CHAPTER – I

PROBLEMS ON HYDROMAGNETIC FLOW OF A SECOND ORDER RIVLIN-ERICKSEN FLUID
CHAPTER – I

Introduction:

This chapter contains three problems concerned with second order (Rivlin-Ericksen) fluid. The first paper considers MHD flow of Rivlin-Ericksen fluid through porous medium between two parallel plates one of which i.e. upper plate is moving and the lower plate is being at rest. The flow is generated due to a time dependent motion of the upper plate and a time dependent pressure gradient. In the second paper of this chapter analytical and numerical results are reported for the unsteady hydromagnetic flow of a dusty visco-elastic (Rivlin-Ericksen) fluid through porous medium between two oscillating plates with given frequencies. The third paper deals with the flow of a conducting dusty Rivlin-Ericksen fluid through porous medium over a rigid horizontal surface in presence of transverse uniform magnetic field. The motion considered has been set up by tangential stress of different types applied at the free surface.

Basic Equations:

The second order approximation of the general constitutive equation given by Rivlin-Ericksen can be written in the following (Coleman and Noll [19] and Noll [61]) form:

\[ \tau_{ij} = -P \delta_{ij} + \mu A^{(1)}_{ij} + \mu_0 A^{(2)}_{ij} + \mu_c A^{(1)}_{ik} A^{(1)}_{kj} \]  

where \( \tau_{ij} \) is the stress tensor, \( P \) is an intermediate pressure and \( \mu > 0, \mu_0 > 0, \mu_c > 0 \) are co-efficients of viscosity, visco-elasticity and cross-viscosity respectively and are in general functions of temperature and material properties. The rate-of-strain tensor \( A^{(1)} \) and the acceleration tensor \( A^{(2)} \) are defined by

\[ A^{(1)}_{ij} = u_{i,j} + u_{j,i} \]  

\[ A^{(2)}_{ij} = a_{i,j} + a_{j,i} + 2u_{m,i}u_{m,j} \]

where \( a_i \)'s are acceleration components given by

\[ a_i = \frac{\partial u_i}{\partial t} + u_j u_{i,j} \]

The equation of continuity for fluid is

\[ u_{i,i} = 0 \]
We assume that the number of density $N$ of the dust particles is constant i.e. $N = N_0$ (Constant). Then the equation of continuity of the dust particles is

$$v_{i,i} = 0 \quad (VI)$$

The motion of a dusty fluid are governed by the coupled equations (Saffman [72]):

$$\rho \left( \frac{\partial u_i}{\partial t} + u_{i,j} u_j \right) = \tau_{ij,j} + K_0 N_0 (v_i - u_i) + (\vec{J} \times \vec{B}) \frac{\mu}{K} u_i \quad (VII)$$

and

$$m_0 \left( \frac{\partial v_i}{\partial t} + v_{i,j} v_j \right) = K_0 (u_i - v_i) \quad (VIII)$$

where $\vec{B} = \mu_0 \vec{H}_e$ is the magnetic induction vector, $\vec{J} = \sigma [\vec{E} + (\vec{u} \times \vec{B})]$ is the density of electric current and $(\vec{J} \times \vec{B})$ is the Lorentz force.

For most of the conducting fluids the magnetic Reynolds number $R_m$ is much less than unity so that the induced magnetic field can be neglected in comparison with the applied magnetic field (Shercliff [81] and Gupta [39]). Since no external electric field is applied, the effect of polarisation of fluid is neglected so that we can take $\vec{E} = (0,0,0)$.

When $N_0 \to 0$, the equation (VII) reduces to

$$\rho \left( \frac{\partial u_i}{\partial t} + u_{i,j} u_j \right) = \tau_{ij,j} + (\vec{J} \times \vec{B}) \frac{\mu}{K} u_i \quad (IX)$$

which is the equation of motion for dust free fluid.
**NOMENCLATURE**

\[ \tau_{ij} = \text{total stress tensor} \]

\[ P = \text{pressure} \]

\[ \mu = \text{co-efficient of viscosity} \]

\[ \mu_0 = \text{co-efficient of visco-elasticity} \]

\[ \mu_c = \text{co-efficient of cross-viscosity} \]

\[ A^{(1)} = \text{rate of strain tensor} \]

\[ A^{(2)} = \text{acceleration tensor} \]

\[ a_i = \text{acceleration component} \]

\[ u_i = \text{velocity component of the fluid} \]

\[ v_i = \text{velocity component of the dust particles} \]

\[ \vec{E} = \text{electric field intensity} \]

\[ \mu_e = \text{magnetic permeability} \]

\[ \sigma = \text{electrical conductivity} \]

\[ \rho = \text{fluid density} \]

\[ \vec{u} = \text{velocity of the conducting medium} \]

\[ N_0 = \text{number of density of the dust particles} \]

\[ m_0 = \text{mass of a dust particle} \]

\[ K = \text{permeability of the porous medium} \]

\[ \vec{H}_e = \text{magnetic field intensity} \]

\[ \vec{B} = \text{magnetic induction vector} \]

\[ K_0 = \text{Stoke's resistance co-efficient which for a spherical particle of radius } \tau \text{ is } 6\pi \mu r \]

\[ \vec{J} = \text{electric current density} \]
1.1 : A NOTE ON GENERALISED PLANE COUETTE FLOW OF VISCOELASTIC FLUID THROUGH POROUS MEDIUM

Introduction:

In physical and biological phenomena it is seen that fluid flows through porous media under different boundary conditions. In many cases these fluids are of visco-elastic character. If such visco-elastic fluid are electrically conducting then these realistic problems are interesting and draw attention of the engineers, geologist and scientists working with animal body problems. Considering their importance, these specific type of problems have been considered here for investigation. The steady flow of a viscous incompressible fluid between two parallel flat plates with constant pressure gradient is well known [Pai [65]]. The unsteady motion of a viscous incompressible fluid due to periodic pressure gradient in different geometries has been considered by Lal and Jhori [52], Gopalan and Raghavan [38] etc. Sen and Roy [75] have considered the unsteady motion of Rivlin-Ericksen fluid between two inclined parallel plates in presence of a uniform magnetic field assuming (i) the upper plate moves with a transient velocity while the lower one remains fixed and (ii) the upper plate performs a steady longitudinal oscillation and the lower one remains fixed. Dube [26] has investigated the unsteady flow of a visco-elastic incompressible fluid in a channel bounded by two parallel flat plates with time varying pressure gradient using Laplace transform technique. The author has considered two cases : firstly, when the pressure gradient is varying linearly with time and, secondly when it is decreasing exponentially with time.

Alfvén [4], Cowling [21], Ferraro and Plumpton [29] and Jeffrey [45] gave a brief introduction to Geo-physical, Astrophysical applications of magnetohydrodynamics and to elementary plasma Physics. The plane MHD Couette flow has been discussed by Pai [66] for an incompressible fluid between non-conducting walls. The generalised plane Couette flow with uniform suction (injection at the stationary plate) has been considered by Bansal [8]. Sengupta and Ray [77] have discussed the

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flow of a conducting \( n \)-immiscible visco-elastic incompressible fluids between two inclined parallel plates under a uniform transverse magnetic field with time varying pressure gradient. Considering two particular types of pressure gradient viz. the transient pressure gradient and periodic pressure gradient. The explicit expression for velocity distribution as well as interface velocity have been derived. Particular case for two fluids has also been derived. Singh and Sharma [85] have discussed unsteady MHD flow between two parallel flat plates with time varying pressure gradient while the upper plate is moving and the lower plate is being at rest. However, the monograph of Kapur, Bhatt and Sachetty [46] have been consulted. Also some related papers has been consulted.

The study of flow through porous media is of fundamental importance in the study of oil and gas, migration of underground water and filtration process etc. Flow through porous medium basically depends upon Darcy’s Law which is as follows:

\[
\mathbf{\dot{v}}_p = -\frac{\mu}{K} \mathbf{\nabla} \cdot \mathbf{v}.
\]

After conducting many experiments Brinkman [15] generalised the Darcy’s law to study the flow through highly porous media which is given by

\[
\mathbf{\dot{v}}_p = -\frac{\mu}{K} \mathbf{\nabla} \cdot \mathbf{v} + \mu \mathbf{\nabla}^2 \mathbf{v}.
\]

Lahiri and Ganguli [51] have investigated the unsteady flow of two immiscible visco-elastic Maxwell conducting fluids between two inclined parallel plates under a uniform magnetic field. Two particular cases have been considered here firstly when the upper plate is given a transient velocity, while the lower one is kept fixed and secondly the upper plate performs a steady longitudinal oscillation, while the lower one remains fixed.

The aim of the present paper is to investigate the unsteady flow of an electrically conducting, incompressible visco-elastic fluid through porous medium between two nonconducting infinite parallel plates with time varying pressure gradient when the upper plate is moving and the lower plate is being at rest. The velocity distribution and expression for shear stress at the stationary plate have been obtained.
by using Laplace transform technique for three different cases viz. the pressure gradient and the upper plate velocity are (i) exponential function of time of the form $A(1 - e^{-mt})$, $m > 0$, $A > 0$, (ii) impulsive type and (iii) constant acting for a finite time. The effect of magnetic field, permeability of the porous medium, elastic parameter and time on the velocity distribution and that on the shear stress at the lower plate have been studied with the help of graphs and tables.

**Formulation of the Problem:**

We consider the motion of a visco-elastic (Rivlin-Ericksen) fluid through porous medium between two infinite parallel plates with time varying pressure gradient when the upper plate is moving and the lower one is being at rest. The continuity and the momentum equations obtained from (V) and (IX) by using (I)–(IV) for two dimensional unsteady visco-elastic incompressible MHD flow, after neglecting the effect of induced magnetic fields when the x-axis is taken along the stationary plate and y-axis is measured normal to it, are

\[
\frac{\partial u}{\partial x} = 0 \tag{1}
\]

\[
\frac{\partial u}{\partial t} - \frac{1}{\rho} \frac{\partial P}{\partial x} + \left( \alpha + \beta \frac{\partial}{\partial x} \right) \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2}{\rho} u - \alpha u
\]

\[
0 = -\frac{1}{\rho} \frac{\partial p}{\partial y} \tag{2}
\]

where $\alpha = \frac{\nu}{\mu}$ is the Kinematic co-efficient of viscosity of the fluid, $\beta = \frac{\mu_p}{\rho}$ is the Kinematic visco-elasticity, $B_0$ is the magnetic induction vector along y-axis.

Introducing the non-dimensional quantities

\[
x' = \frac{x}{h}, \quad y' = \frac{y}{h}, \quad u' = \frac{uh}{\alpha}, \quad t' = \frac{t\alpha}{h^2}, \quad P' = \frac{Ph^2}{\alpha^2 \rho}, \quad K' = \frac{K}{h^2},
\]

(h being the distance between the plates),

in equations (1), (2) and (3), we have (omitting dashes),

\[
\frac{\partial u}{\partial x} = 0 \Rightarrow u = u(y, t) \tag{4}
\]
\[
\frac{\partial u}{\partial t} = -\frac{\partial P}{\partial x} + (1 + E \frac{\partial}{\partial t}) \frac{\partial^2 u}{\partial y^2} - H^2 u \quad (5)
\]
and \[
\frac{\partial P}{\partial y} = 0 \quad (6)
\]
where \( E = \frac{\beta}{h^2} \) (elastic parameter), \( M^2 = B_0 h \sqrt{\frac{\sigma}{\rho \alpha}} \) (Hartmann number), \( H^2 = M^2 + \frac{1}{K} \).

Equations (4), (5) and (6) suggest that \( \frac{\partial P}{\partial x} \) must be a constant or a function of time only, hence we assume
\[
-\frac{\partial P}{\partial x} = p(t) \quad (7)
\]

The boundary and initial conditions of the problem, in non-dimensional form, are
\[
\begin{align*}
&u = 0 \quad \text{at} \quad y = 0 \\
u &= U(t) \quad \text{at} \quad y = 1
\end{align*} \quad (8)
\]
and
\[
u(y, 0) = \frac{p(0)}{H^2} \left[ 1 - \frac{\cosh H(y - \frac{1}{2})}{\cosh H^2} \right] + U(0) \frac{\sinh H y}{\sinh H}, \quad (9)
\]
using the steady-state solution of equation (5).

**Method of Solution**: Applying Laplace transform
\[
\tilde{u} = \int_0^\infty e^{-st} dt,
\]
where \( \text{Re}(s) > 0 \), to the equation (5), we get with the help of (9)
\[
\frac{d^2 \tilde{u}}{dy^2} \left( \frac{H^2 + s}{1 + sE} \right) \tilde{u} = \frac{p(0)(1 - EH^2) \cosh H(y - \frac{1}{2})}{(1 + sE)H^2 \cosh H^2} \cdot \frac{U(0)(1 - EH^2)}{(1 + sE)} \cdot \frac{\sinh H y}{\sinh H}
\]
\[
-\frac{1}{(1 + sE)} \left[ \tilde{p}(s) + \frac{p(0)}{H^2} \right] \quad (10)
\]
Applying Laplace transform to the equation (8) we get

\[
\begin{align*}
\bar{u} &= 0 \quad \text{at} \quad y = 0 \\
\bar{u} &= \bar{U}(s) \quad \text{at} \quad y = 1
\end{align*}
\] (11)

The solution of the differential equation (10) with the help of (11) is

\[
\tilde{u} = \frac{U(0) \sinh H y}{s \sinh H} - \frac{p(0) \cosh H (y - \frac{1}{2})}{sH^2 \cosh H^2} + \frac{1}{(H^2 + s)} \left[ \frac{p(0)}{s} - p(s) \right] \frac{\cosh m (y - \frac{1}{2})}{\cosh m^2}
\]

\[
+ \left[ \frac{\bar{U}(s) - \frac{U(0)}{s}}{\sinh m} \right] \frac{\sinh m y}{\sinh m} + \frac{1}{(H^2 + s)} \left[ \frac{p(0)}{H^2} + p(s) \right]
\] (12)

where \( m^2 = \frac{H^2 + s}{1 + sE} \)

Taking inverse transform of (12) and using convolution theorem, we have

\[
\begin{align*}
\tilde{u}(y, t) &= \frac{U(t) \sinh H y}{\sinh H} + \frac{p(t)}{H^2} \left[ 1 - \frac{\cosh H (y - \frac{1}{2})}{\cosh H^2} \right] - \frac{4}{\pi} \frac{1}{\lambda} \sum_{n=1,3,5,\ldots} \sin n \pi y \frac{e^{-s_1 (1 - \lambda)}}{r(r^2 \pi^2 + H^2)} d\lambda \\
+ 2\pi \frac{1}{s_1} \int_0^1 \frac{\bar{U}(\lambda) \sum_{n=0}^{\alpha_n} (-1)^n \frac{n(1 - EH^2) \sin n \pi y}{(1 + En^2 \pi^2)(H^2 + n^2 \pi^2)} e^{-s_1 (1 - \lambda)}}{1 + n^2 \pi^2} d\lambda
\end{align*}
\] (13)

where \( s_1 = \frac{i^2 \pi^2 + H^2}{1 + Ei^2 \pi^2} \)

When \( E \to 0 \) and \( K \to \infty \), the expression (13) agrees to the result of Singh and Sharma [85].

**Shear Stress**: The equation of shear stress is

\[
\tau = \rho (\alpha + \beta \frac{\partial}{\partial t}) \frac{\partial u}{\partial y}
\] (14)

Introducing the non-dimensional quantities

\[
y' = \frac{y}{h}, \quad u' = \frac{uh}{\alpha}, \quad t' = \frac{t\alpha}{h^2}, \quad \text{and} \quad \tau' = \frac{\tau h^2}{\alpha^2 \rho}
\]

in equation (14), we get (omitting dashes)

\[
\tau' = (1 + E \frac{\partial}{\partial t}) \frac{\partial u}{\partial y}
\] (15)

where \( E \) (elastic parameter) = \( \frac{\beta}{h^2} \), as defined earlier.
PARTICULAR CASES

Case (1). The motion of the upper plate and the pressure gradient are of the form

\[ A(1 - e^{-mt}), \quad (m > 0,\ A > 0) : \]

In this case, we take

\[ U(t) = u_0 (1 - e^{-\lambda_1 t}), \quad \lambda_1 > 0 \]

and \[ p(t) = p_0 (1 - e^{-\lambda_2 t}), \quad \lambda_2 > 0 \]

Putting the values of \( U(t) \) and \( p(t) \) in the equation (13), one can derive

\[
u = \frac{u_0 (1 - e^{-\lambda_1 t}) \sinh H y}{\sinh H} + \frac{p_0 (1 - e^{-\lambda_2 t})}{H^2} \left[ \frac{\cosh H(y - 1/2)}{\cosh H^2} \right]
\]

\[ + 2\pi u_0 \lambda_1 (1 - H^2 E) \sum_{n=0}^{\infty} \frac{(-1)^n n\sin n\pi y (e^{-\lambda_1 t} - e^{-\delta_n t})}{n^2 \pi^2 + H^2} \frac{(1 - \lambda_1 E)n^2 \pi^2 + (H^2 - \lambda_1)}{(1 - \lambda_1 E)n^2 \pi^2 + (H^2 - \lambda_1)} \]

\[ - \frac{4p_0 \lambda_2}{\pi} \sum_{r=1,3,5,...} \frac{(1 + Er^2 \pi^2) \sin r\pi y (e^{\lambda_2 t} - e^{-\delta_n t})}{r^2 \pi^2 + H^2} \frac{(1 - \lambda_2 E)r^2 \pi^2 + (H^2 - \lambda_2)}{(1 - \lambda_2 E)r^2 \pi^2 + (H^2 - \lambda_2)} \]

(1.1)

In steady case i.e. when \( t \to \infty \) in (1.1), one may have

\[
u = \frac{u_0 \sinh H y}{\sinh H} + \frac{p_0}{H^2} \left[ \frac{\cosh H(y - 1/2)}{\cosh H^2} \right] \]

(1.2)

From the equation (1.2) it is clear that in steady case the velocity does not depend on the elastic parameter \( E \).

