates is 2h and the origin is taken midway between the plates. Let us assume that the magnetic field is uniform and it is acting (along z-axis) transversely to the flow. Let u be the velocity in x-direction and u is parallel to the body force X applied at time t = 0. The flow is assumed to be slow. Under these assumptions, the components of velocity field are

\[ [u, 0, 0] \] and that of the body force per unit mass are \([X, 0, 0]\).

And the magnetic induction vector \( \vec{B} = [0,0,B_0] \).

Now, equation (V) with the help of (I) – (III) can be written as

\[
\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \frac{\partial u}{\partial t} = X + \nu \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \frac{\partial^2 u}{\partial y^2} - \left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \frac{\sigma I B_0^2}{\rho} \left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \nu u
\]

where \( \nu = \frac{\mu}{\rho} \) is the Kinematic coefficient of viscosity.

The initial and boundary conditions are

\[
\begin{align*}
\text{when } t = 0 \text{ for all } y \\
\frac{\partial u}{\partial t} = 0
\end{align*}
\]

and

\[
\begin{align*}
u = 0 \quad \text{when } y = \pm h, \ t > 0.
\end{align*}
\]

**Solution of the problem:**

Equation (1) can be written as

\[
\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \frac{\partial u}{\partial t} = X + \nu \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \frac{\partial^2 u}{\partial y^2} - \left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \frac{\sigma I B_0^2}{\rho} + \frac{\nu}{K} \left(1 + \lambda_1 \frac{\partial}{\partial t}\right) u
\]

Introducing the non-dimensional quantities

\[
\begin{align*}
\nu' &= \frac{u \lambda_2}{h}, \quad X' = \frac{X \lambda_1 \lambda_2}{h}, \quad \nu' = \frac{t}{\lambda_1}, \quad y' = \frac{y}{\sqrt{\nu \lambda_1}}, \quad K' = \frac{K}{\lambda_1}, \\
M^2 &= \frac{\sigma I B_0^2}{\rho \lambda_1}, \quad \sigma = \frac{\lambda_2}{\lambda_1} \ < 1
\end{align*}
\]
by omitting dashes, the equation (4) can be written as

\[
\left(1 + \frac{\partial}{\partial t}\right) \frac{\partial u}{\partial t} = X + \left(1 + \sigma \frac{\partial}{\partial t}\right) \frac{\partial^2 u}{\partial y^2} - G \left(1 + \frac{\partial}{\partial t}\right) u. \tag{5}
\]

where

\[ G = M^2 + 1/\kappa \]

The initial and boundary conditions transform to

\[
\begin{aligned}
&u = 0 \\
&\frac{\partial u}{\partial t} = 0
\end{aligned}
\]

when \( t = 0 \) (omitting dashes) \( \text{(6)} \)

and

\[
u : 0 \quad \text{when} \quad y = \pm h = \pm \frac{h}{\sqrt{\nu \lambda_1}}. \tag{7}
\]

We make use of the Laplace transform

\[
\tilde{u} = \int_0^\infty u e^{-st} \, dt, \tag{8}
\]

where \( \text{Re}(s) > 0 \), in solving the differential equation (5).

Using the above Laplace transform in (5), we get

\[
\frac{d^2 \tilde{u}}{dy^2} - m^2(s) \tilde{u} = -\frac{X}{s(1 + \sigma s)}. \tag{9}
\]

where

\[
m^2(s) = \frac{s^2 + (1 + G)s + G}{1 + \sigma s} \quad \text{(9a)}
\]

The solution of the differential equation (9) is

\[
\tilde{u} = A \cosh my + B \sinh my + C. \tag{10}
\]

where \( A \) and \( B \) are constants of integration and

\[
C = \frac{X}{m^2 s(1 + \sigma s)}.
\]

Applying the transformed boundary condition (7) in (10), we get

\[
\tilde{u} = C \left[1 - \frac{\cosh my}{\cosh mh}\right]. \tag{11}
\]
By inverse Laplace transform, we obtain from (11),

\[ u = \frac{X}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{1}{m^2 s(1+\sigma s)} \left[ \frac{\cosh \sqrt{G} y}{\cosh \sqrt{G} h} \right] e^{st} \, dt. \]  

(12)

where \( \gamma \) is greater than the real part of the singularities of the integrand. We shall evaluate (12) by using Bromwich contour.

The integrand of (12) is a single valued function with poles at \( s = 0 \) and at the roots of

\[ \cosh \sqrt{G} h = 0 \quad \text{and} \quad s^2 + (1 + G)s + G = 0. \]

The roots of \( \cosh \sqrt{G} h = 0 \) are \( m = iD_n \),

where \( D_n = (2n+1) \frac{\pi}{2h} \), \( n = 0, 1, 2, 3, \ldots \).

When \( m = iD_n \), we get from (9a),

\[ s^2 - (1 + G + D_n^2 \sigma)s - (G + D_n^2) = 0. \]

(13)

Let the roots of the equation (13) be \( s_n^{(1)} \) and \( s_n^{(2)} \). Performing contour integration, we get, from (12),

\[ u = \frac{X}{G} \left[ 1 - \frac{\cosh \sqrt{G} y}{\cosh \sqrt{G} h} \right] + \frac{2X}{h} \sum (-1)^n \cos D_n y \]

\[ \times \left[ \frac{(1+\sigma s_n^{(1)}) e^{s_n^{(1)} t}}{(\sigma(1+G+D_n^2 \sigma)^2 - (3D_n^2 \sigma + 2G\sigma + G + 1)s_n^{(1)} + \{\sigma(1+G+D_n^2 \sigma)- 2\}(G+D_n^2)} \right. \]

\[ + \frac{(1+\sigma s_n^{(2)}) e^{s_n^{(2)} t}}{(\sigma(1+G+D_n^2 \sigma)^2 - (3D_n^2 \sigma + 2G\sigma + G + 1)s_n^{(2)} + \{\sigma(1+G+D_n^2 \sigma)- 2\}(G+D_n^2)} \]

(14)

where \( (s_n^{(1)}, s_n^{(2)}) = \frac{-(1+G+D_n^2 \sigma) \pm \sqrt{(1+G+D_n^2 \sigma)^2 - 4(G+D_n^2)}}{2} \).