Putting \( u_0 \to 0 \) in the equation (1.1), one can get the flow between two fixed plates due to pressure gradient. Then

\[
u = \frac{p_0 (1 - e^{-\lambda_2 t})}{H^2} \left[ \frac{\cosh H(y - 1/2)}{\cosh H^2} \right]
\]

\[ - \frac{4p_0 \lambda_2}{\pi} \sum_{r=1,3,5,...} \frac{(1 + Er^2 \pi^2) \sin r\pi y (e^{-\lambda_2 t} - e^{-\delta_n t})}{r^2 \pi^2 + H^2} \frac{(1 - \lambda_2 E)r^2 \pi^2 + (H^2 - \lambda_2)}{(1 - \lambda_2 E)r^2 \pi^2 + (H^2 - \lambda_2)} \]

(1.3)

Putting \( p_0 \to 0 \) in the equation (1.1), one may have the flow due to the motion of upper plate. Then

\[
u = \frac{u_0 (1 - e^{-\lambda_1 t}) \sinh H y}{\sinh H} \quad + 2\pi u_0 \lambda_1 (1 - H^2 E) \sum_{n=0}^{\infty} \frac{(-1)^n n\sin n\pi y (e^{-\lambda_1 t} - e^{-\delta_n t})}{n^2 \pi^2 + H^2} \frac{(1 - \lambda_1 E)n^2 \pi^2 + (H^2 - \lambda_1)}{(1 - \lambda_1 E)n^2 \pi^2 + (H^2 - \lambda_1)} \]

(1.4)
Expressions (1.3) and (1.4) describe respectively plane Poiseuille flow and Couette flow.

Shear Stress: The dimensionless shearing stress at the lower plate is given by

\[ \tau = \left(1 + E \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial y} \right) \right)_{y=0} \]

With the help of equation (1.1) one can easily derive

\[ \tau = \frac{u_0 H}{\sinh H} \left[ 1 + (E \lambda_1 - 1) e^{-\frac{\pi^2}{H^2} t} \right] + \frac{p_0}{H} \left[ 1 + (E \lambda_2 - 1) e^{-\frac{\lambda_2}{H^2} t} \right] \tanh \frac{H}{2} \]

\[ + 2\pi^2 u_0 \lambda_1 (1 - E H^2) \sum_{n=0}^{\infty} \frac{(-1)^n n^2 [((1 - E \lambda_1)(1 + E \pi^2) e^{-\frac{\pi^2}{H^2} t} - (1 - E H^2) e^{-\lambda_2 t}]}{(n^2 \pi^2 + H^2)(1 + E \pi^2)} \left[ (H^2 - \lambda_1) + (1 - \lambda_1 E) n^2 \pi^2 \right] \]

\[ - 4p_0 \lambda_2 \sum_{r=1,3,5,\ldots} \frac{[((1 - E \lambda_2)(1 + E r^2 \pi^2)) e^{-\frac{r^2 \pi^2}{H^2} t} - (1 - E H^2) e^{-\lambda_2 t}]}{(r^2 \pi^2 + H^2)(1 + E r^2 \pi^2)} \left[ (H^2 - \lambda_2^2) + (1 - \lambda_2 E) r^2 \pi^2 \right] \] (1.5)

Putting \( u_0 \to 0 \) and \( p_0 \to 0 \) separately in the equation (1.5), one may have the shear stress at the lower plate for Poiseuille flow and Couette flow respectively.

Similarly putting \( E \to 0 \) in (1.5), one can get the shear stress at the lower plate for ordinary viscous fluid.

Case (II). The pressure gradient and the motion of the upper plate are impulsive type:

In this case,

\[ U(t) = U_0 \delta(t) \] \hspace{1cm} (2.1)

and \( p(t) = p_0 \delta(t) \) \hspace{1cm} (2.2)

where \( \delta(t) \) is the Dirac delta function defined by

\[ \delta(t) = \begin{cases} 0, & t \leq 0 \\ \sqrt{\frac{1}{\varepsilon}}, & 0 < t < \varepsilon \\ 0, & t \geq \varepsilon \end{cases} \] \hspace{1cm} (2.3)

and no matter how small \( \varepsilon (>0) \) is.
Then from the equations (12), (2.1) and (2.2), $\bar{u}$ has been calculated as

$$
\bar{u} = u_0 \sinh m y + \frac{p_0}{H^2 + s} \left[ 1 - \frac{\cosh(y - \frac{1}{2})}{\cosh m \frac{1}{2}} \right] (2.4)
$$

With the help of inverse Laplace transform from equation (2.4)

$$
u = 2u_0 \pi (1 - E H^2) \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \sinh n \pi y e^{-s_n t}}{(1 + n E \pi^2)^2} + \frac{4p_0}{\pi} \sum_{r=1.3, 5, \ldots} \frac{\sin \pi y e^{-s_r t}}{r(1 + E r^2 \pi^2)} (2.5)
$$

Putting $u_0 \rightarrow 0$, i.e. for plane Poiseuille flow equation (2.5) becomes

$$
u = 4p_0 \sum_{r=1.3, 5, \ldots} \frac{\sin \pi y e^{-s_r t}}{r(1 + E r^2 \pi^2)} (2.6)
$$

When $p_0 \rightarrow 0$ the expression for velocity from equation (2.5) is

$$
u = 2u_0 \pi (1 - E H^2) \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \sinh n \pi y e^{-s_n t}}{(1 + n E \pi^2)^2} (2.7)
$$

which gives Couette flow.

When $t \rightarrow \infty$ from equation (2.5), (2.6) and (2.7), one can show that the velocity of the corresponding flow dies out.

Similarly when $E \rightarrow 0$ from equation (2.5), (2.6) and (2.7), one can get the corresponding flow for ordinary viscous fluid.

Also it is clear from the equation (2.5) that when $H^2 \rightarrow \frac{1}{E}$, the motion of the upper plate has no contribution to the fluid flow. From equation (2.7), one can conclude that the velocity of Couette flow in this case vanishes.

**Shear Stress**: The expression for shear stress $\tau$ at the lower plate is given by

$$
\tau = \left( 1 + E \frac{\partial}{\partial t} \right) \left( \frac{\partial u}{\partial y} \right)_{y=0} (2.8)
$$

It is then obtained as

$$
\tau = (1 - E H^2)[2\pi^2 u_0 (1 - E H^2) \sum_{n=0}^{\infty} \frac{(-1)^{n+1} n^2 e^{-s_n t}}{(1 + n E \pi^2)^3} + 4p_0 \sum_{r=1.3, 5, \ldots} \frac{e^{-s_r t}}{(1 + E r^2 \pi^2)^2}] (2.9)
$$
When \( u_0 \to 0 \) in (28), we get the shear stress for plane Poiseuille flow as

\[
\tau = 4p_0(1 - EH^2) \sum_{r=1,3.5,\ldots} \frac{e^{-s_t^2}}{(1 + Er^{-2} + \frac{2}{\pi^2})^2}
\]  

(2.10)

From (2.9), we see that as \( EH^2 \) increases the shear stress at the lower plate decreases. So we can conclude that for a fixed value of the permeability of the porous medium, the shear stress decreases with the increase of the magnetic field and for a fixed value of the magnetic field, the shear stress decreases with the decrease of the permeability of the porous medium. The most interesting case occurs when \( EH^2 \to 1 \); in this case the shear stress on the lower plate becomes negligibly small. But it may be noted that since the elastic parameter \( E \) is small, \( H^2 \) has to be moderately large. This can happen when the Hartman number \( M \) is large in which case, the approximation may not be valid. We also see from (2.9) that \( \tau \) decreases as \( t \) increases.

When \( p_0 \to 0 \) in (2.9), we get the shear stress for Couette flow. When \( E \to 0 \) in (2.9), we have

\[
\tau = 2\pi^2 u_0 \sum_{n=0}^{\infty} (-1)^{n+1} n^2 e^{-(n^2\pi^2 + H^2)t} + 4p_0 \sum_{r=1,3.5,\ldots} e^{-(n^2\pi^2 - H^2)t}
\]  

(2.11)

This is the expression for shear stress at the lower plate for ordinary viscous fluid.

Case (III). Flow under a discontinuous pressure gradient and a discontinuous motion of the plate:

In this circumstance, we have

\[
U(t) = u_0[H(t) - H(t - T)]
\]

and

\[
p(t) = p_0[H(t) - H(t - T)]
\]

where \( H(t) \) is the Heaviside unit function defined by

\[
H(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1 & \text{for } t > 0 \end{cases}
\]

Here the motion of the upper plate and the pressure gradient are acting for a finite time \( T \).

Now from (12), we have

\[
\bar{u} = \left[ \frac{u_0 \sinh \mu y}{\sinh \mu} + \frac{p_0}{(H^2 + s)} \left\{ 1 - \frac{\cosh \mu(y - \frac{1}{2})}{\cosh \mu \frac{1}{2}} \right\} \left( 1 - e^{-sT} \right) \right] \frac{1}{s}
\]  

(3.1)
By inverse Laplace transform, we get from (3.1)

\[
\frac{u}{\sinh H} = \frac{u_0 \sinh Hy}{\sinh H} + \frac{p_0}{H^2} \left[ 1 - \frac{\cosh H(y - \frac{1}{2})}{\cosh H^2} \right] - \frac{4p_0}{\pi} \sum_{r=1,3,5,\ldots} \frac{\sin r\pi y e^{-s_r t}}{r(H^2 + r^2 \pi^2)} + 2\pi u_0 (1 - EH^2) \sum_{n=0}^{\infty} \frac{(-1)^n \sin n\pi y e^{-s_n t}}{(1 + E n^2 \pi^2)(H^2 + n^2 \pi^2)}
\]

when \(0 < t \leq T\) \hspace{1cm} (3.2)

\[
= 2\pi u_0 (1 - EH^2) \sum_{n=0}^{\infty} \frac{(-1)^n \sin n\pi y (1 - e^{s_n T}) e^{-s_n t}}{(1 + E n^2 \pi^2)(H^2 + n^2 \pi^2)} - \frac{4p_0}{\pi} \sum_{r=1,3,5,\ldots} \frac{\sin r\pi y (1 - e^{s_r T}) e^{-s_r t}}{r(H^2 + r^2 \pi^2)}
\]

when \(t > T\) \hspace{1cm} (3.3)

We note that if \(U(t) = u_0\) and \(P(t) = p_0\) continue indefinitely, the velocity as \(t \to \infty\), asymptotically attains the steady state and the fluid velocity assumes the form

\[
\frac{u}{\sinh H} = \frac{u_0 \sinh Hy}{\sinh H} + \frac{p_0}{H^2} \left[ 1 - \frac{\cosh H(y - \frac{1}{2})}{\cosh H^2} \right] - \frac{4p_0}{\pi} \sum_{r=1,3,5,\ldots} \frac{\sin r\pi y e^{-s_r t}}{r(H^2 + r^2 \pi^2)}
\]

This velocity is independent of the elastic parameter \(E\). The result (3.4) agrees with the result (1.2).

When \(u_0 \to 0\) in (3.2) and (3.3), we get respectively the corresponding plane Poiseuille flow given by

\[
\frac{u}{\sinh H} = \frac{p_0}{H^2} \left[ 1 - \frac{\cosh H(y - \frac{1}{2})}{\cosh H^2} \right] - \frac{4p_0}{\pi} \sum_{r=1,3,5,\ldots} \frac{\sin r\pi y e^{-s_r t}}{r(H^2 + r^2 \pi^2)} \quad \text{when} \quad 0 < t \leq T
\]

(3.5)

\[
- \frac{4p_0}{\pi} \sum_{r=1,3,5,\ldots} \frac{\sin r\pi y (1 - e^{s_r T}) e^{-s_r t}}{r(H^2 + r^2 \pi^2)} \quad \text{when} \quad t > T
\]

(3.6)

When \(p_0 \to 0\) in (3.2) and (3.3), we get respectively

\[
u = \frac{u_0 \sinh Hy}{\sinh H} + 2\pi u_0 (1 - EH^2) \sum_{n=0}^{\infty} \frac{(-1)^n \sin n\pi y e^{-s_n t}}{(1 + E n^2 \pi^2)(H^2 + n^2 \pi^2)}
\]

\[
= 2\pi u_0 (1 - EH^2) \sum_{n=0}^{\infty} \frac{(-1)^n \sin n\pi y (1 - e^{s_n T}) e^{-s_n t}}{(1 + E n^2 \pi^2)(H^2 + n^2 \pi^2)}
\]

\[\text{when} \quad 0 < t \leq T\]

(3.7)

\[\text{when} \quad t > T\]

(3.8)

which are the corresponding Couette flow.

If we make \(u_0 \to \infty, \quad p_0 \to \infty\) and \(T \to 0\) in (3.3) in such a manner that \(u_0/\sinh H = u_0'\) and \(p_0 T = p_0'\), we have

\[
u = 2\pi u_0' (1 - EH^2) \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \sin n\pi y e^{-s_n t}}{(H^2 + n^2 \pi^2)^2} + \frac{4p_0'}{\pi} \sum_{r=1,3,5,\ldots} \frac{\sin r\pi y e^{-s_r t}}{r(1 + E n^2 \pi^2)}
\]

(3.9)
Expression (3.9) agrees with (2.5). So expression (3.9) represents the velocity of the fluid due to simultaneous effect of impulsive pressure gradient $p'_{0}\delta(t)$ and impulsive motion $u'_{0}\delta(t)$ of the upper plate.

**Discontinuity in the velocity**: We observe that when the pressure gradient and the motion of the upper plate are ceased at $t = T$, there is a jump discontinuity of amount

$$J_u = \frac{u_0 \sinh H y}{\sinh H} + \frac{p_0}{H^2} \left[ 1 - \frac{\cosh H (y - \frac{1}{2})}{\cosh \frac{H}{2}} \right] + 2\pi u_0 (1 - EH^2) \sum_{n=0}^{\infty} \frac{(-1)^n \sin \pi y}{n \pi} \sum_{r=1,3,\ldots} \frac{\sin \pi y}{r (H^2 + r^2 \pi^2)}$$

in the velocity.

The amount of discontinuity in case of plane Poiseuille flow and Couette flow are obtained respectively by putting $u_0 \to 0$ and $p_0 \to 0$ in (3.10). It is interesting to note that the amount of discontinuity in case of Poiseuille flow (i.e. when $u_0 \to 0$) does not depend on the visco-elastic parameter $E$. It is evident from (3.10) that the amount of discontinuity $J_u$ is independent of time $T$.

**Table - 3.1**

**Numerical values of $J_u$ for different values of $E$ and $y$**

<table>
<thead>
<tr>
<th>$y$</th>
<th>$E = 0.1$</th>
<th>$E = 0.2$</th>
<th>$E = 0.3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0000000</td>
<td>0.0000000</td>
<td>0.0000000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0409041</td>
<td>0.0731302</td>
<td>0.0910922</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0859320</td>
<td>0.1499320</td>
<td>0.1852290</td>
</tr>
<tr>
<td>0.3</td>
<td>0.1396250</td>
<td>0.2342620</td>
<td>0.2855580</td>
</tr>
<tr>
<td>0.4</td>
<td>0.2073940</td>
<td>0.3303540</td>
<td>0.3954310</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2960780</td>
<td>0.4430330</td>
<td>0.5185220</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4146110</td>
<td>0.5779550</td>
<td>0.6589460</td>
</tr>
<tr>
<td>0.7</td>
<td>0.5749650</td>
<td>0.7418960</td>
<td>0.8213950</td>
</tr>
<tr>
<td>0.8</td>
<td>0.7932890</td>
<td>0.9430870</td>
<td>1.0113000</td>
</tr>
<tr>
<td>0.9</td>
<td>1.0916000</td>
<td>1.1916300</td>
<td>1.2350100</td>
</tr>
<tr>
<td>1.0</td>
<td>1.5000000</td>
<td>1.5000000</td>
<td>1.5000000</td>
</tr>
</tbody>
</table>
Table – 3.2

Numerical values of $J_u$ for different values of $H$ and $y$

when $E = 0.1$, $u_0 = 1.5$ and $p_0 = 1.5$.