The corresponding flow of a viscous conducting incompressible fluid through porous medium under uniform transverse magnetic field and a uniform body force in dimensional form is
\[
u = \frac{X}{G} \left[ \cosh \left( \frac{G}{\nu} \sqrt{\frac{y}{v}} \right) \right] - 2X \sum_{n=0}^{\infty} \frac{(-1)^n \cos(D_n y)}{h} \frac{e^{-(G+D_n^2)v)t}}{D_n (G + vD_n^2)} \tag{15}\]

where \( G = \frac{\sigma B_0^2}{\rho} + \frac{v}{K} \).

The corresponding non-MHD flow without porous medium between two parallel plates is obtained from (15) by passing to the limit \( \frac{\sigma B_0^2}{\rho} \to 0 \) and \( K \to \infty \), i.e. \( G \to 0 \) and this limiting value in dimensional form is

\[
u = \frac{X(h^2 - y^2)}{2\nu} - 2X \sum_{n=0}^{\infty} \frac{(-1)^n \cos D_n y}{h} \frac{e^{-(G+D_n^2)v)t}}{D_n^3} \tag{16}\]

where \( D_n = (2n + 1) \frac{\pi}{2h} \).

This result is in agreement with the motion of a viscous fluid which is set in motion by an uniform body force between two parallel plates as presented by Carslaw and Jaeger [16].

The non-MHD flow of Oldroyd type fluid without porous medium between two parallel plates under a uniform body force is obtained from (14) by passing to the limit as \( G \to 0 \) which gives

\[
u = \frac{X}{2} (h^2 - y^2) + \frac{2X}{h} \sum_{n=0}^{\infty} \frac{(-1)^n \cos D_n y}{D_n}
\times \left[ \frac{(1 + \sigma s_n^{(1)}) e^{s_n^{(1)} t}}{(1 + D_n^2 \sigma^2)^2 - (3D_n^2 \sigma + 1)s_n^{(1)} + (\sigma(1 + D_n^2 \sigma^2) - 2D_n^2)} \right. \\
\left. + \frac{(1 + \sigma s_n^{(2)}) e^{s_n^{(2)} t}}{(1 + D_n^2 \sigma^2)^2 - (3D_n^2 \sigma + 1)s_n^{(2)} + (\sigma(1 + D_n^2 \sigma^2) - 2D_n^2)} \right] \tag{17}\]

where \( s_n^{(1)} \cdot s_n^{(2)} = \frac{-(1 + D_n^2 \sigma) \pm \sqrt{(1 + D_n^2 \sigma)^2 - 4D_n^2}}{2} \).
Shear Stress at the Plates:

The shear stress at the lower plate for Oldroyd liquid is written by the following equation:

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \tau = \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \left(\frac{\partial u}{\partial y}\right)_{y = -h}.$$ \hspace{1cm} (18)

Using the non-dimensional quantities

$$u' = \frac{u\lambda_2}{h}, \quad t' = \frac{t}{\lambda_1}, \quad y' = \frac{y}{\sqrt{\lambda_1}}, \quad \tau' = \frac{\sqrt{\lambda_1} \lambda_2 \mu}{h} \tau$$

in (15), we get (omitting the dashes)

$$\frac{\partial \tau}{\partial t} + \tau = \left(1 + \sigma \frac{\partial}{\partial t}\right) \left(\frac{\partial u}{\partial y}\right)_{y = -h}.$$ \hspace{1cm} (19)

where \( \sigma = \lambda_2 / \lambda_1 \).

From (14) and (19), we get

$$\frac{\tau}{X} = \frac{\tanh \sqrt{G} h}{\sqrt{G}} + \frac{2}{h} \sum_{n=0}^{\infty} \left[ \frac{\{1 - \sigma^2 (G + D_n^2) + \sigma s_n^{(1)} (2 - \sigma \xi)\} e^{s_n^{(1)}}}{\{-\sigma \xi^3 + \sigma \xi^2 + (G + D_n^2 \sigma) \eta + (\sigma \xi - 2)(G + D_n^2) s_n^{(1)} + (G + D_n^2) \eta\}} \right]$$

$$+ \frac{\{1 - \sigma^2 (G + D_n^2) + \sigma s_n^{(2)} (2 - \sigma \xi)\} e^{s_n^{(2)}}}{\{-\sigma \xi^3 + \sigma \xi^2 + (G + D_n^2 \sigma) \eta + (\sigma \xi - 2)(G + D_n^2) s_n^{(2)} + (G + D_n^2) \eta\}}$$ \hspace{1cm} (20)

where \( \xi = 1 + G + D_n^2 \sigma \) and \( \eta = 3D_n^2 \sigma + 2G \sigma + G + 1 \).

Similarly, we can calculate the shear stress at the upper plate in dimensionless form.

Discussion:

In order to study the effects of magnetic field, permeability of porous medium, elastic parameter and time on velocity profiles, we have drawn graphs for velocity \( \left\{ \frac{u}{X} \right\} \) against \( y \). From figs. 1, 2, 3 and 4 it is clear that the velocity distribution is
symmetrical about the middle plane $y = 0$ of the channel. From fig. 1 we see that the magnetic field is not in favour of the flow of the fluid i.e., the fluid velocity decreases with the increase of the intensity of the magnetic field. This phenomenon is in conformity with the concept that the magnetic lines of force always drag the fluid perpendicular to it. Magnetic lines of force are as if frozen in the fluid. This decrease is more pronounced in the central region of the channel, therefore, the effect of magnetic field in the flow is to flatten the velocity profiles. From fig. 2 it is seen that as the permeability of the porous medium increases the velocity of the fluid element increases. From fig. 3 it is noted that the fluid velocity increases with the increase of time. From fig. 4 it is clear that the velocity of the fluid element diminishes with the increase of the value of the elastic parameter. For different values of the elastic parameter the velocity attains a common maximum value in the central region of the channel. The effect of the elastic parameter is more pronounced in the regions $y = 0.5$ to $y = 0.9$ and $y = -0.5$ to $y = -0.9$. From figs. 1, 2, 3 and Fig. 4 it is evident that for prescribed values of the flow parameters the velocity becomes uniform in the central region of the channel.

The effects of magnetic field, permeability of the porous medium and elastic parameter on skin-friction or shear stress on the lower plate have been shown in figs. 5, 6, 7. From fig. 5 we observe that the value of the shear stress decreases with the increase of the magnetic field. We see from fig. 6 that the value of the shear stress increases with the increase of the permeability of the porous medium. It is evident from figs. 5, 6 that for prescribed values of the flow parameters the skin-friction or shear stress on the lower plate sensibly remains the same although a small variation of decrease with the increase of time. From fig. 7 it is noted that the value of the shear stress decreases with the increase of elastic parameter. It is also noted from fig. 7 that the skin-friction initially decreases rapidly and then decreases very slowly.

An examination of the solution (14) reveals that its transient components decay very rapidly to zero as $t \to \infty$. Consequently, an ultimate steady state is set up in the limit.
Fig. 1: Velocity profiles for various values of M when \( K = 1, t = 0.2 \) and \( \sigma = 0.3 \).
Fig. 2: Velocity profiles for various values of K when $M = 1$, $t = 0.2$ and $\sigma = 0.3$. 
Fig. 3 : Velocity profiles for various values of $T$ when $M = 1$, $K = 0.2$ and $\sigma = 0.3$. 
Fig. 4: Velocity profiles for various values of $\sigma$ when $M = 1$, $K = 0.5$ and $t = 0.2$. 
Fig. 5: Shear stress profiles at the lower plate against \( t \) for various values of \( M \) when \( \kappa = 1 \) and \( \sigma = 0.3 \).
Fig. 6: Shear stress profiles at the lower plate against t for various values of K when M = 1 and σ = 0.3.
Fig. 7: Shear stress profiles at the lower plate against $t$ for various values of $\sigma$ when $M = 1$ and $K = 0.2$. 
3.2: MHD UNSTEADY FLOW OF A VISCO-ELASTIC FLUID IN A LONG POROUS EQUILATERAL TRIANGULAR TUBE

Introduction:

Viscoelastic fluids have assumed wide importance in the present technology and industries with the growing use of such fluids in the form of molten plastics, paints, polymer solutions and petroleum products. MHD steady and unsteady flows of visco-elastic fluids through porous medium have received much attention for quite sometime. Different model of visco-elastic fluid are employed. Singh et al. [84] discussed the unsteady flow of visco-elastic fluid through porous medium between two parallel plates. Flow through porous medium basically depends upon Darcy’s law. After conducting many experiments Brinkman generalised the Darcy’s law to study the flow through highly porous media. Singh et al. [82] discussed the unsteady flow of viscous fluid in a porous equilateral triangular tube. Darcy’s law was latter generalised by many investigators viz. [82], [74].

In the present paper the attempt is to investigate MHD flow of visco-elastic Oldroyd fluid through equilateral triangular tube with impermeable boundary and with axial pressure gradient. The pressure gradient has two parts – one is constant and the other is time dependent. Time dependent part is of different types which are discussed separately –

(I) Firstly this part is of the type \( A(1 - e^{-\beta t}) \), \( A, \beta > 0 \).

(II) Secondly time dependent part is periodic.

(III) Thirdly this part of the pressure gradient is constant which acts for a finite time.

Towards solving the problem, Laplace transform technique has been applied. The effect of the permeability of the porous medium, magnetic parameter and elastic parameter on velocity profile have been shown with the help of graphs and tables.

Formulation of the Problem:

We consider the hydromagnetic flow of an incompressible visco-elastic Oldroyd fluid through a porous equilateral triangular tube with impermeable

boundary. Let \((x, y, z)\) be the rectangular co-ordinates in cartesian system. Also let the flow of the fluid be along the \(z\)-axis which is the axis of the tube. So we take the component of velocity field as

\[ [0, 0, u]. \]

Now, from (IV) we have

\[ \frac{\partial u}{\partial z} = 0 \Rightarrow u = u(x, y, t) \]  

(1)

If the liquid is assumed to be of low conductivity i.e. the magnetic Reynolds number be very small, the induced magnetic field produced by the motions of electrically conducting fluid is neglected in comparison with the applied magnetic field so that we can take \( \vec{B} = (B_1, B_2, 0) \), \( B_1 \) and \( B_2 \) being constants. Since no external electric field is applied, the effect of polarisation of fluid is neglected so that we have assumed \( E = (0, 0, 0) \).

The electromagnetic body force i.e. Lorentz force

\[ \vec{J} \times \vec{B} = \sigma_1 (\vec{v} \times \vec{B}) \times \vec{B} = -i_x \sigma_1 B_0^2 u \]  

(2)

where \( i_x \) is the unit vector along \( z \)-axis and \( B_0^2 = B_1^2 + B_2^2 \).

The equation of motion (V) with the help of (I) – (III), (1) and (2) can be written as

\[ \left( 1 + \lambda_1 \frac{\partial}{\partial t} \right) \frac{\partial u}{\partial t} = \frac{1}{\rho} \left( 1 + \lambda_1 \frac{\partial}{\partial t} \right) \frac{\partial p}{\partial t} + \alpha \left( 1 + \lambda_2 \frac{\partial}{\partial t} \right) \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \left( 1 + \lambda_1 \frac{\partial}{\partial t} \right) \left( \frac{\alpha}{K} + \frac{\sigma_1 B_0^2}{\rho} \right) u \]  

(4)

where \( \alpha = \mu/\rho \) is the Kinematic co-efficient of viscosity.