<table>
<thead>
<tr>
<th>$y$</th>
<th>$H = 3$</th>
<th>$H = 4$</th>
<th>$H = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.000000</td>
<td>0.000000</td>
<td>0.000000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0409041</td>
<td>0.0409031</td>
<td>0.0409030</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0859320</td>
<td>0.0859302</td>
<td>0.0859301</td>
</tr>
<tr>
<td>0.3</td>
<td>0.1396250</td>
<td>0.1396230</td>
<td>0.1396229</td>
</tr>
<tr>
<td>0.4</td>
<td>0.2073940</td>
<td>0.2073940</td>
<td>0.2073940</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2960780</td>
<td>0.2960780</td>
<td>0.2960780</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4146110</td>
<td>0.4146180</td>
<td>0.4146181</td>
</tr>
<tr>
<td>0.7</td>
<td>0.5749650</td>
<td>0.5749652</td>
<td>0.5749660</td>
</tr>
<tr>
<td>0.8</td>
<td>0.7932890</td>
<td>0.7932900</td>
<td>0.7932910</td>
</tr>
<tr>
<td>0.9</td>
<td>1.0916000</td>
<td>1.0916100</td>
<td>1.0916101</td>
</tr>
<tr>
<td>1.0</td>
<td>1.5000000</td>
<td>1.5000000</td>
<td>1.5000000</td>
</tr>
</tbody>
</table>

From table 3.1 and table 3.2 it is seen that as $y$ increases the amount of jump discontinuity increase. It can be concluded from table 3.1 that the amount of discontinuity increases with the increase of the elastic parameter $E$. Table 3.2 reveals that as $H$ increases $J_u$ decreases. This happens upto a certain value of $y$ whereafter the opposite happens. So we can conclude that for fixed $K$ the permeability of the porous medium $J_u$ decreases with the increase of the intensity of the magnetic field and for fixed $M$ the magnetic parameter $J_u$ decreases with the decrease of $K$. This takes place upto a certain value of $y$ whereafter the opposite occurs.

Shear Stress:

The expression for shear stress at the lower plate is given by

$$
\tau = \left(1 + E \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial y} \right) \right)_{y=0}.
$$

It is then obtained as

$$
\tau = \frac{u_0 H}{\sinh H} + \frac{p_0 \tanh H}{H} \frac{H^2}{2} + 2\pi^2 u_0 (1 - EH^2)^2 \sum_{n=0}^{\infty} \frac{(-1)^n n^2 e^{-s_{n,1}}}{(1 + En^2\pi^2)^2 (H^2 + n^2\pi^2)}
- 4p_0 (1 - EH^2) \sum_{r=1}^{3} \frac{e^{-s_{n,1}}}{(1 + Er^2\pi^2)(H^2 + r^2\pi^2)}
$$

when $0 < t \leq T$ (3.11)
\[ z = 2\pi^2 u_0 (1 - EH^2)^2 \sum_{n=0}^{\infty} \frac{(-1)^n n^2 (1 - e^{-s_n T}) e^{-s_n t}}{(1 + En^2 \pi^2)^2 (H^2 + n^2 \pi^2)} \]

\[ -4p_0 (1 - EH^2) \sum_{r=1,3,5,\ldots} \frac{(1 - e^{-s_{rT}}) e^{-s_{rt}}}{(1 + Er^2 \pi^2)(H^2 + r^2 \pi^2)} \quad \text{when } t > T \]  

(3.12)

When \( u_0 \to 0 \) in (3.11) and (3.12), we get the shear stress at lower plate for plane Poiseuille flow within the prescribed interval of time. By putting \( p_0 \to 0 \) in (3.11) and (3.12), we can get the shear stress at the lower plate for Couette flow within the prescribed interval of time.

Putting \( p_0 \to \infty \), \( u_0 \to \infty \) and \( T \to 0 \) in (3.12) in such a manner that \( p_0 T = p_0' \) and \( u_0 T = u_0' \), we get

\[ T = (1 - EH^2) \left[ 2\pi^2 u_0' (1 - EH^2)^2 \sum_{n=0}^{\infty} \frac{(-1)^n n^2 e^{-s_n t}}{(1 + En^2 \pi^2)^3} + 4p_0' \sum_{r=1,3,5,\ldots} \frac{e^{-s_{rT}}}{(1 + Er^2 \pi^2)^2} \right] \]  

(3.13)

which is in full agreement with (2.9). So (3.13) represents the shear stress at the lower plate when the pressure gradient and the motion of the upper plate are impulsive type.

**Discontinuity in the shear stress:**

When the pressure gradient and the motion of the upper plate are ceased, at time \( t = T \), there is a jump discontinuity in the shear stress of amount

\[ J_t = \frac{u_0 H}{\sinh H} + \frac{p_0 \tanh(H^2)}{H^2} + 2\pi^2 u_0 (1 - EH^2)^2 \sum_{n=0}^{\infty} \frac{(-1)^n n^2}{(1 + En^2 \pi^2)(H^2 + n^2 \pi^2)} \]

\[ -4p_0 (1 - EH^2) \sum_{r=1,3,5,\ldots} \frac{1}{(1 + Er^2 \pi^2)(H^2 + r^2 \pi^2)} \]  

(3.14)

It is interesting to note that this amount of discontinuity is independent of time \( T \).

The amount of discontinuity in case of plane Poiseuille flow and Couette flow are respectively obtained by putting \( u_0 \to 0 \) and \( p_0 \to 0 \) in (3.11).

**Table 3.3**

**Numerical values of \( J_t \) for different values of \( E \) when \( M = 2, K = 0.2, u_0 = 1.5 \) and \( p_0 = 1.5 \).**

<table>
<thead>
<tr>
<th>( E )</th>
<th>( 0.1 )</th>
<th>( 0.2 )</th>
<th>( 0.3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J_t )</td>
<td>0.8803</td>
<td>0.7696</td>
<td>0.2644</td>
</tr>
</tbody>
</table>
Table 3.4

Numerical values of $J_\tau$ for different values of $H$
when $E = 0.1$, $u_0 = 1.5$ and $p_0 = 1.5$.

<table>
<thead>
<tr>
<th>$H$</th>
<th>2.5</th>
<th>4.0</th>
<th>5.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_\tau$</td>
<td>0.9740</td>
<td>0.5462</td>
<td>0.0850</td>
</tr>
</tbody>
</table>

Table 3.3 describes that the amount of jump discontinuity decreases with the increases of $E$. From table 3.4 it is observed that as $H$ increases the amount of jump discontinuity decreases. As $H^2 = M^2 + \frac{1}{\kappa}$ we can conclude that for fixed $\kappa$ the amount of discontinuity decreases with the increase of $M$ and for fixed $M$ the amount the amount is discontinuity decreases with the decrease of $\kappa$.

Table 3.5

Numerical values of $\tau$ for different values of $t$ and $H$ when $E = 0.1$, $T = 1.0$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\tau(H=2.0)$</th>
<th>$\tau(H=2.5)$</th>
<th>$\tau(H=3.0)$</th>
<th>$\tau(H=4.5)$</th>
<th>$\tau(H=5.0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>1.12013</td>
<td>1.01187</td>
<td>0.88364</td>
<td>0.40394</td>
<td>0.22194</td>
</tr>
<tr>
<td>0.12</td>
<td>1.26743</td>
<td>1.08057</td>
<td>0.89539</td>
<td>0.46707</td>
<td>0.38218</td>
</tr>
<tr>
<td>0.23</td>
<td>1.33688</td>
<td>1.10892</td>
<td>0.89953</td>
<td>0.47553</td>
<td>0.39731</td>
</tr>
<tr>
<td>0.34</td>
<td>1.36954</td>
<td>1.12060</td>
<td>0.90098</td>
<td>0.47623</td>
<td>0.39764</td>
</tr>
<tr>
<td>0.45</td>
<td>1.38488</td>
<td>1.12541</td>
<td>0.90149</td>
<td>0.47612</td>
<td>0.39727</td>
</tr>
<tr>
<td>0.56</td>
<td>1.39206</td>
<td>1.12740</td>
<td>0.90167</td>
<td>0.47604</td>
<td>0.39712</td>
</tr>
<tr>
<td>0.67</td>
<td>1.39542</td>
<td>1.12821</td>
<td>0.90174</td>
<td>0.47601</td>
<td>0.39707</td>
</tr>
<tr>
<td>0.78</td>
<td>1.39699</td>
<td>1.12855</td>
<td>0.90176</td>
<td>0.47600</td>
<td>0.39706</td>
</tr>
<tr>
<td>0.89</td>
<td>1.39772</td>
<td>1.12869</td>
<td>0.90177</td>
<td>0.47600</td>
<td>0.39706</td>
</tr>
<tr>
<td>0.98</td>
<td>1.39802</td>
<td>1.12874</td>
<td>0.90177</td>
<td>0.47600</td>
<td>0.39706</td>
</tr>
<tr>
<td>1.0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>1.09</td>
<td>0.16069</td>
<td>0.06137</td>
<td>0.00848</td>
<td>0.01632</td>
<td>0.03089</td>
</tr>
<tr>
<td>1.11</td>
<td>0.14009</td>
<td>0.05224</td>
<td>0.00701</td>
<td>0.01096</td>
<td>0.01913</td>
</tr>
<tr>
<td>1.12</td>
<td>0.06579</td>
<td>0.02153</td>
<td>0.00247</td>
<td>0.00069</td>
<td>0.00072</td>
</tr>
</tbody>
</table>
Table 3.6
Numerical values of $\tau$ for different values of $t$ and $E$ when $M = 2$, $K = 0.2$, $T = 1.0$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\tau(E=0.06)$</th>
<th>$\tau(E=0.1)$</th>
<th>$\tau(E=0.2)$</th>
<th>$\tau(E=0.3)$</th>
<th>$\tau(E=0.4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.729541</td>
<td>0.883636</td>
<td>0.893652</td>
<td>0.794376</td>
<td>0.697373</td>
</tr>
<tr>
<td>0.12</td>
<td>0.853603</td>
<td>0.895394</td>
<td>0.898669</td>
<td>0.840740</td>
<td>0.771003</td>
</tr>
<tr>
<td>0.23</td>
<td>0.888378</td>
<td>0.899529</td>
<td>0.900743</td>
<td>0.867321</td>
<td>0.818422</td>
</tr>
<tr>
<td>0.34</td>
<td>0.898065</td>
<td>0.900983</td>
<td>0.901543</td>
<td>0.882485</td>
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</tr>
<tr>
<td>0.45</td>
<td>0.900750</td>
<td>0.901494</td>
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<td>0.891085</td>
<td>0.868381</td>
</tr>
<tr>
<td>0.56</td>
<td>0.901491</td>
<td>0.901674</td>
<td>0.901838</td>
<td>0.895928</td>
<td>0.880824</td>
</tr>
<tr>
<td>0.67</td>
<td>0.901694</td>
<td>0.901737</td>
<td>0.901841</td>
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<td>0.888730</td>
</tr>
<tr>
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<td>0.901847</td>
<td>0.900122</td>
<td>0.893728</td>
</tr>
<tr>
<td>0.89</td>
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<td>0.901767</td>
<td>0.901853</td>
<td>0.900933</td>
<td>0.896868</td>
</tr>
<tr>
<td>0.98</td>
<td>0.901769</td>
<td>0.901769</td>
<td>0.901855</td>
<td>0.901307</td>
<td>0.898537</td>
</tr>
<tr>
<td>1.0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>1.09</td>
<td>0.068223</td>
<td>0.008447</td>
<td>0.004190</td>
<td>0.071035</td>
<td>0.145849</td>
</tr>
<tr>
<td>1.11</td>
<td>0.054096</td>
<td>0.007012</td>
<td>0.003421</td>
<td>0.064086</td>
<td>0.134475</td>
</tr>
<tr>
<td>1.12</td>
<td>0.015049</td>
<td>0.002466</td>
<td>0.001160</td>
<td>0.036233</td>
<td>0.085857</td>
</tr>
<tr>
<td>1.33</td>
<td>0.004166</td>
<td>0.000867</td>
<td>0.000279</td>
<td>0.020325</td>
<td>0.054592</td>
</tr>
</tbody>
</table>

Discussion and Conclusion:

Velocity distribution derived in equation (13) agrees well with equation of Singh and Sharma [85] when elastic parameter tends to zero and permeability of the porous medium tends to infinity. But for obtaining this equation here Laplace transform technique is used. Similarly shear stress equation is obtained in better way. Three particular cases, considered here, are interesting. In first case, velocity does not depend on the elastic parameter when $t \to \infty$. When $u_0 \to 0$ flow between two fixed plates due to pressure gradient and for $p_0 \to 0$ flow due to the motion of upper plate calculated, are plane Poiseuille flow and Couette flow respectively. For $u_0 \to 0$ and $p_0 \to 0$ shear stress at the lower plate for Poiseuille flow and Couette flow can be easily calculated from the derived equation. Similarly if elastic parameter ($E$) tends to zero then the shear stress at the lower plate for ordinary viscous fluid can be
calculated. In second case it is revealed that when the pressure gradient and the motion of the upper plate are of impulsive type the results are interesting. In third case some interesting results are obtained. One may observe that the velocity and the shear stress at the lower plate are discontinuous at time \( t = T \). It is shown that the results of 2nd case may be obtained from 3rd case. The velocity and the shear stress at the lower plate for constant pressure gradient and constant velocity of the upper plate may be obtained from 3rd case by making \( T \to \infty \). For numerical results, it is assumed \( u_0 = p_0 = 1.5 \). In order to study the effects of magnetic field, permeability of the porous medium, elastic parameter and time on velocity profiles and shear stress at the lower plates graphs and tables are appended for different cases.

**Case (I).** From fig. 1.1 it is seen that as \( y \) increases the velocity increases. It is also seen from fig. 1.1 that the magnetic field is not in favour of the flow of the fluid i.e. the fluid velocity decreases with the increase of the intensity of the magnetic field. Fig. 1.2 shows that as \( t \) increases the velocity increases and it ultimately attains to a steady state which is given by equation (17). Fig. 1.2 also shows that as \( K \) increases the velocity increases. From fig. 1.3 it is evident that the velocity field increases with the increasing value of the elastic parameter \( E \). From fig. 1.1, 1.2 and 1.3 we can draw the conclusion that the magnetic field reduces the effect of the elastic parameter and the permeability of the porous medium on the velocity profiles.

Fig. 1.4 shows that the shear stress at the lower plate increases with the increasing value of \( t \) the time and as \( t \to \infty \) an ultimate steady state is set up. Fig. 1.4 also shows that the shear stress decreases with the increase of the magnetic field. Fig. 1.5 reveals that the shear stress increases with the increase of \( K \) the permeability of the porous medium. It is evident from fig. 1.6 that as \( E \) the elastic parameter increases the shear stress increases. From Fig. 1.4, 1.5 and 1.6 one may have the conclusion that the magnetic field reduces the effect of the elastic parameter and the permeability of the porous medium on the shear stress at the lower plate.

**Case (II).** From fig. 2.1 it seen that as \( M \) increases the velocity field decreases. It has been found from fig. 2.1 that for fixed \( K \) and \( E \) there exists a critical value of \( M \) for
which the flow field is divided into two regions having distinct flow characteristics. For critical value of $M$ the direction of the flow is positive near the lower plate and its sense is reverse in the remaining region. If the value of $M$ above the critical value, the flow reverses in direction. From fig. 2.2 it is observed that the effect of the elastic parameter is just like the magnetic parameter. From table 2.1, we see that there exists a critical value of $K$. If the value of $K$ above the critical value, the shear stress increases with the increases of $K$ the permeability of the porous medium. If the value of $K$ below the critical value of its own, the shear stress decreases with the decrease of $K$ for a small interval of time from the beginning. And thereafter the shear stress increases with the decrease of $K$. From table 2.1, we see that the shear stress decreases with the increase of $t$ the time.