Governing equation of motion (4) in trilinear co-ordinates system is transformed as follows –

Let LMN be equilateral triangle whose centroid is G. The line parallel to MN is taken as x-axis and the line perpendicular to MN is taken as y-axis. Let the length of each side of the equilateral triangle be 2a, so that we have
Fig. 1a : Cross-section of the equilateral triangular tube

Let $Y_1$, $Y_2$, $Y_3$ be the length of the perpendiculars from any point $F(x, y)$ within the triangle LMN on MN, LM and LN respectively. Let the radius of the inscribed circle be $R$ (fig. 1a).

Then we have

$$
Y_1 = R - y
$$
$$
Y_2 = R + \frac{y + \frac{x\sqrt{3}}{2}}{2}
$$
$$
Y_3 = R + \frac{y + \frac{x\sqrt{3}}{2}}{2}
$$

$$
\therefore Y_1 + Y_2 + Y_3 = 3R = \sqrt{3} a
$$

Now it follows that

$$
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial Y_1^2} + \frac{\partial^2}{\partial Y_2^2} + \frac{\partial^2}{\partial Y_3^2} - \frac{\partial^2}{\partial Y_1 \partial Y_2} - \frac{\partial^2}{\partial Y_2 \partial Y_3} - \frac{\partial^2}{\partial Y_3 \partial Y_1}
$$

Now the equation (4) transforms to the form

$$
\left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \frac{\partial u}{\partial t} = -\frac{1}{\rho} \left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \frac{\partial p}{\partial z} + \alpha \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \left(\frac{\partial^2 u}{\partial Y_1^2} + \frac{\partial^2 u}{\partial Y_2^2} + \frac{\partial^2 u}{\partial Y_3^2} - \frac{\partial^2 u}{\partial Y_1 \partial Y_2} - \frac{\partial^2 u}{\partial Y_2 \partial Y_3} - \frac{\partial^2 u}{\partial Y_3 \partial Y_1}\right) + \left(1 + \lambda_1 \frac{\partial}{\partial t}\right) \left(\frac{\alpha + \sigma}{K} \frac{\partial u}{\partial Y_1}\right)
$$

Introducing the non-dimensional quantities

$$
Y'_1 = \frac{Y_1}{\sqrt{\alpha \lambda_1}}, \quad Y'_2 = \frac{Y_2}{\sqrt{\alpha \lambda_1}}, \quad Y'_3 = \frac{Y_3}{\sqrt{\alpha \lambda_1}}, \quad z' = \frac{z}{\sqrt{\alpha \lambda_1}}, \quad u' = \frac{u \lambda_1}{a},
$$

$$
t' = \frac{t}{\lambda_1}, \quad K' = \frac{K}{\alpha \lambda_1}, \quad p = \frac{p \lambda_1}{\alpha \lambda_1} \sqrt{\frac{\lambda_1}{\alpha}}
$$
in (8), we get (omitting dashes)

\[
\left(1 + \frac{\partial}{\partial t}\right) \frac{\partial u}{\partial t} = \left(1 + \frac{\partial}{\partial t}\right) \frac{\partial p}{\partial t} + \left(1 + \sigma \frac{\partial}{\partial t}\right) \left(\frac{\partial^2 u}{\partial Y_1^2} + \frac{\partial^2 u}{\partial Y_2^2} + \frac{\partial^2 u}{\partial Y_3^2} - \frac{\partial^2 u}{\partial Y_1 \partial Y_2} - \frac{\partial^2 u}{\partial Y_2 \partial Y_3} - \frac{\partial^2 u}{\partial Y_3 \partial Y_1}\right) - \frac{\partial^2 u}{\partial Y_3 \partial Y_1} \left(1 + \frac{\partial}{\partial t}\right) G u
\]

(9)

where \( M^2 \) (magnetic parameter) = \( \frac{\sigma B_0^2}{\rho} \lambda_1 \)

\( \sigma \) (elastic parameter) = \( \frac{\lambda_2}{\lambda_1} \) (\(< 1)\)

\( G = M^2 + \frac{1}{K} \).

Equations (5) and (6) in non-dimensional form are (omitting dashes)

\[
\begin{align*}
Y_1 &= R - y \\
Y_2 &= R + \frac{y}{2} - \frac{x \sqrt{3}}{2} \\
Y_3 &= R + \frac{y}{2} + \frac{x \sqrt{3}}{2}
\end{align*}
\]

(9a)

and \( Y_1 + Y_2 + Y_3 = 3R = \sqrt{3} a = d \) (say)

where

\[
\begin{align*}
Y'_1 &= \frac{Y_1}{\sqrt{\alpha \lambda_1}}, \quad Y'_2 = \frac{Y_2}{\sqrt{\alpha \lambda_1}}, \quad Y'_3 = \frac{Y_3}{\sqrt{\alpha \lambda_1}}, \quad x' = \frac{x}{\sqrt{\alpha \lambda_1}}, \quad y' = \frac{y}{\sqrt{\alpha \lambda_1}}, \quad z' = \frac{z}{\sqrt{\alpha \lambda_1}}.
\end{align*}
\]

\( a' = \frac{a}{\sqrt{\alpha \lambda_1}} \) and \( R' = \frac{R}{\sqrt{\alpha \lambda_1}} \).

We assume \( -\frac{\partial p}{\partial z} = P + f(t) \) (10)

and \( u(Y_1, Y_2, Y_3, t) = u_1(Y_1, Y_2, Y_3) + u_2(Y_1, Y_2, Y_3, t) \) (11)

where \( P \) is a constant and \( f(t) \) is an arbitrary function of time.