As $H^2 = M^2 + \frac{1}{K}$, we can conclude from fig. 2.3 that:

(i) For fixed $K$ and $E$, as $M$ increases the shear stress decreases. This happens up to a certain value called critical value of $M$ thereafter the opposite takes place for a certain interval of time from the beginning. (ii) For fixed $M$ and $E$, the shear stress decreases with the decreasing of $K$ to a certain extent. Thereafter the opposite occurs for a certain interval of time from the beginning. Fig. 2.4 narrates that as $E$ increases the shear stress decreases. This happens up to a certain value of $E$ thereafter the opposite happens i.e. the shear stress increases with the increase of $E$, the elastic parameter. From fig. 2.3 and fig. 2.4 it is seen that as $t$ increases the shear stress decreases rapidly and ultimately it falls to zero asymptotically.

Case (III). From figs. 3.1, 3.3, 3.1a and 3.3a we see that the flow pattern is like as in case (I) and as in case (II) for $t < T$ and for $t > T$ respectively. The effect of $M$ and that of $E$ on the velocity profile is like as in case (I) when $t < T$ and that as in case (II) when $t > T$. From fig. 3.2 we see that for fixed $M$ and $E$, there exists some critical value of $K$ for which the velocity is constant form the beginning when $t < T$. If the value of $K$ is above the critical value, the velocity initially increases and then reaches to a constant. When $t < T$. If $K$ is below the critical value, the velocity initially decreases and then reaches to a constant with the increase of $t$ and it ultimately falls
to zero asymptotically. For $t > T$ the direction of velocity is negative when $K$ takes value below the value critical one of its own. From fig. 3.2 we also see that as $K$ decreases the velocity decreases when $t < T$. When $K$ assumes higher than the critical value, the velocity decreases with the decrease of $K$ for $t > T$. For $t > T$, if $K$ assumes lower than the critical value, the velocity decreases with the decrease of $K$. And this happens upto a certain value of $t$ whereafter the opposite occurs i.e. the velocity increases with the decrease of $K$.

As $H^2 = M^2 + \sqrt{K}$, we may conclude from table 3.5 that –

(a) For fixed $K$ and $E$: The shear stress decreases with the increase of $M$, the Hartmann number, when $t < T$. The shear stress increases with the increase of the time when $t < T$. This occurs upto a certain value called critical value of $M$. When the value of $M$ above the critical value of its own, the shear stress initially increases and thereafter slightly decreases when $t < T$. For $t > T$, the shear stress decreases with the increase value of $M$ and this happens upto the critical value of $M$. When the value of $M$ above the critical value of $M$, the shear stress increases with the increase of $M$ when $t > T$. The shear stress decreases as $t$ increases when $t > T$.

(b) For fixed $M$ and $E$: The shear stress decreases with the decrease of $K$, the permeability of the porous medium, when $t < T$. The shear stress increases with the increase of the time when $t < T$. This occurs upto a certain value called critical value of $K$. When the value of $K$ below the critical value of its own, the shear stress initially increases and thereafter slightly decreases when $t < T$. For $t > T$, the shear stress decreases with the decrease of $K$ and this happens upto the critical value of $K$. When the value of $K$ below the critical value of $K$, the shear stress increases with the decreases of $K$ when $t > T$.

From table 3.6 we see that as $E$ the elastic parameter increases the shear stress increases when $t < T$. This happens upto a certain value called critical value of $E$. When the value of $E$ above the critical value of its own, the shear stress decreases with the increases of $E$ when $t < T$. For $t > T$, the shear stress decreases with the increase of $E$ and this happens upto the critical value of $E$. When the value of $E$ above the critical value of its own, the shear stress increases with the increase of $E$. 
Fig. 1.1 (Case I): Velocity Profiles for various values of $M$ with fixed $K$, $E$, $t$, $\lambda_1$, and $\lambda_2$.

Fig. 2.1 (Case II): Velocity Profiles for various values of $M$ with fixed $K$, $E$, and $t$.

Fig. 3.1 (Case III for $t < T$): Velocity Profiles for various values of $M$ with fixed $K$, $E$, $t$, and $T$.

Fig. 3.1a (Case III for $t > T$): Velocity Profiles for various values of $M$ with fixed $K$, $E$, $t$, and $T$. 
Fig. 3.2 (Case III): Velocity Profiles for various values of $K$ with fixed $M$, $E$, $T$ and $\gamma$.

Fig. 1.2 (Case I): Velocity Profiles for various values of $K$ with fixed $M$, $E$, $\gamma$, $\lambda_1$ and $\lambda_2$. 
Fig. 1.3 (Case I): Velocity Profiles for various values of $E$ with fixed $K$, $M$, $t$, $\lambda_1$ and $\lambda_2$.

Fig. 2.2 (Case II): Velocity Profiles for various values of $E$ with fixed $K$, $M$ and $t$.

Fig. 3.3 (Case III for $t < T$): Velocity Profiles for various values of $E$ with fixed $K$, $M$, $t$ and $T$.

Fig. 3.3a (Case III for $t > T$): Velocity Profiles for various values of $E$ with fixed $K$, $M$, $t$ and $T$. 

<table>
<thead>
<tr>
<th>$K$</th>
<th>$M$</th>
<th>$t$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
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<td>1</td>
<td>0.1</td>
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<table>
<thead>
<tr>
<th>$K$</th>
<th>$M$</th>
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</table>
Fig. 2.3 (Case II) : Shear stress profiles for various values of $H$ with fixed $K$ and $E$

Fig. 2.4 (Case II) : Shear stress profiles for various values of $E$ with fixed $K$ and $M$.

Fig. 1.6. (Case I) : Shear stress profiles for various values of $E$ with fixed $K$, $M$, $\lambda_1$ and $\lambda_2$. 
Table 2.1 (Case II)
Numerical values of $u$ for different values of $t$ and $K$ when $M=1.5$, $E=0.1$ and $y=0.5$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$u$ (K=0.57143)</th>
<th>$u$ (K=0.25)</th>
<th>$u$ (K=0.14815)</th>
<th>$u$ (K=0.07272)</th>
<th>$u$ (K=0.05555)</th>
<th>$u$ (K=0.04396)</th>
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<td>0.01</td>
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<td>1582560</td>
<td>1021530</td>
<td>-0.334040</td>
<td>-1.107480</td>
<td>-1.929680</td>
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<tr>
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<td>0.656253</td>
<td>0.360367</td>
<td>-0.074160</td>
<td>-0.191442</td>
<td>-0.245010</td>
</tr>
<tr>
<td>0.23</td>
<td>0.457729</td>
<td>0.271567</td>
<td>0.127162</td>
<td>-0.016010</td>
<td>-0.031026</td>
<td>-0.025990</td>
</tr>
<tr>
<td>0.34</td>
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<td>-0.003290</td>
<td>-0.004370</td>
<td>-0.001160</td>
</tr>
<tr>
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<td>0.003234</td>
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<td>1.22</td>
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<td>0.000092</td>
<td>0.000011</td>
<td>0.000000</td>
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</table>

Fig. 1.4 (Case I) : Shear stress profiles for various values of $M$ with fixed $K$, $E$, $\lambda_1$ and $\lambda_2$.

Fig. 1.5 (Case I) : Shear stress profiles for various values of $K$ with fixed $M$, $E$, $\lambda_1$ and $\lambda_2$. 
1.2 : MHD FLOW OF A DUSTY VISCO-ELASTIC CONDUCTING FLUID THROUGH POROUS MEDIUM BETWEEN TWO OSCILLATING PLATES

Introduction:

Interest in problems of flow of dusty fluid have increased in recent years. Model equations describing the motion of such fluid has been given by Saffman [72]. A. K. Ghosh and L. Debnath [31] have studied hydromagnetic Stokes flow in rotating fluid with suspended small particles. P. Mitra and P. Bhattacharyya [56] have studied hydromagnetic flow of a dusty fluid between two parallel plates, one being stationary and the other oscillating. P. Mitra and P. Bhattacharyya [57] have studied unsteady hydromagnetic laminar flow of a conducting dusty fluid between two parallel plates started impulsively from rest. M. M. Stanisic, B. H. Fetz, H. P. Mickelsen, Jr. and F. M. Czumak [88] have investigated the flow of a hydromagnetic fluid between two oscillating flat plates. H. T. Yang and J. V. Healy [99] have investigated the Stokes problem for a conducting fluid with a suspension of particles. Recently L. Debnath and A. K. Ghosh [24] have studied unsteady hydromagnetic flows of a dusty fluid between two oscillating plates with different frequencies.

In the present paper we consider the unsteady hydromagnetic flow of a dusty visco-elastic (Rivlin-Ericksen) fluid through porous medium between two oscillating plates with different given frequencies.

Mathematical formulation of the problem:

We take the rectangular Cartesian co-ordinates (x, y, z) so that x and z-axes lie on the plate y = 0 and y-axis normal to it. Consider the motion of a dusty visco-elastic fluid through porous medium between two oscillating plates whose equations are y = 0 and y = h, where ‘h’ is the distance between the plates. The components of velocities of the fluid and the dust particles are given by

\[ [u(y, t), 0, 0] \text{ and } [v(y, t), 0, 0]. \]

* Published in Indian Journal of Theoretical Physics, Vol. 47, No. 2, p. 113 (1999).
And the components of magnetic induction vector $\mathbf{B}$ are $[0, B_0, 0]$.

After neglecting the induced magnetic field in absence of pressure gradient, we have from equation (VII) and (VIII) by using (I) - (IV):

$$\frac{\partial u}{\partial t} = \left( \alpha + \beta \frac{\partial}{\partial t} \right) \frac{\partial^2 u}{\partial y^2} + \frac{K_0 N_0}{\rho} (v - u) - \frac{\alpha}{K} u - \frac{\sigma B_0^2}{\rho} u \tag{1}$$

and $m_0 \frac{\partial v}{\partial t} = K_0 (u - v) \tag{2}$

Where $\alpha = \frac{\mu}{\rho}$ is the Kinematic co-efficient of viscosity of the fluid and $\beta = \frac{\mu_0}{\rho}$ is the Kinematic visco-elasticity.

Introducing the non-dimensional variables:

$$y' = \frac{y}{h}, \quad t' = \frac{\alpha t}{h^2}, \quad u' = \frac{uh}{\alpha}, \quad v' = \frac{vh}{\alpha}, \quad t'' = \frac{m_0 N_0}{\rho} \text{ (mass concentration)}, \quad (2a)$$

$$K' = \frac{K}{h^2}, \quad M = B_0 h \sqrt{\frac{\sigma}{\alpha \rho}} \text{ (Hartmann number)}, \quad E = \frac{\beta}{h^2} \text{ (elastic parameter)}$$

in equations (1) and (2), we have (omitting dashes)

$$\frac{\partial u}{\partial t} = \left( 1 + E \frac{\partial}{\partial t} \right) \frac{\partial^2 u}{\partial y^2} + \lambda (v - u) - Hu \tag{3}$$

and $\frac{\partial v}{\partial t} = L(u - v) \tag{4}$

where $\lambda = \frac{\mu_0}{\rho}, \quad H = M^2 + \frac{1}{K}, \quad L = \frac{K_0 h^2}{m_0 \alpha} \text{ (reciprocal of time relaxation of dust)}$.

The boundary conditions (no-slip) of the problem are

$$u = a e^{i \omega_1 t} + b e^{-i \omega_1 t} \text{ at } y = 0, \ t > 0 \tag{5}$$

$$u = c e^{i \omega_2 t} + d e^{-i \omega_2 t} \text{ at } y = h, \ t > 0 \tag{6}$$

where $a, b, c, d$ are complex constants, such that $u$ becomes real on the plates.

The initial conditions of the problem are

$$u = v = 0 \text{ at } t \leq 0 \text{ for all } y \text{ in } (0, h). \tag{7}$$
Introducing the non-dimensional variables given by equation (2a) and parameters

\[(\sigma_1, \sigma_2) = \frac{h^2}{\alpha} \left(\omega_1, \omega_2\right), \quad (a', b', c', d') = \frac{h}{\alpha} \left(a, b, c, d\right)\]

to the boundary and initial conditions (5), (6) and (7), we have (omitting dashes)

\[u = ae^{i\sigma_1 t} + be^{-i\sigma_1 t}\] \(\text{on } y = 0, \ t > 0\) \hspace{1cm} (8)

\[u = ce^{i\sigma_2 t} + de^{-i\sigma_2 t}\] \(\text{on } y = 1, \ t > 0\) \hspace{1cm} (9)

where \(a, b, c, d\) are complex constants, such that \(u\) becomes real on the plates.

and \(u = v = 0\) at \(t \leq 0\) for all \(y\) in \((0, 1)\). \hspace{1cm} (10)

Applying Laplace transform to equations (3) and (4) with the initial conditions (10), we obtain

\[\frac{d^2 \tilde{u}}{dy^2} - m^2 (s) \tilde{u} = 0\] \hspace{1cm} (11)

where

\[m^2 (s) = \frac{\frac{s^2}{2} + (H + L + \lambda)s + HL}{(s + L)(1 + sE)}\] \hspace{1cm} (12)

and \(s\) is Laplace transform variable,

and \(\tilde{v} = \frac{Lu}{s + L}\) \hspace{1cm} (13)

Applying the same Laplace transform to the boundary condition (8) and (9), we get

\[\tilde{u} = \frac{a}{s - i\sigma_1} + \frac{b}{s + i\sigma_1}\] \(\text{on } y = 0, \ t > 0\) \hspace{1cm} (14)

\[\tilde{u} = \frac{c}{s - i\sigma_2} + \frac{d}{s + i\sigma_2}\] \(\text{on } y = 1, \ t > 0\) \hspace{1cm} (15)

The solution of (11) with the help of (14) and (15) is

\[\tilde{u} = \left(\frac{a}{s - i\sigma_1} + \frac{b}{s + i\sigma_1}\right) \frac{\sinh m(1-y)}{\sinh m} + \left(\frac{c}{s - i\sigma_2} + \frac{d}{s + i\sigma_2}\right) \frac{\sinh my}{\sinh m}\] \hspace{1cm} (16)

By applying Inverse Laplace transform to (16) we get
\[
\begin{align*}
 u &= a e^{i \gamma_1 t} \frac{\sinh \{ (\mu_1 + i \mu_2) (1 - y) \}}{\sinh(\mu_1 + i \mu_2)} + b e^{-i \gamma_1 t} \frac{\sinh \{ (\mu_1 - i \mu_2) (1 - y) \}}{\sinh(\mu_1 - i \mu_2)} \\
 &\quad + c e^{i \gamma_2 t} \frac{\sinh \{ (\mu_3 + i \mu_4) y \}}{\sinh(\mu_3 + i \mu_4)} + d e^{-i \gamma_2 t} \frac{\sinh \{ (\mu_3 - i \mu_4) y \}}{\sinh(\mu_3 - i \mu_4)} \\
 &= -2 \pi \sum_{n=0}^{\infty} (-1)^n \sin n\pi (1 - y) [\Delta_n^{(1)} e^{-s_n^{(1)} t} + \Delta_n^{(2)} e^{-s_n^{(2)} t}] \\
 &= -2 \pi \sum_{n=0}^{\infty} (-1)^n \sin n\pi y [\Delta_n^{(3)} e^{-s_n^{(3)} t} + \Delta_n^{(4)} e^{-s_n^{(4)} t}]
\end{align*}
\]

where
\[
\begin{align*}
 A_j &= \frac{[(HL - \sigma_j^2)(L - E\sigma_j^2) + \sigma_j^2 (H + L + \lambda)(1 + LE)]}{(L - E\sigma_j^2)^2 + (1 + LE)^2 \sigma_j^2}, \quad (j = 1, 2) \\
 B_j &= \frac{\sigma_j [(H + L + \lambda)(L - E\sigma_j^2) - (HL - \sigma_j^2)(1 + LE)]}{(L - E\sigma_j^2)^2 + (1 + LE)^2 \sigma_j^2}, \quad (j = 1, 2)
\end{align*}
\]

\[
\begin{align*}
 (\mu_1, \mu_2) &= \left( \frac{\sqrt{A_1^2 + B_1^2} \pm \Delta_1}{2} \right)^2, \quad (\mu_3, \mu_4) &= \left( \frac{\sqrt{A_2^2 + B_2^2} \pm \Delta_2}{2} \right)^2
\end{align*}
\]

\[
\Delta_n^{(j)} = \begin{bmatrix}
a - b \\ -s_n^{(j)} - i \sigma_j \\
\end{bmatrix} \begin{bmatrix}
\{E(s_n^{(j)} - 1)(1 + LE)s_n^{(j)} + L\}^2 \\
(1 + \lambda E - HE)s_n^{(j)} - 2L(1 - HE)s_n^{(j)} + L(L + \lambda - HEL)
\end{bmatrix}
\]

\[
\begin{align*}
 (s_n^{(1)}, s_n^{(2)}) &= -\frac{[H + L + \lambda + (1 + LE)n^2 \pi^2]}{2(1 + n^2 \pi^2 \lambda)} \\
 &\quad \pm \sqrt{[H + L + \lambda + (1 + LE)n^2 \pi^2]^2 - 4(1 + n^2 \pi^2 \lambda)(HL + n^2 \pi^2 L)}
\end{align*}
\]

\(\Delta_3^{(3)}\) and \(\Delta_4^{(4)}\) will be obtained respectively from \(\Delta_n^{(1)}\) and \(\Delta_n^{(2)}\) by replacing \(a\) and \(b\) by \(c\) and \(d\).

Applying convolution for Laplace transform in (13), we get
\[
\begin{align*}
 v &= L \left\{ a \sinh \{ (\mu_1 + i \mu_2) (1 - y) \} \left( e^{i \gamma_1 t} - e^{-\gamma_1 t} \right) + b \sinh \{ (\mu_1 - i \mu_2) (1 - y) \} \left( e^{-i \gamma_1 t} - e^{i \gamma_1 t} \right) \right\}
\end{align*}
\]
The results (17) and (23) describe respectively the fluid and the particle velocities. Many particular results can be obtained from these results. If $K \rightarrow \infty$, we shall get a dusty visco-elastic flow in presence of uniform magnetic field between two oscillating plates with different frequencies. When $K \rightarrow \infty$, $E \rightarrow 0$, $a = b = 0$, $c = d = \frac{c}{2}$ and $\omega_2 = \omega$ the results obtained from (17) and (23) agree with the results of P. Mitra and P. Bhattacharyya [56]. If $K \rightarrow \infty$, $E \rightarrow 0$, and $\omega_1 = \omega_2 = 0$, the result corresponding to [57] will follow. However if $K \rightarrow \infty$, $E \rightarrow 0$, $(a, b) = \left(\frac{u_0}{2i}, \frac{-u_0}{2i}\right)$ and $(c, d) = \left(\frac{u_1}{2i}, \frac{-u_1}{2i}\right)$ are substituted in (17) and (23) with $\sigma_2 = \sigma_1 = \sigma$, the results provide an extension of the work [88] for the two phase fluid particle system. Moreover when $K \rightarrow \infty$ and $E \rightarrow 0$, (17) and (23) reduce to those given in [24].

Test of convergence of the series (17) and (23):

We can express $-s_n^{(1)}$ and $-s_n^{(2)}$ as

$$-s_n^{(1)} = -L + 0 \left(\frac{1}{n^2}\right) \quad \text{and} \quad -s_n^{(2)} = -\frac{1}{E} + 0 \left(\frac{1}{n^2}\right)$$

Now $\Delta_n^{(1)} = 0 \left(\frac{1}{n^4}\right)$, $\Delta_n^{(2)} = 0 \left(\frac{1}{n^4}\right)$, $\Delta_n^{(3)} = 0 \left(\frac{1}{n^4}\right)$ and $\Delta_n^{(4)} = 0 \left(\frac{1}{n^4}\right)$.

Consequently, the series in (17) and (23) converge for $t \geq 0$. 

$$+ \frac{c \sinh \{(\mu_3 + i\mu_4)y\}}{(L + i\sigma_2) \sinh(\mu_3 + i\mu_4)} (e^{i\sigma_2 t} - e^{-1t}) + \frac{d \sinh \{(\mu_3 - i\mu_4)y\}}{(L - i\sigma_2) \sinh(\mu_3 - i\mu_4)} (e^{-i\sigma_2 t} - e^{-1t})$$

$$- 2\pi \sum_{n=0}^{\infty} (-1)^n \sin n\pi (1 - y) \left[ \frac{e^{-s_n^{(1)}t} - e^{-1L t}}{-s_n^{(1)} + L} \Delta_n^{(1)} + \frac{e^{-s_n^{(2)}t} - e^{-1L t}}{-s_n^{(2)} + L} \Delta_n^{(2)} \right]$$

$$- 2\pi \sum_{n=0}^{\infty} (-1)^n \sin n\pi y \left[ \frac{e^{-s_n^{(1)}t} - e^{-1L t}}{-s_n^{(1)} + L} \Delta_n^{(3)} + \frac{e^{-s_n^{(2)}t} - e^{-1L t}}{-s_n^{(2)} + L} \Delta_n^{(4)} \right]$$

(23)
The steady state solutions:

Proceeding to the limit $t \to \infty$ in (17) we find that the transient effects die out for all the frequencies $\sigma_1$, $\sigma_2$ and the ultimate steady state is attained and the fluid velocity assumes the form

$$u = ae^{i\sigma_1 t} \frac{\sinh\{(\mu_1 + i\mu_2)(1-y)\}}{\sinh(\mu_1 + i\mu_2)} + be^{-i\sigma_1 t} \frac{\sinh\{(\mu_1 - i\mu_2)(1-y)\}}{\sinh(\mu_1 - i\mu_2)}$$

$$+ ce^{i\sigma_2 t} \frac{\sinh\{(\mu_3 + i\mu_4)\}y}{\sinh(\mu_3 + i\mu_4)} + de^{-i\sigma_2 t} \frac{\sinh\{(\mu_3 - i\mu_4)y\}}{\sinh(\mu_3 - i\mu_4)}$$

(24)

The result (24) represents Stokes-Hartmann boundary layers in a Rivlin Ericksen fluid particle system combination on the plates. The thickness of these layers $\delta_1$ on $y = 0$ and $\delta_2$ on $y = 1$ in non-dimensional form are of the order

$$\delta_j = \sqrt{\frac{2}{\left(\sqrt{A_j^2 + B_j^2 + C_j}\right)^2}} \approx \frac{\sqrt{2}(L_j^2 + \sigma_j^2)(1 + E_j^2\sigma_j^2)}{\sqrt{[\left[H + \sigma_j^2\right]^2 + \sigma_j^2(H + L + \lambda)^2]}(L_j^2 + \sigma_j^2)(1 + E_j^2\sigma_j^2)}$$

$$+ (H + E\sigma_j^2)(L_j^2 + \sigma_j^2) + \lambda\sigma_j^2(1 + L\sigma_j^2)}^{1/2}$$

(25)

Table - 1

<table>
<thead>
<tr>
<th>E</th>
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<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
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<td>0.3232574</td>
<td>0.3299962</td>
<td>0.338410</td>
</tr>
</tbody>
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With the help of the table (1) we can conclude that the thickness of the boundary layer on the plate $y = 0$ increases with the increase of the elastic parameter $E$. We observe the same result on the plate $y = 1$. It is easily seen from equation (25) that the thickness of the layers decrease with the increase in concentration of the dust particle and the magnetic field. But the thickness of the layers increase with the increase in the permeability of the porous medium. Consequently, the attainment of the steady state is much quicker in an electrically conducting dusty elastico-viscous fluid in porous medium compared to the corresponding dust free hydromagnetic situation in porous medium. The thickness of the layers formed on the respective plates becomes identical when $\sigma_1 = \sigma_2 = \sigma$ (oscillating with same frequency). When $K \to \infty$ and $E \to 0$, the
The thickness of the boundary layers are same with that of [24]. The steady state is slower in porous medium compared to the corresponding non-porous medium as the presence of porous medium increase the thickness of the boundary layers. In the limit \( h \to \infty \), the steady state solution (omitting dashes) for the non-dimensional visco-elastic fluid velocity corresponding to the single oscillating plate is given by

\[
\begin{align*}
    u &= a \exp \left[ it - (\alpha_1 + i\alpha_2)y \right] + b \exp \left[ -it - (\alpha_1 - i\alpha_2)y \right] \\
    A' &= \left[ \sqrt{A'^2 + B'^2 + A'} \right]^2, \\
    A' &= \frac{[(HL-1)(L-E) + (H+L+\lambda)(1+LE)]}{(L-E)^2 + (1+LE)^2}, \\
    B' &= \frac{[(H+L+\lambda)(1-E) - (HL-1)(1+LE)]}{(L-E)^2 + (1+LE)^2}, \\
    E &= \frac{\beta \omega_1}{\alpha}, \\
    \lambda &= \frac{\beta_0}{\alpha}, \\
    f &= \frac{m_0 N_0}{\rho}, \\
    L &= \frac{K_0}{m_0 \omega_1}, \\
    M &= B_0 \sqrt{\frac{\sigma}{\rho \omega_1}}, \\
    K' &= \frac{K \omega_1}{\alpha}, \\
    \alpha &= \left( \frac{B_0^2 \sigma}{\rho} + \frac{\alpha}{K} \right) \frac{1}{\omega_1} = M^2 + \frac{1}{K'}. \\
\end{align*}
\]

\( (a', b') = \frac{1}{\sqrt{\alpha \omega_1}} (a, b) \) and the non-dimensional quantities are

\[
\begin{align*}
    u' &= \frac{u}{\sqrt{\alpha \omega_1}}, \\
    v' &= \frac{v}{\sqrt{\alpha \omega_1}}, \\
    y' &= \sqrt{\frac{\omega_1}{\alpha}} y \quad \text{and} \quad t' = \omega_1 t.
\end{align*}
\]

When \( E \to 0 \) and \( K \to \infty \) the result coincides with the result of [24].

However if \( E \to 0, K \to \infty \) and \( a = b = \frac{1}{2} \), (26) reduces to the form

\[
\begin{align*}
    u &= e^{-\alpha_1 y} \cos(t - \alpha_2 y) \\
\end{align*}
\]  

This is the steady state result for the fluid velocity given in [99] in then analysis of Stokes problem for a conducting fluid with suspension of particles.

Finally, the steady state solution for the particle velocity becomes

\[
\begin{align*}
    v &= \frac{\left( a e^{i\sigma_1 t} \sinh\left( \mu_1 + i\mu_2 \right) \right) e^{i\sigma_1 t} \sinh\left( \mu_1 + i\mu_2 \right) + be^{-i\sigma_1 t} \sinh\left( \mu_1 - i\mu_2 \right) e^{-i\sigma_1 t} \sinh\left( \mu_1 - i\mu_2 \right)}{(L + i\sigma_1) \sinh(\mu_1 + i\mu_2) + (L - i\sigma_1) \sinh(\mu_1 - i\mu_2)} \\
    &\quad + \frac{ce^{i\sigma_2 t} \sinh\left( \mu_3 + i\mu_4 \right) e^{i\sigma_2 t} \sinh\left( \mu_3 + i\mu_4 \right) + de^{-i\sigma_2 t} \sinh\left( \mu_3 - i\mu_4 \right) e^{-i\sigma_2 t} \sinh\left( \mu_3 - i\mu_4 \right)}{(L + i\sigma_2) \sinh(\mu_3 + i\mu_4) + (L - i\sigma_2) \sinh(\mu_3 - i\mu_4)} \\
\end{align*}
\]  

(28)
This solution shows that in the steady state condition fluid moves faster than the particles when \( \sigma_1 \neq 0 \) and \( \sigma_2 \neq 0 \). But in the non-oscillatory case \( (\sigma_1 = \sigma_2 = 0) \) the particles move together with the fluid. Hence the effect of dust particles appear in the steady state solution only in presence of oscillation. When \( K \to \infty \) and \( E \to 0 \), this expression (28) for velocity of particles is in agreement with the velocity profile of L. Debnath and A. K. Ghosh [24] in steady case.

In particular, if \( \sigma_1 = \sigma_2 \) the steady state results for the fluid and the particle velocities are

\[
u = e^{i\sigma_1 t} \left[ \frac{a \sinh\left\{ (\mu_1 + i\mu_2)(1 - y) \right\} + c \sinh\left( \mu_1 + i\mu_2 \right) y}{\sinh(\mu_1 + i\mu_2)} \right] + \frac{b \sinh\left( \mu_1 - i\mu_2 \right) (y - 1)}{\sinh(\mu_1 - i\mu_2)} \sinh(\mu_1 + i\mu_2) \sinh(\mu_1 - i\mu_2)
\]

\[
u = \frac{Le^{-i\sigma_1 t}}{L + i\sigma_1} \left[ \frac{a \sinh\left\{ (\mu_1 + i\mu_2)(1 - y) \right\} + c \sinh\left( \mu_1 + i\mu_2 \right) y}{\sinh(\mu_1 + i\mu_2)} \right] + \frac{b \sinh\left( \mu_1 - i\mu_2 \right) (y - 1)}{\sinh(\mu_1 - i\mu_2)} \sinh(\mu_1 + i\mu_2) \sinh(\mu_1 - i\mu_2)
\]

When \( \sigma_1 = 0 \), \( u = v \). These are stated earlier.

**Some special cases:**

**(i) Hydromagnetic clean fluid flow through porous medium:**

In the limit \( L \to \infty \) and \( f \to 0 \) the fluid velocity corresponding to the flow of a clean visco-elastic fluid through porous medium between two oscillating plates is given by

\[
u = ae^{i\sigma_1 t} \sinh\left\{ (\alpha_1 + i\alpha_2)(1 - y) \right\} \sinh(\alpha_1 + i\alpha_2) + be^{-i\sigma_1 t} \sinh\left\{ (\alpha_1 - i\alpha_2)(1 - y) \right\} \sinh(\alpha_1 - i\alpha_2)
\]

\[+ ce^{i\sigma_2 t} \sinh\left\{ (\alpha_3 + i\alpha_4) \right\} \sinh(\alpha_3 + i\alpha_4) + de^{-i\sigma_2 t} \sinh\left\{ (\alpha_3 - i\alpha_4) \right\} \sinh(\alpha_3 - i\alpha_4)
\]

\[- 2\pi \sum_{n=0}^{\infty} (-1)^n \left[ D_n^{(1)} \sin n\pi(1 - y) + D_n^{(2)} \sin n\pi y \right] e^{\lambda_n t}
\]

(30)
where \( A = \frac{H + \sigma_1^2}{1 + \sigma_1^2}, \quad B = \frac{\sigma_1(1 - HE)}{1 + \sigma_1^2}, \quad G = \frac{H + \sigma_2^2}{1 + \sigma_2^2}, \quad F = \frac{\sigma_2(1 - HE)}{1 + \sigma_2^2} \)

\((\alpha_1, \alpha_2) = \left( \frac{\sqrt{A^2 + B^2 - 4A}}{2} \right)^2, \quad (\alpha_3, \alpha_4) = \left( \frac{\sqrt{G^2 + F^2 - 4G}}{2} \right)^2, \quad \lambda_n = -\frac{(H + n^2 \pi^2)}{1 + n^2 \pi^2} \)

\( D_n^{(1)} = \left[ \frac{a}{\lambda_n - i\sigma_1} + \frac{b}{\lambda_n + i\sigma_1} \right] \left[ \frac{(1 + \lambda_n E)^2}{(1 - HE)} \right], \quad D_n^{(2)} = \left[ \frac{c}{\lambda_n - i\sigma_2} + \frac{d}{\lambda_n + i\sigma_2} \right] \left[ \frac{(1 + \lambda_n E)^2}{(1 - HE)} \right] \)

When \( K \to \infty \) and \( E \to 0 \) the expression (30) coincides with the special case (i) of L. Debnath and A. K. Ghosh [24]. However, when \( \sigma_2 = \sigma_1 \) (plates oscillates with same frequency), (30) reduces to a simpler form.