From (9), (10) and (11), we get

\[
\frac{\partial^2 u_1}{\partial Y_1^2} + \frac{\partial^2 u_1}{\partial Y_2^2} + \frac{\partial^2 u_1}{\partial Y_3^2} - \frac{\partial^2 u_1}{\partial Y_1 \partial Y_2} - \frac{\partial^2 u_1}{\partial Y_2 \partial Y_3} - \frac{\partial^2 u_1}{\partial Y_3 \partial Y_1} - Gu_1 + P = 0
\]

(12)

and
\[ \left( 1 + \frac{\partial}{\partial t} \right) \frac{\partial u_2}{\partial t} = \left( 1 + \sigma \frac{\partial}{\partial t} \right) f(t) + \left( 1 + \sigma \frac{\partial}{\partial t} \right) \left( \frac{\partial^2 u_2}{\partial Y_i^2} + \frac{\partial^2 u_2}{\partial Y_2^2} + \frac{\partial^2 u_2}{\partial Y_3^2} - \frac{\partial^2 u_2}{\partial Y_2 \partial Y_3} - \frac{\partial^2 u_2}{\partial Y_2 \partial Y_1} \right) \]

The boundary and initial conditions of the problem are

\[ u_1 = u_2 = 0 \quad \text{at} \quad Y_1 = Y_2 = Y_3 = 0 \]  \hspace{1cm} (14)

and

\[ u_2 = 0 \quad \text{at} \quad t \leq 0 \]
\[ \frac{\partial u_2}{\partial t} = 0 \quad \text{at} \quad t \leq 0 \]  \hspace{1cm} (15)

Solution:

We take the solution of the equation (12) and (13) satisfying the condition (14) are of the form

\[ u_1 = \sum_{n=1}^{\infty} \sum_{i=1}^{3} C_n \sin(p_n Y_i) \]  \hspace{1cm} (16)

and

\[ u_2 = \sum_{n=1}^{\infty} \sum_{i=1}^{3} A_n(t) \sin(p_n Y_i) \]  \hspace{1cm} (17)

where

\[ p_n = \frac{2n\pi}{d} \]  \hspace{1cm} (18)

and

\[ A_n(t) = \frac{dA_n}{dt} = 0 \quad \text{at} \quad t \leq 0. \]

Substituting (16) in (12), we get

\[ \sum_{n=1}^{\infty} \sum_{i=1}^{3} \left( p_n^2 + G \right) C_n \sin(p_n Y_i) = P. \]  \hspace{1cm} (19)

we can write, by virtue of \( Y_1 + Y_2 + Y_3 = d \)

\[ P = \frac{P}{d} [((d - 2Y_1) + (d - 2Y_2) + (d - 2Y_3))] \]

Now \((d - 2Y_i) \ (i = 1, 2, 3)\) can be easily expressed by Fourier sine series in the interval \(0 \leq Y_i < d\), and hence it follows that

\[ P = \frac{2P}{\pi} \sum_{n=1}^{\infty} \sum_{i=1}^{3} \frac{1}{n} \sin(p_n Y_i) \]  \hspace{1cm} (20)
From equation (19) and (20), we have

\[ C_n = \frac{2P}{n\pi(p_n^2 + G)} \]  \hspace{1cm} \text{(21)}

\[ \therefore u_1(Y_1, Y_2, Y_3) = \frac{2P}{\pi} \sum_{n=1}^{\infty} \sin(p_nY_i) \sum_{i=1}^{3} \frac{1}{n(p_n^2 + G)} \]  \hspace{1cm} \text{(22)}

Putting the value of \( u_2 \) from (17) in (13) and using Fourier sine series as used in equation (19) and then comparing the coefficients, we get

\[ \frac{d^2 A_n}{dt^2} + (1 + G + p_n^2\sigma) \frac{dA_n}{dt} + (G + p_n^2)A_n = \frac{2}{n\pi} \left( 1 + \frac{\partial}{\partial t} \right) f(t) \]  \hspace{1cm} \text{(23)}

Laplace transform of the equation (23) yields

\[ \bar{A}_n(s) = \frac{2}{n\pi} \frac{(1+s)f(s) - f(0)}{s^2 + (1 + G + p_n^2\sigma)s + (G + p_n^2)} \]  \hspace{1cm} \text{(24)}

Inverting (24) by convolution theorem, we get

\[ A_n(t) = \frac{2}{n\pi} \int_0^t \left[ \frac{(1+s_n^{(1)})e^{s_n^{(1)\lambda}} - (1+s_n^{(2)})e^{s_n^{(2)\lambda}}}{(s_n^{(1)} - s_n^{(2)})} \right] f(t-\lambda)d\lambda \]

\[ -\frac{2}{n\pi} f(0) \left[ \frac{(1+s_n^{(1)})e^{s_n^{(1)\lambda}} - (1+s_n^{(2)})e^{s_n^{(2)\lambda}}}{(s_n^{(1)} - s_n^{(2)})} \right] \]  \hspace{1cm} \text{(25)}

where \( s_n^{(1)} \) and \( s_n^{(2)} \) are the roots of the equation

\[ s^2 + (1 + G + p_n^2\sigma)s + (G + p_n^2) = 0. \]  \hspace{1cm} \text{(26)}

\[ \therefore u_2(Y_1, Y_2, Y_3, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{i=1}^{3} \frac{1}{n} \int_0^t \left[ \frac{(1+s_n^{(1)})e^{s_n^{(1)\lambda}} - (1+s_n^{(2)})e^{s_n^{(2)\lambda}}}{(s_n^{(1)} - s_n^{(2)})} \right] f(t-\lambda)d\lambda \]

\[ -f(0) \left[ \frac{(1+s_n^{(1)})e^{s_n^{(1)\lambda}} - (1+s_n^{(2)})e^{s_n^{(2)\lambda}}}{(s_n^{(1)} - s_n^{(2)})} \right] \sin(p_nY_i) \]  \hspace{1cm} \text{(27)}

Therefore, the complete solution of the equation of motion satisfying the boundary conditions is obtained as

\[ \therefore u = \frac{2P}{\pi} \sum_{n=1}^{\infty} \sum_{i=1}^{3} \frac{\sin(p_nY_i)}{n(p_n^2 + G)} + \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{i=1}^{3} \frac{1}{n} \int_0^t \left[ \frac{(1+s_n^{(1)})e^{s_n^{(1)\lambda}} - (1+s_n^{(2)})e^{s_n^{(2)\lambda}}}{(s_n^{(1)} - s_n^{(2)})} \right] \]

\[ \times f(t-\lambda)d\lambda - f(0) \left[ \frac{(1+s_n^{(1)})e^{s_n^{(1)\lambda}} - (1+s_n^{(2)})e^{s_n^{(2)\lambda}}}{(s_n^{(1)} - s_n^{(2)})} \right] \sin(p_nY_i) \]  \hspace{1cm} \text{(28)}
**Particular Case:**

**Case (I):**

Let us now stress on to the particular case of pressure gradient in which the pressure gradient is a function of time which increases numerically with time and ultimately tends to a constant value. In other words we assume

\[ f(t) = A (1 - e^{\beta t}) \]  \hspace{1cm} (1.1)

where \( A \) and \( \beta \) are positive constants.

Now, from (28) and (1.1), we have

\[
\begin{align*}
\frac{u}{2} &= 2 \pi \sum_{n=1}^{\infty} \sum_{i=1}^{3} \frac{\sin(p_n Y_i)}{n(p_n^2 + G)} + 2A \pi \sum_{n=1}^{\infty} \sum_{i=1}^{3} \frac{1}{n} \left[ \frac{(1 + s_n^{(1)}) \{s_n^{(1)}(1 - e^{-\beta t}) + \beta(1 - e^{-\beta t})\}}{s_n^{(1)}(s_n^{(1)} + \beta)} \right] \\
&\quad \times \frac{\sin(p_n Y_i)}{(s_n^{(1)} - s_n^{(2)})}
\end{align*}
\]  \hspace{1cm} (1.2)

Proceeding to the limit \( t \to \infty \) in (1.2), we find that the transient effects die out for all values of \( \beta \) \((\beta > 0)\) and the ultimate steady state is attained and the fluid velocity assumes the form

\[
\frac{u}{2} = \frac{2(P + A)}{\pi} \sum_{n=1}^{\infty} \sum_{i=1}^{3} \frac{\sin(p_n Y_i)}{n(p_n^2 + G)}
\]  \hspace{1cm} (1.3)

It is interesting to note that in the steady state the velocity does not depend on elastic parameter \( \sigma \).

When \( \beta \to \infty \) in (1.2), we have

\[
\begin{align*}
\frac{u}{2} &= 2 \pi \sum_{n=1}^{\infty} \sum_{i=1}^{3} \frac{\sin(p_n Y_i)}{n(p_n^2 + G)} + A \pi \sum_{n=1}^{\infty} \sum_{i=1}^{3} \frac{1}{n} \left[ \frac{1}{G + p_n^2} + \frac{(1 + s_n^{(1)}) e^{s_n^{(1)t}}}{s_n^{(1)}(s_n^{(1)} - s_n^{(2)})} - \frac{(1 + s_n^{(2)}) e^{s_n^{(2)t}}}{s_n^{(2)}(s_n^{(1)} - s_n^{(2)})} \right] \\
&\quad \times \sin(p_n Y_i)
\end{align*}
\]  \hspace{1cm} (1.4)

Equation (1.4) gives the velocity of the fluid when the arbitrary part of the pressure gradient is a constant.
Case (II):

Let us now consider the case in which the arbitrary part of the pressure gradient is a periodic function of time. In other words, we assume

\[ f(t) = A \sin \omega t \]

where \( A \) and \( \omega \) are constants.

Then from (28), we have

\[
\begin{align*}
  u = & \frac{2P}{\pi} \sum_{n=1}^{\infty} \frac{3}{n} \frac{\sin(p_n Y_n)}{n(p_n^2 + G)} + \frac{2A}{\pi} \sum_{n=1}^{\infty} \sum_{i=1}^{n-1} \frac{1}{n} \left[ \frac{(1+s_n^{(1)})(\omega e^{s_n^{(1)} t} - s_n^{(1)} \sin \omega t - \cos \omega t)}{s_n^{(1)}^2 + \omega^2} ight] \\
  & - \frac{(1+s_n^{(2)})(\omega e^{s_n^{(2)} t} - s_n^{(2)} \sin \omega t - \cos \omega t)}{s_n^{(2)}^2 + \omega^2} \sin(p_n Y_n) \\
  & \left( s_n^{(1)} - s_n^{(2)} \right) \\
\end{align*}
\]

Equation (2.2) gives the velocity of the fluid in the present case.

Case (III):

Let us now consider the case in which the arbitrary part of the pressure gradient is constant which acts for a finite time 'T'. In this case, we shall take

\[ f(t) = A[H(t) - H(t - T)] \]

where \( A \) is a constant and \( H(t) \) is Heaviside unit function defined as

\[ H(t) = \begin{cases} 
0 & \text{when } t \leq 0 \\
1 & \text{when } t > T 
\end{cases} \]

Here

\[ \bar{f}(s) = \frac{A(1-e^{-sT})}{s} \]

From (24), (3.1) and (3.2), we have

\[
\bar{A}_n(s) = \frac{A(1+s)(1-e^{-sT})}{s^2 + (1+G+p_n^2 \sigma)s + (G+p_n^2)}, \quad \frac{1}{s} \cdot \frac{2}{n\pi} 
\]