If the plates oscillates in phase, we can put \( a = b = \frac{1}{2} \) and \( c = d = \frac{1}{2} \) and it turns out from (30) that

\[ u = \frac{1}{2} \left[ e^{i\gamma_1 t} \sinh\left\{ \frac{1}{2} (\alpha_1 + i\alpha_2)(1 - y) \right\} + e^{-i\gamma_1 t} \sinh\left\{ \frac{1}{2} (\alpha_1 - i\alpha_2)(1 - y) \right\} \right] \]

\[ + e^{i\gamma_2 t} \sinh\left\{ \frac{1}{2} (\alpha_3 + i\alpha_4)y \right\} + e^{-i\gamma_2 t} \sinh\left\{ \frac{1}{2} (\alpha_3 - i\alpha_4)y \right\} \]

\[ - 2\pi \sum_{n=0}^{\infty} \frac{n(-1)^n \lambda_n (1 + \lambda_n E)^2}{(1 - HE)} \left[ \frac{\sin n\pi (1 - y)}{\lambda_n^2 + \sigma_1^2} + \frac{\sin n\pi y}{\lambda_n^2 + \sigma_2^2} \right] e^{\lambda_n t} \]  

(31)

Further, if the plates oscillates \( 90^\circ \) out of phase but with same frequency in which case \( a = b = \frac{1}{2} \) and \( c = \frac{1}{2i}, \quad d = -\frac{1}{2i} \), we find from (30) that the fluid velocity takes the form

\[ u = (A_1 + B_1)\cos \gamma_1 t + (A_1 - B_1)\sin \gamma_1 t \]

\[ - 2\pi \sum_{n=0}^{\infty} \frac{n(-1)^n \lambda_n (1 + \lambda_n E)^2}{(1 - HE)} \times \left[ \frac{\sin n\pi (1 - y) + \sigma_1}{\lambda_n} \sin n\pi y \right] e^{\lambda_n t} \]  

(32)

where \( A_1 = \frac{1}{\ell} \left[ \ell_1 \cos \alpha_2 (1 - y) \sinh \alpha_1 (1 - y) + \ell_2 \cosh \alpha_1 (1 - y) \sin \alpha_2 (1 - y) \right] \]
\[
B_3 = \frac{1}{t_3} \left[ t_1 \cosh \alpha_1 (1 - y) \sinh \alpha_2 (1 - y) - t_2 \sinh \alpha_1 (1 - y) \cos \alpha_2 (1 - y) \right]
\]

\[
t_1 = \sinh \alpha_1 \cos \alpha_2, \quad t_2 = \cosh \alpha_1 \sin \alpha_2, \quad t_3 = t_1^2 + t_2^2
\]

Finally, when \( \sigma_1 = \sigma_2 \), \( a = \frac{u_{01}}{2i} \), \( b = -\frac{u_{01}}{2i} \) and \( c = \frac{u_1}{2i} \). \( d = -\frac{u_1}{2i} \)

We obtain from (30)

\[
u = (u_0 + u_1)(B_3 \cos \sigma_1 t + A_1 \sin \sigma_1 t)
\]

\[
-2\pi \sigma_1 \sum_{n=0}^{\infty} n(-1)^n \left[ u_{01} \lambda_n \sin n\pi (1 - y) + u_1 \sigma_1 \sin n\pi y \right] \frac{(1 + \lambda_n E)^2}{(1 - HE)} \cdot \frac{e^{\lambda_n t}}{(\lambda_n^2 + \sigma_1^2)}
\]

when \( K \to \infty \) and \( E \to 0 \), (33) reduces to the form which coincides with the result for the fluid velocity given in [88].

**(ii) Non-oscillatory two-phase hydromagnetic flow through porous medium:**

In the non-oscillatory situation \( \sigma_1 = \sigma_2 = 0 \). Accordingly, the fluid velocity in this case as obtained from (17) is given by

\[
u = (a + b) \frac{\sinh \sqrt{H} (1 - y)}{\sinh \sqrt{H}} + (c + d) \frac{\sinh \sqrt{Hy}}{\sinh \sqrt{H}}
\]

\[
-2\pi \sum_{n=0}^{\infty} n(-1)^n \sin n\pi (1 - y) \left[ F_n^{(1)} e^{-s_n^{(1)} t} + F_n^{(2)} e^{-s_n^{(2)} t} \right]
\]

\[
-2\pi \sum_{n=0}^{\infty} n(-1)^n \sin n\pi y \left[ F_n^{(3)} e^{-s_n^{(3)} t} + F_n^{(4)} e^{-s_n^{(4)} t} \right]
\]

(34)

where \( F_n^{(i)} = \frac{a + b}{s_n^{(i)}} \left[ \frac{(E s_n^{(i)} - (1 + \lambda E) s_n^{(i)} + L)^2}{(1 - \lambda E - HE) s_n^{(i)} - 2L (1 - HE) s_n^{(i)} + L (L + \lambda - HEL)} \right] \), \( i = 1, 2 \)

\( F_n^{(3)} \) and \( F_n^{(4)} \) will be obtained from \( F_n^{(1)} \) and \( F_n^{(2)} \) respectively by replacing \( a + b \)

by \( c + d \) and \( s_n^{(1)} \), \( s_n^{(2)} \) are given in (22).

The first two terms of (34) represent the steady Hartmann layer flow of a viscoelastic fluid through porous medium similar to that in a clean fluid while the unsteady parts contain the effect of dust particles, magnetic field and porous medium. This clearly indicates that the effect of dust particles appear only in the transient part of the
fluid velocity when the motion is due to the motion of non-oscillatory movement of
the plates. Hence in such a situation, the ultimate velocity field becomes independent
of the effects of the particles and the flow becomes identical to that of clean fluid
motion through porous medium in presence of uniform magnetic field.

The exact expression for the particle velocity in the non-oscillatory case has the
form

\[ v = (a + b)(1 - e^{-Lt}) \frac{\sinh \sqrt{H}(1 - y)}{\sinh \sqrt{H}} + (c + d)(1 - e^{-L_t}) \frac{\sin \sqrt{H}y}{\sinh \sqrt{H}} \]

\[
- 2\pi l \sum_{n=0}^{\infty} n(-1)^n \sin \pi y (1 - y) \left[ F_n^{(1)} \left( e^{-s_n^{(1)} t} - e^{-L_t} \right) - s_n^{(1)} + L \right] + F_n^{(2)} \left( e^{-s_n^{(2)} t} - e^{-L_t} \right) - s_n^{(2)} + L \]

\[
- 2\pi l \sum_{n=0}^{\infty} n(-1)^n \sin \pi y \left[ F_n^{(3)} \left( e^{-s_n^{(1)} t} - e^{-L_t} \right) - s_n^{(1)} + L \right] + F_n^{(4)} \left( e^{-s_n^{(2)} t} - e^{-L_t} \right) - s_n^{(2)} + L \]  

(35)

Obviously, in the non-oscillatory case, particles move together with the fluid when
steady state is reached, while in the non-stationary situation fluid moves faster than
the particles. When \( K \to \infty, E \to 0 \), the fluid and particles velocity are in agreement to
those of L. Debnath and A. K. Ghosh [24] in “Non-oscillatory two phase hydro-
magnetic flow” situation.

The wall shear stresses:

The shear stress at the plate \( y = 0 \).

We know for Rivlin-Ericksen fluid

\[ \tau = \rho \left( \alpha + \beta \frac{\partial}{\partial t} \right) \left( \frac{\partial u}{\partial y} \right)_{y=0} \]  

(36)

where \( \tau \) is the shear stress at the plate \( y = 0 \).

Using the non-dimensional variables

\[
y' = \frac{y}{h}, \quad u' = \frac{uh}{\alpha}, \quad t' = \frac{\alpha t}{h^2} \quad \text{and} \quad \tau' = \frac{\tau h^2}{\rho \alpha^2}, \quad E = \frac{\beta}{h^2} \quad \text{(elastic parameter)}
\]

in (36), we get (omitting dashes),

\[ \tau \approx \left( 1 + E \frac{\partial}{\partial t} \right) \left( \frac{\partial u}{\partial y} \right)_{y=0} \]
From (17) and (37), we get

\[
\tau = a\left\{ (L_1 - EM_1\sigma_1)\cos\sigma_1 t - (M_1 + EL_1\sigma_1)\sin\sigma_1 t \right\} + \mu_1 (L_1 - EM_1\sigma_1)\cos\sigma_1 t
\]

\[
+ (L_1 - EM_1\sigma_1)\sin\sigma_1 t \right\} + b\left\{ (L_1 - EM_1\sigma_1)\cos\sigma_1 t - (M_1 + EL_1\sigma_1)\sin\sigma_1 t \right\}
\]

\[
- i\{(M_1 + EL_1\sigma_1)\cos\sigma_1 t + (L_1 - EM_1\sigma_1)\sin\sigma_1 t \right\} + c\left\{ (L_2 - EM_2\sigma_2)\cos\sigma_2 t
\]

\[
- (M_2 + EL_2\sigma_2)\sin\sigma_2 t \right\} + i\{(M_2 + EL_2\sigma_2)\cos\sigma_2 t + (L_2 - EM_2\sigma_2)\sin\sigma_2 t \right\}
\]

\[
+ d\left\{ (L_2 - EM_2\sigma_2)\cos\sigma_2 t - (M_2 + EL_2\sigma_2)\sin\sigma_2 t \right\} - i\{(M_2 + EL_2\sigma_2)\cos\sigma_2 t
\]

\[
+ (L_2 - EM_2\sigma_2)\sin\sigma_2 t \right\}
\]

\[
+ 2\pi^2 \sum_{n=0}^{\infty} \frac{n^2}{(1 - E n(1)) \Lambda_n \sinh n^2 t} + (1 - E n(2)) \Lambda_n \cosh n^2 t]
\]

\[
- 2\pi^2 \sum_{n=0}^{\infty} \frac{n^2 (-1)^n}{(1 - E n(1)) \Lambda_n \cosh n^2 t} + (1 - E n(2)) \Lambda_n \sinh n^2 t]
\]

where

\[
L_1 = \frac{1}{K_3} (\mu_1 \sinh \mu_1 \cosh \mu_1 + \mu_2 \sinh \mu_2 \cos \mu_2)
\]

\[
L_2 = \frac{1}{K_3} (\mu_2 \sinh \mu_1 \cosh \mu_1 - \mu_1 \sinh \mu_2 \cos \mu_2)
\]

\[
M_2 = \frac{1}{K_3} (K_4 \mu_3 + K_2 \mu_4)
\]

\[
K_1 = \sinh \mu_1 \cos \mu_2, \quad K_2 = \cosh \mu_1 \sin \mu_2, \quad K_3 = K_1^2 + K_2^2
\]

\[
K_4 = \sinh \mu_3 \cos \mu_4, \quad K_5 = \cosh \mu_3 \sin \mu_4, \quad K_6 = K_4^2 + K_5^2
\]

When \(E \to 0\) in (38), we shall get the shear stress at the plate \(y = 0\) for ordinary viscous dusty fluid moves through porous medium bounded by two oscillating plates with different frequencies. When \(E \to 0\) and \(K \to \infty\) in (38), we get the expression for shear stress which coincides to that of [24] at the plate \(y = 0\). When \(L \to \infty\) in (38) we shall get the shear stress at the wall \(y = 0\) for clean visco-elastic fluid moves through porous in presence of transverse magnetic field. Similarly, we can calculate the shear stress at the upper oscillating plate in dimensionless form.
Table 2: Numerical values of \( u \) and \( v \) for different values of \( y \) and \( f \) when
\[ M = 1, K = 0.1, E = 0.1, t = 0.1 \text{ and } L = 6.67 \]

<table>
<thead>
<tr>
<th>( y )</th>
<th>( u (f = 0.1) )</th>
<th>( u (f = 0.15) )</th>
<th>( u (f = 0.2) )</th>
<th>( v (f = 0.1) )</th>
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</table>

Results and Discussion:

Numerical discussions are made for \( a = c = \frac{1}{2i}, \ b = d = -\frac{1}{2i} \) and \( \sigma_1 = \sigma_2 = 5.0 \).

From fig. 1, 2 and table 2, it is seen that the velocities of the fluid and the dust particles are maximum near the plates. It is also seen that when the frequencies of both the plates are same, the velocity profiles of the fluid and the dust are symmetrical about the plane \( y = 0.5 \) and those are minimum on the plane \( y = 0.5 \). From fig. 1 it is observed that as the time relaxation increases, the velocities of both the fluid and the dust particles decrease. The velocity of the fluid only at the central region is slightly effected due to increase of time relaxation. But the effect of time relaxation on the velocity of the dust is pronounced everywhere. From fig. 2 it is clear that as \( E \) increases the velocities of the fluid and the dust particles decrease. It is interesting to note that when \( E = 0.2 \) and \( E = 0.3 \), the dust takes reverse motion in the region \( y = 0.3 \) to \( y = 0.7 \) and \( y = 0.2 \) to \( y = 0.8 \) respectively. From fig. 3 it is observed that the velocities of the fluid and the dust particles are periodic with respect to time. It is also observed that as \( H \) increases the amplitude of the velocities of the fluid and the dust particles decrease. So we can conclude that the magnetic field is not in favour of the
velocity of the fluid-particle system. We can also conclude that as the permeability of the porous medium increases, the amplitude of the velocity of the fluid and the particles increase. From table 2 it is seen that when \( f \) increases the velocity of the fluid and the particles increase.

From fig. 4, 5 and 6 we see that the shear stress is periodic with respect to time. From fig. 4 it is seen that the amplitude of the shear stress increases with the increase of \( H \). So we can draw the conclusion that the amplitude of the shear stress increases with the increase and decrease of the intensity of the magnetic field and permeability of the porous medium respectively. From fig. 5 and 6 it is observed that as \( f \) and \( E \) increase the amplitude of the shear stress increases.
Fig. 1: Velocity profiles for various values of $L$ with fixed $M$, $K$, $E$, $f$ and $t$. 

\begin{tabular}{|c|c|c|c|c|}
\hline
$M$ & $K$ & $E$ & $t$ & $f$ \\
\hline
1 & 0.1 & 0.1 & 0.1 & 0.1 \\
\hline
\end{tabular}
Fig. 2: Velocity profiles for various values of $E$ with fixed $M, K, L, f$ and $t.$
Fig. 3 : Velocity profiles for various values of $H$ with fixed $y$, $E$, $f$ and $L$. 

<table>
<thead>
<tr>
<th>$y$</th>
<th>$E$</th>
<th>$f$</th>
<th>$L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>0.1</td>
<td>0.1</td>
<td>6.67</td>
</tr>
</tbody>
</table>
Fig. 4: Shear stress at the plate against t for various values of H with fixed E, f and L.
Fig. 5: Shear stress at the plate against t for various values of F with fixed M, K, E and L.
Fig. 6: Shear stress at the plate against $t$ for various values of $E$ with fixed $M$, $K$, $f$ and $L$.
1.3 : MHD FLOW OF A CONDUCTING DUSTY VISCO-ELASTIC FLUID OF FINITE DEPTH THROUGH POROUS MEDIUM DUE TO TIME DEPENDENT TANGENTIAL STRESS APPLIED AT THE FREE SURFACE

Introduction :

The motion of a conducting visco-elastic dusty fluid in presence of magnetic field set up by time dependent shearing forces applied on the surface is considered in connection with the study of generation of current by the action of wind. The presence of dust particles with transverse magnetic field and porous medium renders this problem more complex, but not intractable. A lot of problems on viscous fluid are discussed by many researchers. On incompressible visco-elastic fluid in presence of dust, magnetic field and porous medium, a few problems only are discussed.

The formulation of Saffman [72] has been applied by different authors to solve various problems containing dust particles. Lamb [53] investigated the unidirectional motion of viscous incompressible fluid of finite depth due to the action of uniform tangential stress acting at the surface. A similar problem has been treated by Batchelor [11] for semi infinite viscous incompressible fluid by taking into account the effects of the rotating earth on the surface drift motion. The unsteady flow generated by the pulse of a tangential force acting for a prescribed period has been studied by Ray, Samad and Chaudhury [69]. Das [22], [23] in his papers has tried to solve the problems of unidirectional motion of conducting viscous fluid of finite depth due to tangential shearing stress on the free surface in presence of magnetic field. The influence of dust particles on visco-elastic flows has a great importance in petroleum industry and paper technology in purification of crude oil and several geophysical situations.

In the present paper, we consider the MHD flow of a conducting dusty visco-elastic fluid due to time dependent tangential stress of different types applied at the free surface.

Formulation of the Problem:

We consider a magnetohydrodynamic basic problem in which an unlimited mass of conducting dusty visco-elastic (RIVLIN-ERICKSEN) incompressible fluid of finite depth initially at rest is set into motion by the action of time dependent tangential stress applied at the horizontal surface in presence of transverse uniform magnetic field and porous medium.

The co-ordinate system is chosen in such a manner that the origin lies on the fixed base. x-axis is chosen in the direction of flow, y-axis being taken perpendicular to the base and pointing upwards. Assuming the motion to be slow, in absence of pressure gradient the governing equations of motion obtained from (VII) and (VIII) by using (I) – (IV), are:

\[
\frac{\partial u}{\partial t} = (\alpha + \beta \frac{\partial}{\partial t}) \frac{\partial^2 y}{\partial y^2} + \frac{K_0 N_0}{\rho} (v - u) - \frac{\sigma B_0^2}{\rho} u - \frac{\alpha}{K} u
\]

and

\[
m_0 \frac{\partial v}{\partial t} = K_0 (u - v)
\]

where \(\alpha = \mu/\rho\) is the Kinematic co-efficient of viscosity, \(\beta = \mu_0/\rho\) the Kinematic co-efficient of visco-elasticity, \(T_1 = m_0/K_0\) is the time relaxation parameter and \(B_0\) is the magnetic induction vector along y-axis.

The initial and boundary conditions of the problem are:

\[
u = v = 0 \text{ when } t \leq 0 \text{ for all } y \text{ in } [0, h]
\]

and

\[
u = v = 0 \text{ when } y = 0 \text{ for all } t,
\]

\[
\rho(\alpha + \beta \frac{\partial}{\partial t}) \frac{\partial u}{\partial y} = F(t) \text{ when } y = h \text{ for all } t,
\]

where \(F(t)\) being a prescribed function of time, different in different case of the problem.