By inverse Laplace transform, we have

\[
A_n(t) = \frac{2A}{(G+p_n^2)n\pi} + \left( \frac{1+s_n^{(1)}e^{s_n^{(1)} t}}{s_n^{(1)}} - \frac{(1+s_n^{(2)}e^{s_n^{(2)} t})}{s_n^{(2)}} \right) \left[ \frac{2A}{(s_n^{(1)} - s_n^{(2)})n\pi} \right] \\
\text{when } 0 < t \leq T
\]
From (3.4), (3.5) and (28), we have

$$u = \frac{2P}{\pi} \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{\sin(p_n Y_i)}{n (p_n^2 + G)} + \frac{2A}{\pi} \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{1}{G + p_n^2} \left[ \frac{1}{s_n^{(1)}} \frac{(1 + s_n^{(1)}) e^{s_n^{(1)} t}}{s_n^{(1)} (s_n^{(1)} - s_n^{(2)})} - \frac{(1 + s_n^{(2)}) e^{s_n^{(2)} t}}{s_n^{(2)} (s_n^{(1)} - s_n^{(2)})} \right] \sin(p_n Y_i)$$

when $0 < t \leq T$  \hspace{1cm} (3.6)

$$= \frac{2P}{\pi} \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{\sin(p_n Y_i)}{n (p_n^2 + G)} + \frac{2A}{\pi} \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{1}{G + p_n^2} \left[ \frac{(1 + s_n^{(1)}) \psi(s_n^{(1)}) e^{s_n^{(1)} t}}{s_n^{(1)} (s_n^{(1)} - s_n^{(2)})} - \frac{(1 + s_n^{(2)}) \psi(s_n^{(2)}) e^{s_n^{(2)} t}}{s_n^{(2)} (s_n^{(1)} - s_n^{(2)})} \right] \sin(p_n Y_i)$$

when $t > T$  \hspace{1cm} (3.7)

where $\psi(s) = 1 - e^{-sT}$

We note that if $f(t) = A$ continues indefinitely, the velocity, as $t \to \infty$, asymptotically attains the steady state and the fluid velocity assumes the form

$$u = \frac{2(A + P)}{\pi} \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{\sin(p_n Y_i)}{n (G + p_n^2)}$$

This velocity is independent of the elastic parameter $\sigma$. 
Table – 1 (Case I)
Numerical Values of $u$ for different values of $t$ and $\sigma$ when $x = 0, y = 0, K = 0.1, M = 3$ and $\beta = 0.5$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$u (\sigma = 0.1)$</th>
<th>$u (\sigma = 0.2)$</th>
<th>$u (\sigma = 0.3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.048095</td>
<td>0.048095</td>
<td>0.048095</td>
</tr>
<tr>
<td>0.2</td>
<td>0.051335</td>
<td>0.051396</td>
<td>0.051445</td>
</tr>
<tr>
<td>0.4</td>
<td>0.055515</td>
<td>0.055620</td>
<td>0.055699</td>
</tr>
<tr>
<td>0.6</td>
<td>0.059439</td>
<td>0.059541</td>
<td>0.059616</td>
</tr>
<tr>
<td>0.8</td>
<td>0.063052</td>
<td>0.063127</td>
<td>0.063180</td>
</tr>
<tr>
<td>1.0</td>
<td>0.066346</td>
<td>0.066386</td>
<td>0.066413</td>
</tr>
<tr>
<td>1.2</td>
<td>0.069330</td>
<td>0.069337</td>
<td>0.069337</td>
</tr>
<tr>
<td>1.4</td>
<td>0.072024</td>
<td>0.072001</td>
<td>0.071978</td>
</tr>
<tr>
<td>1.6</td>
<td>0.074448</td>
<td>0.074402</td>
<td>0.074359</td>
</tr>
<tr>
<td>1.8</td>
<td>0.076626</td>
<td>0.076562</td>
<td>0.076505</td>
</tr>
<tr>
<td>2.0</td>
<td>0.078682</td>
<td>0.078505</td>
<td>0.078438</td>
</tr>
<tr>
<td>2.2</td>
<td>0.080336</td>
<td>0.080253</td>
<td>0.080178</td>
</tr>
<tr>
<td>2.4</td>
<td>0.081911</td>
<td>0.081823</td>
<td>0.081746</td>
</tr>
<tr>
<td>2.6</td>
<td>0.083324</td>
<td>0.083236</td>
<td>0.083157</td>
</tr>
<tr>
<td>2.8</td>
<td>0.084592</td>
<td>0.084506</td>
<td>0.084427</td>
</tr>
<tr>
<td>3.0</td>
<td>0.085732</td>
<td>0.085648</td>
<td>0.085572</td>
</tr>
</tbody>
</table>

Table – 2 (Case II)
Numerical Values of $u$ for different values of $t$ and $\sigma$ when $x = 0, y = 0, K = 0.1, M = 3$ and $\omega = 6.0$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$u (\sigma = 0.1)$</th>
<th>$u (\sigma = 0.2)$</th>
<th>$u (\sigma = 0.3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.048095</td>
<td>0.048095</td>
<td>0.048095</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0623488</td>
<td>0.0625818</td>
<td>0.0627860</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0822525</td>
<td>0.0829056</td>
<td>0.0834263</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0923863</td>
<td>0.0931265</td>
<td>0.0936721</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0877639</td>
<td>0.0881666</td>
<td>0.0884201</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0696771</td>
<td>0.0694648</td>
<td>0.0692416</td>
</tr>
<tr>
<td>0.6</td>
<td>0.0443156</td>
<td>0.0434619</td>
<td>0.0427699</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0204574</td>
<td>0.0191894</td>
<td>0.0182201</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0063759</td>
<td>0.0050883</td>
<td>0.0041454</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0069433</td>
<td>0.0060552</td>
<td>0.0054449</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0219257</td>
<td>0.0217295</td>
<td>0.0216508</td>
</tr>
<tr>
<td>1.1</td>
<td>0.0460619</td>
<td>0.0466182</td>
<td>0.0470915</td>
</tr>
<tr>
<td>1.2</td>
<td>0.0709003</td>
<td>0.0720138</td>
<td>0.0728717</td>
</tr>
<tr>
<td>1.3</td>
<td>0.0877493</td>
<td>0.0890351</td>
<td>0.0899796</td>
</tr>
<tr>
<td>1.4</td>
<td>0.0907126</td>
<td>0.0917291</td>
<td>0.0924348</td>
</tr>
<tr>
<td>1.5</td>
<td>0.0787479</td>
<td>0.0791498</td>
<td>0.0793766</td>
</tr>
<tr>
<td>1.6</td>
<td>0.0560305</td>
<td>0.0556885</td>
<td>0.0553647</td>
</tr>
<tr>
<td>1.7</td>
<td>0.0304938</td>
<td>0.0295392</td>
<td>0.0287862</td>
</tr>
<tr>
<td>1.8</td>
<td>0.0110576</td>
<td>0.0098358</td>
<td>0.0089254</td>
</tr>
<tr>
<td>1.9</td>
<td>0.0045118</td>
<td>0.0034614</td>
<td>0.0027204</td>
</tr>
<tr>
<td>2.0</td>
<td>0.0131443</td>
<td>0.0126436</td>
<td>0.0123393</td>
</tr>
</tbody>
</table>
### Table – 3 (Case III)

**Numerical Values of \( u \) for different values of \( t \) and \( \sigma \) when \( x = 0, y = 0, K = 0.1, M = 3 \) and \( T = 1.0 \).**

<table>
<thead>
<tr>
<th>( t )</th>
<th>( u (\sigma = 0.1) )</th>
<th>( u (\sigma = 0.2) )</th>
<th>( u (\sigma = 0.3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0480952</td>
<td>0.0480952</td>
<td>0.0480952</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0865447</td>
<td>0.087336</td>
<td>0.0879926</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0935465</td>
<td>0.0943632</td>
<td>0.0949575</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0953861</td>
<td>0.0959181</td>
<td>0.0962768</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0962338</td>
<td>0.0964989</td>
<td>0.0966662</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0977769</td>
<td>0.0968396</td>
<td>0.0968663</td>
</tr>
<tr>
<td>0.6</td>
<td>0.0971536</td>
<td>0.0970709</td>
<td>0.0969958</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0974108</td>
<td>0.0972274</td>
<td>0.0970803</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0975767</td>
<td>0.0973268</td>
<td>0.0971307</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0976723</td>
<td>0.0973818</td>
<td>0.0971544</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0540641</td>
<td>0.0516861</td>
<td>0.0501136</td>
</tr>
<tr>
<td>1.1</td>
<td>0.0524527</td>
<td>0.0507189</td>
<td>0.0495511</td>
</tr>
<tr>
<td>1.2</td>
<td>0.0511814</td>
<td>0.0499472</td>
<td>0.0490989</td>
</tr>
<tr>
<td>1.3</td>
<td>0.0501852</td>
<td>0.0493355</td>
<td>0.0487380</td>
</tr>
<tr>
<td>1.4</td>
<td>0.0494108</td>
<td>0.0488546</td>
<td>0.0484525</td>
</tr>
<tr>
<td>1.5</td>
<td>0.0488148</td>
<td>0.0484803</td>
<td>0.0482290</td>
</tr>
<tr>
<td>1.6</td>
<td>0.0483617</td>
<td>0.0481925</td>
<td>0.0480563</td>
</tr>
<tr>
<td>1.7</td>
<td>0.0480226</td>
<td>0.0479746</td>
<td>0.0479251</td>
</tr>
<tr>
<td>1.8</td>
<td>0.0477741</td>
<td>0.0478130</td>
<td>0.0478276</td>
</tr>
<tr>
<td>1.9</td>
<td>0.0479971</td>
<td>0.0476965</td>
<td>0.0477573</td>
</tr>
<tr>
<td>2.0</td>
<td>0.0474762</td>
<td>0.0476159</td>
<td>0.0477090</td>
</tr>
</tbody>
</table>

**Discussion:**

For numerical discussion it is assumed \( A = 1, P = 1 \) and \( a = \sqrt{3} \). In order to study the effects of magnetic field, permeability of the porous medium, elastic parameter and time on velocity profiles graphs and tables are made for different cases. Three particular cases, considered here, are interesting. In first case, it interesting to note that the velocity does not depend on the elastic parameter \( \sigma \) when \( t \to \infty \). Figures 1, 2 and 3 show that the velocity distribution is symmetrical about the plane \( x = 0 \) and the velocity of the fluid decreases with the increase of \( G \). As \( G = M^2 + \frac{1}{K} \), we can conclude that for fixed \( K \), the velocity decreases with the increase of \( M \) i.e. the magnetic field impedes the flow and for fixed \( M \), the velocity of the fluid increases with the increase of \( K \), the permeability of the porous medium. Figures 1, 2 and 3 show that the effects of the magnetic field and the porous medium are pronounced in the central region of the channel. Figures 1, 2 and 3 also show that near the wall of the
tube the velocity increases rapidly with $x$ and at the central region the velocity slightly decreases. From fig. 3 we see that the velocity decreases notably after ceasing the arbitrary part of the pressure gradient at time $t = T$. Figures 1, 2 and 3 show that the velocity profile type has similarity in all the considered three cases. Tables 1, 2 and 3 show that as $\sigma$ the elastic parameter increases the velocity increases up to a certain value of $t$ whereafter the opposite happens. From the tables and graphs we can draw the conclusion that the magnetic field initially reduces the effect of elastic parameter. Table 1 describes that in first case, as $t$ increases the velocity increases. Table 2 shows that the flow is periodic w. r. t. time in case (II). Table 3 reveals that for $0 < t < T$ and $t \geq T$ the velocity increases and decreases respectively with the increase of time $t$ in case (III).

<table>
<thead>
<tr>
<th>$y$</th>
<th>$\sigma$</th>
<th>$\beta$</th>
<th>$t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.1</td>
<td>0.5</td>
<td>0.1</td>
</tr>
</tbody>
</table>

![Graph](image)

**Fig. 1 (Case I): Velocity profiles for various values of $G$ with fixed $y$, $\sigma$, $\beta$ and $t$.**
Fig. 2 (Case II): Velocity profiles for various values of $G$ with fixed $y$, $\sigma$, $\omega$ and $t$. 
Fig. 3 (Case III): Velocity profiles for various values of $G$ with fixed $y$, $\sigma$ and $T$. 

<table>
<thead>
<tr>
<th>$y$</th>
<th>$\sigma$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.1</td>
<td>0.3</td>
</tr>
</tbody>
</table>