Method of Solution:

Introducing the non-dimensional quantities

\[
y' = \frac{y}{h}, \quad u' = \frac{uh}{\alpha}, \quad v' = \frac{vh}{\alpha}, \quad t' = \frac{\alpha t}{h^2}, \quad K' = \frac{K}{h^2}
\]
in (1) and (2), we get (omitting dashes) respectively
\[ \frac{\partial u}{\partial t} = (1 + E) \frac{\partial}{\partial t} \left( \frac{\partial^2 u}{\partial y^2} + \lambda(v - u) - Gu \right) \tag{6} \]
and
\[ \frac{\partial u}{\partial t} = L(u - v) \tag{7} \]
where \( E \) (elastic parameter) = \( \frac{\beta}{h^2} \), \( f \) (mass concentration) = \( \frac{m_0 N_0}{\rho} \).

\[ L = \frac{h^2}{\alpha q_1} = \frac{h^2 K_0}{\alpha m_0}, \quad M \) (Hartmann number) = \( B_0 h \sqrt{\frac{\sigma}{\rho \alpha}}, \quad \lambda = fL, \quad G = M^2 + \frac{1}{K} \]

The initial and boundary conditions in non-dimensional form are (omitting dashes)
\[ u = v = 0 \text{ when } t \leq 0 \text{ for all } y \in [0, 1] \tag{8} \]
and \[ u = v = 0 \text{ when } y = 0 \text{ for all } t. \tag{9} \]

\[ (1 + E) \frac{\partial}{\partial t} \frac{\partial u}{\partial y} = f(t) \text{ when } y = 1, \text{ where } f(t) = \frac{h^2}{\alpha^2 \rho} F(t' h^2) \tag{10} \]

By applying Laplace transform
\[ \bar{u} = \int_0^t u e^{-st} dt, \text{ where } \text{Re}(s) > 0 ; \]
to (6) and (7), we get respectively
\[ \frac{d^2 \bar{u}}{dy^2} - p^2(s) \bar{u} = 0, \quad \text{where } p^2(s) = \frac{s^2 + (\lambda + G + L)s + GL}{(s + L)(1 + sE)} \tag{11} \]
and
\[ \bar{v} = \frac{L \bar{u}}{s + L} \tag{12} \]

The transformed boundary conditions are
\[ \bar{u} = \bar{v} = 0 \text{ when } y = 0 \tag{13} \]
and \[ (1 + sE) \frac{d \bar{u}}{dy} = f(s), \text{ when } y = 1 \tag{14} \]

The solution of (11) with the help of (13) and (14) is
\[ \bar{u} = \frac{F(s) \sinh py}{p (1 + sE) \cosh p} \tag{15} \]
From (12) and (15), we get

$$v = -\frac{L f(s) \sinh \eta v}{p (s + L) (1 + sE) \cosh p}$$

(16)

**SPECIAL CASES**

**Case (I). Flow Due to Transient Shearing Stress:**

Let us consider the case in which

$$F(t) = Se^{-mt}$$

where $S > 0$ and $m > 0$ are constants.

Then from (15) and (16)

$$u = \frac{S \sinh \eta v}{p (s + m) (1 + sE) \cosh p}$$

(1.1)

and

$$v = \frac{LS \sinh \eta v}{p (s + m) (1 + sE) (s + L) \cosh p}$$

(1.2)

By inverse Laplace transform, we have respectively from (1.1) and (1.2)

$$u = \frac{e^{-mt} \sinh qy}{S q(1 - mE) \cosh q} + 2 \sum_{n=0}^{\alpha} (-1)^n \sin (2n + 1) \frac{\pi y}{2} \times \left[ \frac{\Delta_n^{(1)} e_s^{(1)} + \Delta_n^{(2)} e_s^{(2)}}{s_n^{(1)} + m} \right]$$

(1.3)

and

$$v = \frac{e^{-mt} L \sinh qy}{S q(1 - mE)(1 - m) \cosh q} + 2L \sum_{n=0}^{\alpha} (-1)^n \sin (2n + 1) \frac{\pi y}{2} \times$$

$$\left[ \frac{\Delta_n^{(1)} e_s^{(1)}}{s_n^{(1)} + m} + \frac{\Delta_n^{(2)} e_s^{(2)}}{s_n^{(2)} + m} \right]$$

(1.4)

where

$$q^2 = \frac{m^2 - (\lambda + G + L)m + GL}{Em^2 - (1 + LE)m + L}$$

and $s_n^{(1)}$, $s_n^{(2)}$ are the roots of the equation

$$(1 + ED_n^2)s^2 + \left[ \lambda + G + L + (1 + LE)D_n^2 \right] s + L(G + D_n^2) = 0.$$
\[ D_n = \frac{(2n + 1) \pi}{2} \]

\[ \Delta_n^{(i)} = \frac{(1 + s_n^{(i)} E)(s_n^{(i)} + L)^2}{[1 - E(\lambda + G)]s_n^{(i)} - 2L(GE - 1)s_n^{(i)} + L(\lambda + L - GLE)} \quad i = 1, 2 \quad (1.5a) \]

The velocity of the fluid and the dust particles due to constant shearing stress \( S \) may be obtained respectively from the expressions (1.3) and (1.4) by putting \( m = 0 \).

For \( G \to 0 \) in (1.3) and (1.4), we shall get the corresponding non-MHD flow when no resistance is offered by the porous medium. When \( E \to 0 \) in (1.3) and (1.4), we get the corresponding flow for dusty viscous fluid.

The dimensionless shearing stress \( \tau \) at the fixed plate given by

\[ \tau = \left(1 + E \frac{\partial}{\partial \lambda} \left( \frac{\partial u}{\partial \lambda} \right) \right) \quad \lambda = 0 \]

is then obtained as

\[ \frac{\tau}{S} = \frac{e^{-mt}}{\cosh q} + \pi \sum_{n=0}^{\infty} (-1)^n (2n + 1) \left[ \frac{\Delta_n^{(1)} (1 + Es_n^{(1)}) e^{s_n^{(1)} t}}{(s_n^{(1)} + m)} - \frac{\Delta_n^{(2)} (1 + Es_n^{(2)}) e^{s_n^{(2)} t}}{(s_n^{(2)} + m)} \right] \quad (1.6) \]

When \( m \to 0 \) in (1.6), we have

\[ \frac{\tau}{S} = \frac{1}{\cosh \sqrt{G}} + \pi \sum_{n=0}^{\infty} (-1)^n (2n + 1) \left[ \frac{\Delta_n^{(1)} (1 + Es_n^{(1)}) e^{s_n^{(1)} t}}{s_n^{(1)}} + \frac{\Delta_n^{(2)} (1 + Es_n^{(2)}) e^{s_n^{(2)} t}}{s_n^{(2)}} \right] \quad (1.7) \]

Equation (1.7) describes the shear stress at the fixed base due to constant shearing stress applied at the free surface of the fluid.

When \( t \to \infty \) in (1.7) i.e. in steady state

\[ \tau = \frac{S}{\cosh \sqrt{G}} \quad (1.8) \]

It is very interesting to see that the shear stress at the fixed base in steady state does not depend on the viscoelastic parameter \( F \) but on the permeability of the porous medium and Hartmann number when the applied shear stress at the free surface is constant.

When \( G \to 0 \) in (1.8), we have

\[ \tau = S. \]
It is also very interesting to note that the shear stress at the fixed base in steady state is the constant $S$, the applied shear stress at the free surface of the fluid when the magnetic field and the porous medium are ceased.

**Case (II). Flow Due to Periodic Shearing Stress:**

In this case we shall take

$$f(t) = S \cos \omega t,$$

where $S$ and $\omega$ are constants.

Then from (15) and (16)

$$u = \frac{S s \sinh py}{p \left(s^2 + \omega^2\right)(1 + sE) \cosh p}, \quad (2.1)$$

and

$$v = \frac{L \cdot S s \sinh py}{p \left(s^2 + \omega^2\right)(s + L)(1 + sE) \cosh p}, \quad (2.2)$$

By inverse Laplace transform, we respectively get from (2.1) and (2.2)

$$\frac{u}{S} = A_1(y) \cos \omega t + B_1(y) \sin \omega t + 2 \sum_{n=0}^{\infty} (-1)^n \sin(2n + 1) \frac{\pi y}{2} \cdot$$

$$\left[ \frac{\Delta_n^{(1)} s_n^{(1)} e^{s_n^{(1)} t}}{\left(s_n^{(1)} + \omega^2\right)(s_n^{(1)} + L)} + \frac{\Delta_n^{(2)} s_n^{(2)} e^{s_n^{(2)} t}}{\left(s_n^{(2)} + \omega^2\right)(s_n^{(2)} + L)} \right], \quad (2.3)$$

and

$$\frac{v}{S} = \frac{A_1(y)(L^2 - L \omega) \cos \omega t + B_1(y)(L^2 + L \omega) \sin \omega t}{(L^2 + \omega^2)} + 2L \sum_{n=0}^{\infty} (-1)^n \sin(2n + 1) \frac{\pi y}{2} \cdot$$

$$\left[ \frac{\Delta_n^{(1)} s_n^{(1)} e^{s_n^{(1)} t}}{\left(s_n^{(1)} + \omega^2\right)(s_n^{(1)} + L)} + \frac{\Delta_n^{(2)} s_n^{(2)} e^{s_n^{(2)} t}}{\left(s_n^{(2)} + \omega^2\right)(s_n^{(2)} + L)} \right] \cdot (2.4)$$

where

$$A = \frac{(G \lambda - \omega^2)(L - E \omega^2) + (\lambda + G + L)(1 + LE)}{(L - E \omega^2)^2 + (1 + LE)^2 \omega^2},$$

$$B = \frac{\omega[\lambda + G + L)(L - E \omega^2) + (GL - \omega^2)(1 + LE)]}{(L - E \omega^2)^2 + (1 + LE)^2 \omega^2}.$$
\[(\mu_1, \mu_2) = \left( \frac{\sqrt{A^2 + B^2 + A}}{2} \right)^{1/2}, \text{ } s_n^{(1)} \text{ } \text{and} \text{ } s_n^{(2)} \text{ } \text{are the roots of} \text{ } (1.5). \]

\[K_1 = \cosh \mu_1 \cos \mu_2, \quad K_2 = \sinh \mu_1 \sin \mu_2, \quad K_3 = K_1^2 + K_2^2\]

\[\xi(y) = K_1 \sinh \mu_1 y \cos \mu_2 y + K_2 \cosh \mu_1 y \sin \mu_2 y\]

\[\eta(y) = K_1 \cosh \mu_1 y \sin \mu_2 y - K_2 \sinh \mu_1 y \cos \mu_2 y\]

\[A_1(y) = \frac{\mu_1 - E \omega \mu_2}{K_3} \frac{\xi(y) + (E \omega \mu_1 + \mu_2) \eta(y)}{(\mu_1 - E \omega \mu_2)^2 + (E \omega \mu_1 + \mu_2)^2}\]

\[B_1(y) = \frac{E \omega \mu_1 + \mu_2}{K_3} \frac{\xi(y) - (\mu_1 - E \omega \mu_2) \eta(y)}{(\mu_1 - E \omega \mu_2)^2 + (E \omega \mu_1 + \mu_2)^2}\]

When \(\omega \to 0\) in (2.3) and (2.4), we get respectively

\[u = \frac{\sinh \sqrt{G} y}{S} \frac{\mu_1 - E \omega \mu_2}{\sqrt{G} \cosh \sqrt{G}} + 2 \sum_{n=0}^{\infty} \frac{(-1)^n \sin(2n+1)}{2} \frac{\pi y}{2} \left[ \frac{\Lambda_n^{(1)} e^{s_n^{(1)} t}}{s_n^{(1)} + L} + \frac{\Lambda_n^{(2)} e^{s_n^{(2)} t}}{s_n^{(2)} + L} \right] \tag{2.5}\]

and

\[v = \frac{\sinh \sqrt{G} y}{S} \frac{\mu_1 - E \omega \mu_2}{\sqrt{G} \cosh \sqrt{G}} + 2 L \sum_{n=0}^{\infty} \frac{(-1)^n \sin(2n+1)}{2} \frac{\pi y}{2} \left[ \frac{\Lambda_n^{(1)} e^{s_n^{(1)} t}}{s_n^{(1)} (s_n^{(1)} + L)} + \frac{\Lambda_n^{(2)} e^{s_n^{(2)} t}}{s_n^{(2)} (s_n^{(2)} + L)} \right] \tag{2.6}\]

Equations (2.5) and (2.6) give respectively the velocity of the fluid and the dust particles due to constant shearing stress.

When \(t \to \infty\) in (2.3) and (2.4) i.e. in steady state the transient effects die out. Then the fluid and the dust particles have different velocities. But when \(t \to \infty\) in (2.5) and (2.6), we see that the fluid and the dust particles move together with same velocity. So we can draw the conclusion that in steady state the fluid and the particle have different velocities when the shear stress is periodic and have same velocity when the shear stress is constant.

The dimensionless shearing stress \(\tau\) at the fixed base given by

\[\tau = \left(1 + E \frac{\partial}{\partial \alpha} \left(\frac{\partial u}{\partial \gamma} / y = 0\right)\right)\]
is then obtained as

$$\tau = \frac{(K_1 \cos t + K_2 \sin t)}{S} + \pi \sum_{n=0}^{\infty} (-1)^n (2n+1) \times \Delta_n^{(1)} s_n^{(1)} (1 + E s_n^{(1)}) e^{s_n^{(1)} t} + \Delta_n^{(2)} s_n^{(2)} (1 + E s_n^{(2)}) e^{s_n^{(2)} t} \left( s_n^{(1)} + \omega^2 \right) \left( s_n^{(2)} + \omega^2 \right) \right] \right)$$

(2.7)

When $\omega \rightarrow 0$ in (2.7), we have

$$\tau = \frac{1}{\cosh \sqrt{G}} + \pi \sum_{n=0}^{\infty} (-1)^n (2n+1) \left[ \frac{\Delta_n^{(1)} (1 + E s_n^{(1)}) e^{s_n^{(1)} t}}{s_n^{(1)}} + \frac{\Delta_n^{(2)} (1 + E s_n^{(2)}) e^{s_n^{(2)} t}}{s_n^{(2)}} \right]$$

(2.8)

Equations (2.8) describes the shear stress at the fixed base due constant shear stress applied at the free surface of the fluid. This result agrees with (1.7).

**Case (III). Flow due to transient and periodic shearing stress:**

In this case, we shall take

$$f(t) = S e^{-mt} \cos \omega t$$

where $S$, $m$ ($>0$) and $\omega$ are constants.

Then from (15) and (16), we have

$$\bar{u} = \frac{S (s + m) \sinh py}{p \left[(s + m)^2 + \omega^2 \right] (1 + sE) \cosh p}$$

(3.1)

and

$$\bar{v} = \frac{L S (s + m) \sinh py}{p \left[(s + m)^2 + \omega^2 \right] (1 + sE) (s + L) \cosh p}$$

(3.2)

By inverse Laplace transform, we respectively get from (3.1) and (3.2)

$$u = \frac{A_2(y) \cos \omega t + B_2(y) \sin \omega t}{S} \times e^{-mt} + 2 \sum_{n=0}^{\infty} (-1)^n \sin(2n+1) \frac{\pi y}{2} \times \Delta_n^{(1)} (s_n^{(1)} + m) e^{s_n^{(1)} t} + \Delta_n^{(2)} (s_n^{(2)} + m) e^{s_n^{(2)} t} \left( (s_n^{(1)} + m)^2 + \omega^2 \right) \left( (s_n^{(2)} + m)^2 + \omega^2 \right)$$

(3.3)
\begin{align*}
v &= \left\{ \frac{(L-m)A_2(y) - \omega B_2(y)}{(L-m)^2 + \omega^2} \cos \omega t + \frac{\omega A_2(y) + (L-m)B_2(y)}{(L-m)^2 + \omega^2} \sin \omega t \right\} e^{mt} \\
S &= + 2L \sum_{n=0}^{\infty} (-1)^n \sin(2n+1) \frac{\pi y}{2} \sum_{m=0}^{\infty} \frac{\Delta_{n}^{(1)}(s_{n}^{(1)} + m) e^{s_{n}^{(1)} t}}{(s_{n}^{(1)} + m)^2 + \omega^2} \left( s_{n}^{(1)} + L \right) + \frac{\Delta_{n}^{(2)}(s_{n}^{(2)} + m) e^{s_{n}^{(2)} t}}{(s_{n}^{(2)} + m)^2 + \omega^2} \left( s_{n}^{(2)} + L \right)
\end{align*}

where

\begin{align*}
A' &= \frac{\left[ m^2 - (\lambda + G + L)m + (GL - \omega^2) \right] \left[ E m^2 - m(1 + LE) + (L - E \omega^2) \right]}{\left[ E m^2 - m(1 + LE) + (L - E \omega^2) \right]^2 + \omega^2 \left[ (1 + LE - mE) \right]^2} \\
B' &= \frac{\omega \left[ m^2 - (\lambda + G + L)m + (GL - \omega^2) \right] \left[ (1 + LE - mE) \right]}{\left[ E m^2 - m(1 + LE) + (L - E \omega^2) \right]^2 + \omega^2 \left[ (1 + LE - mE) \right]^2} \\
(\mu_3, \mu_4) &= \left\{ \frac{\sqrt{A'^2 + B'^2} \pm A'}{2} \right\}^{1/2} \\
K'_1 &= \cosh \mu_3 \cos \mu_4 \quad K'_2 = \sinh \mu_3 \sin \mu_4 \quad K'_3 = K'_1^2 + K'_2^2 \\
\xi_l(y) &= K'_1 \sinh \mu_3 y \cos \mu_4 y + K'_2 \cosh \mu_3 y \sin \mu_4 y \\
\eta_l(y) &= K'_1 \cosh \mu_3 y \sin \mu_4 y - K'_2 \sinh \mu_3 y \cos \mu_4 y \\
A_2(y) &= \frac{\left[ (1-mE)\mu_3 + E \omega \mu_4 \right] \xi_l(y) + \left[ (1-mE)\mu_4 + E \omega \mu_3 \right] \eta_l(y)}{\left( \mu_3^2 + \mu_4^2 \right) \left[ (1-mE)^2 + E \omega^2 \right] K'_3} \\
B_2(y) &= \frac{\left[ -((1-mE)\mu_4 + E \omega \mu_3) \xi_l(y) + \left[ (1-mE)\mu_3 + E \omega \mu_4 \right] \eta_l(y) }{\left( \mu_3^2 + \mu_4^2 \right) \left[ (1-mE)^2 + E \omega^2 \right] K'_3}
\end{align*}

$s_{n}^{(1)}$, $s_{n}^{(2)}$ are roots of the equation (1.5). $\Delta_{n}^{(i)}$ ($i = 1, 2$) are given by (1.5a).
When \( L \to \infty \) in (3.3) or (3.4), we have

\[
\frac{u}{S} = [A_3(y) \cos \omega t + B_3(y) \sin \omega t] e^{-\frac{m}{2} t} + 2 \sum_{n=0}^{\infty} (-1)^n \sin(2n + 1) \frac{\pi y}{2} \times
\]

\[
\left( \frac{1 + E \xi_n^{(1)}}{1 - \sqrt{\frac{1}{2} \left( \frac{(\xi_m^{(1)} + m)^2 + N^2}{1 - \sqrt{\frac{1}{2} \left( \frac{(\xi_m^{(1)} + m)^2 + N^2}{1} \right)} \right)}} \right)
\]

where

\[
A^* = \frac{(G - m)(1 - mE) + \omega^2 E}{(1 - mE)^2 + \omega^2 E^2}, \quad B^* = \frac{\omega(1 - GE)}{(1 - mE)^2 + \omega^2 E^2},
\]

\[
(\mu_5, \mu_6) = \left( \sqrt{A^*^2 + B^*^2 \pm A^*} \right)^2
\]

\[
K_4 = \cosh \mu_4 \cos \mu_6, \quad K_5 = \sinh \mu_4 \sin \mu_6, \quad K_6 = K_4^2 + K_5^2,
\]

\[
\xi_2(y) = K_4 \sinh \mu_4 y \cos \mu_6 y + K_5 \cosh \mu_4 y \sin \mu_6 y
\]

\[
\xi_3(y) = \frac{K_4 \sinh \mu_4 y \cos \mu_6 y - K_5 \cosh \mu_4 y \sin \mu_6 y}{K_4^2 + K_5^2}
\]

\[
A_3(y) = \frac{[(1 - mE)\mu_5 + E \omega \mu_6] \xi_2(y) + [(1 - mE)\mu_6 + E \omega \mu_5] \xi_3(y)}{[(\mu_5^2 + \mu_6^2)((1 - mE)^2 + E^2 \omega^2)K_6}
\]

\[
B_3(y) = \frac{-[(1 - mE)\mu_6 + E \omega \mu_5] \xi_2(y) + [(1 - mE)\mu_5 + E \omega \mu_6] \xi_3(y)}{[(\mu_5^2 + \mu_6^2)((1 - mE)^2 + E^2 \omega^2)K_6}
\]

Equation (3.5) describes the velocity of dust free fluid. The velocity of the fluid and the dust particles due to the shearing stress \( S \omega^m \) may be obtained from (3.3) and (3.4) by putting \( \omega = 0 \), which have the conformity with the results already obtained in case (I). Similarly the expressions (2.3) and (2.4) may be obtained from (3.3) and (3.4) by putting \( m = 0 \), which are the velocities of the fluid elements and the dust particles due to periodic shearing stress \( S \cos \omega t \). The shear stress at the fixed base (as in case I) is

\[
\tau = \frac{e^{-\frac{m}{2} t} (K_1 \cos \omega t + K_2 \sin \omega t)}{K_1' \omega} + \pi \sum_{n=0}^{\infty} (-1)^n (2n + 1)
\]

\[
\left[ \frac{\Delta_n^{(1)}(1 + E S_n^{(1)})(S_n^{(1)} + m)c_s^{(1)l}}{S_n^{(1)^2 + m} + \omega^2} + \frac{\Delta_n^{(2)}(1 + E S_n^{(2)})(S_n^{(2)} + m)c_s^{(2)l}}{S_n^{(2)^2 + m} + \omega^2} \right]
\]

(3.6)
When $\omega \to 0$ in (2.7), we have

$$\tau = \frac{e^{-\omega l}}{\cos q} + \pi \sum_{n=0}^{\infty} (-1)^n (2n + 1) \left[ \frac{\Delta_n^{(1)} (1 + E s_n^{(1)}) e^{s_n^{(1)} t}}{(s_n^{(1)} + m)} + \frac{\Delta_n^{(2)} (1 + E s_n^{(2)}) e^{s_n^{(2)} t}}{(s_n^{(2)} + m)} \right]$$

Equation (3.7) describes shear stress at the fixed base due to transient shear stress which is applied at the free surface. This result agrees with (1.6). The expression (2.7) may be obtained from (3.6) by putting $m = 0$, which is the shear stress at the fixed base due to applied periodic shear stress $S \cos \omega t$ at the free surface of the fluid.

**Case (IV). Flow due to impulsive shearing stress:**

In the present circumstance, we take

$$f(t) = S \delta(t),$$

where ‘$S$’ is a constant and $\delta(t)$ is Dirac delta function.

From (15) and (16), we get

$$\bar{u} = \frac{S \sinh py}{\rho (1 + s\varepsilon) \cosh p}$$

and

$$\bar{v} = \frac{S \sinh py}{\rho (1 + s\varepsilon) (s + L) \cosh p}$$

By inverse Laplace transform, we obtain from (3.1) and (3.2) the expression for $u$ and $v$ as

$$\frac{u}{S} = 2 \sum_{n=0}^{\infty} (-1)^n \sin(2n + 1) \frac{\pi y}{2} \left[ \frac{\Delta_n^{(1)} e^{s_n^{(1)} t}}{s_n^{(1)} + m} + \frac{\Delta_n^{(2)} e^{s_n^{(2)} t}}{s_n^{(2)} + m} \right]$$

and

$$\frac{v}{S} = 2L \sum_{n=0}^{\infty} (-1)^n \sin(2n + 1) \frac{\pi y}{2} \left[ \frac{\Delta_n^{(1)} e^{s_n^{(1)} t}}{s_n^{(1)} + L} + \frac{\Delta_n^{(2)} e^{s_n^{(2)} t}}{s_n^{(2)} + L} \right]$$

The shear stress at the fixed base (as previous cases) is

$$\tau = \frac{\pi}{S} \sum_{n=0}^{\infty} (-1)^n (2n + 1) \left[ \frac{\Delta_n^{(1)} (1 + E s_n^{(1)}) e^{s_n^{(1)} t}}{(s_n^{(1)} + m)} + \frac{\Delta_n^{(2)} (1 + E s_n^{(2)}) e^{s_n^{(2)} t}}{(s_n^{(2)} + m)} \right]$$

**Case (V). Flow due to constant shearing stress acting for a finite time ‘$T$’:**

In this case, let us take

$$f(t) = S[H(t) - H(t - T)].$$
where \( 'S' \) is a constant and \( H(t) \) is the Heaviside unit function defined by
\[
H(t) = 0 \quad \text{for} \quad t \leq 0 \\
= 1 \quad \text{for} \quad t > 0
\]

Then from (15) and (16)
\[
\bar{u} = \frac{S(1 - e^{-ST}) \sinh \beta y}{s(1 + sE) \cosh \beta y}
\]
and
\[
\bar{v} = \frac{S(1 - e^{-ST}) \sinh \beta y}{s(1 + sE)(L + s) \cosh \beta y}
\]

By inverse Laplace transform, we have
\[
\bar{u} = \frac{\sinh \sqrt{\beta} y}{\sqrt{\beta} \cosh \sqrt{\beta} y} + 2 \sum_{n=0}^{\infty} (-1)^n \sin(2n+1) \frac{\pi y}{2} \left[ \frac{\Delta_n^{(1)} e^{s_n^{(1)} t}}{s_n^{(1)} + L} + \frac{\Delta_n^{(2)} e^{s_n^{(2)} t}}{s_n^{(2)} + L} \right]
\]
when \( 0 < t \leq T \)
\[
\bar{v} = \frac{\sinh \sqrt{\beta} y}{\sqrt{\beta} \cosh \sqrt{\beta} y} + 2L \sum_{n=0}^{\infty} (-1)^n \sin(2n+1) \frac{\pi y}{2} \left[ \frac{\Delta_n^{(1)} e^{s_n^{(1)} t}}{s_n^{(1)} + L} + \frac{\Delta_n^{(2)} e^{s_n^{(2)} t}}{s_n^{(2)} + L} \right]
\]
and
\[
\bar{v} = 2L \sum_{n=0}^{\infty} (-1)^n \sin(2n+1) \frac{\pi y}{2} \left[ \frac{\Delta_n^{(1)} e^{s_n^{(1)} t}}{s_n^{(1)} + L} + \frac{\Delta_n^{(2)} e^{s_n^{(2)} t}}{s_n^{(2)} + L} \right]
\]
when \( 0 < t \leq T \)

**Discontinuity in the Velocity:**

We observe that when the shear stress is withdrawn there is a jump discontinuity of amount
\[
J_u = \frac{\sinh \sqrt{\beta} y}{\sqrt{\beta} \cosh \sqrt{\beta} y} + 2 \sum_{n=0}^{\infty} (-1)^n \sin(2n+1) \frac{\pi y}{2} \left( \Delta_n^{(1)} + \Delta_n^{(2)} \right)
\]
and
\[
J_v = \frac{\sinh \sqrt{\beta} y}{\sqrt{\beta} \cosh \sqrt{\beta} y} + 2L \sum_{n=0}^{\infty} (-1)^n \sin(2n+1) \frac{\pi y}{2} \left( \frac{\Delta_n^{(1)} e^{s_n^{(1)} t}}{s_n^{(1)} + L} + \frac{\Delta_n^{(2)} e^{s_n^{(2)} t}}{s_n^{(2)} + L} \right)
\]
in the velocity of the fluid and the dust particles respectively.
Shear Stress:

The shear stress at the fixed base (as in case I) is

\[
\tau_S = \text{sech} \sqrt{G} + \pi \sum_{n=0}^{\infty} (-1)^n (2n + 1) \left[ \Delta_n^{(1)} \left( 1 + E \epsilon_{s_n^{(1)}} \right) e^{\epsilon_{s_n^{(1)}} t} + \Delta_n^{(2)} \left( 1 + E \epsilon_{s_n^{(2)}} \right) e^{\epsilon_{s_n^{(2)}} t} \right]
\]

when \( 0 < t \leq T \) \hspace{1cm} (5.9)

\[
= \pi \sum_{n=0}^{\infty} (-1)^n (2n + 1) \left[ \Delta_n^{(1)} \left( 1 + E \epsilon_{s_n^{(1)}} \right) e^{\epsilon_{s_n^{(1)}} t} + \Delta_n^{(2)} \left( 1 + E \epsilon_{s_n^{(2)}} \right) e^{\epsilon_{s_n^{(2)}} t} \right]
\]

when \( t > T \) \hspace{1cm} (5.10)

Discontinuity in the shear stress:

When the surface traction is withdrawn, at time \( t = T \) there is a jump discontinuity of amount

\[
J = \text{sech} \sqrt{G} + \pi \sum_{n=0}^{\infty} (-1)^n (2n + 1) \left[ \Delta_n^{(1)} \left( 1 + E \epsilon_{s_n^{(1)}} \right) e^{\epsilon_{s_n^{(1)}} t} + \Delta_n^{(2)} \left( 1 + E \epsilon_{s_n^{(2)}} \right) e^{\epsilon_{s_n^{(2)}} t} \right]
\]

(5.11)

Table - 3.1

Numerical Values of \( u \) and \( v \) for various values of \( y \) and \( f \) when \( t = 0.1, M = 1, K = 0.1, E = 0.1, L = 6.67, \omega = 2.0 \) and \( m = 1.0 \).

<table>
<thead>
<tr>
<th>( y )</th>
<th>( u ) (( f = 0.02 ))</th>
<th>( u ) (( f = 0.1 ))</th>
<th>( u ) (( f = 0.2 ))</th>
<th>( v ) (( f = 0.02 ))</th>
<th>( v ) (( f = 0.1 ))</th>
<th>( v ) (( f = 0.2 ))</th>
</tr>
</thead>
<tbody>
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<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
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</tr>
<tr>
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<td>0.0041</td>
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<td>0.0022</td>
<td>0.0022</td>
<td>0.0023</td>
</tr>
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</tr>
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<td>0.0163</td>
<td>0.0165</td>
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</tr>
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<td>0.0618</td>
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</tbody>
</table>
Table – 3.2

Numerical Values of \( u \) and \( v \) for various values of \( y \) and \( L \) when \( t = 0.1, M = 1, K = 0.1, E = 0.1, f = 0.1, \omega = 2.0 \) and \( m = 1.0 \).

<table>
<thead>
<tr>
<th>( y )</th>
<th>( u ) (( L = 6.67 ))</th>
<th>( u ) (( L = 50 ))</th>
<th>( u ) (( L = 100 ))</th>
<th>( v ) (( L = 6.67 ))</th>
<th>( v ) (( L = 50 ))</th>
<th>( v ) (( L = 100 ))</th>
</tr>
</thead>
<tbody>
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<td>0.0000</td>
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</tr>
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<td>0.0266</td>
</tr>
<tr>
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<td>0.0407</td>
<td>0.0406</td>
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<td>0.0921</td>
<td>0.1415</td>
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</tr>
</tbody>
</table>

Discussion:

The numerical results for case (III) are given with the help of graphs and tables. From Fig. 3.1 and Fig. 3.2 it is clear that the velocity of the fluid is greater than that of the dust particles for prescribed values of the different parameters and the velocity of both, the fluid and the dust particles increase with the increase of \( y \). From Fig. 3.1 we observe that as the intensity of the magnetic field increases the velocity of the fluid decreases. It is also observed from Fig. 3.1 that the magnetic field does not effect the velocity of the dust up to a certain value of \( y \) whereafter the velocity of the dust decreases with the increase of the magnetic field. Fig. 3.2 shows that the velocity of both, the fluid and the dust particles increase with the increase of elastic parameter \( E \). This takes place up to a certain value of \( y \) whereafter the opposite happens i.e. the velocity profile for a lower value of \( E \) crosses that for a higher value of \( E \). From Fig. 3.3 it is seen that the velocity of both, the fluid and the dust particles are periodic w. r. t. time. From Fig. 3.3 we see that as \( K \) increases the amplitude of the velocity of both, the fluid and the dust particles decrease. Table 3.1 depicts that as \( f \) (mass
concentration) increases the velocity of the fluid decreases. The opposite happens for the dust particles except near to the free surface where shear stress is applied. Near the free surface the velocity of the dust particles decreases with the increase of \( f \), the mass concentration of dust. From table 3.2 it is observed that as \( L \) (reciprocal of time relaxation of dust) increases the velocity of the fluid decreases. The opposite happens for the dust particles. This implies that for coarse or fine particles the velocity of the fluid increases or decreases respectively and for coarse or fine particles the velocity of the dust particles decreases or increases respectively.

From Fig. 3.4 and Fig. 3.5, we see that the shear stress at the fixed base is periodic w. r. t. time and the amplitude of the shear stress decreases as the time goes on increasing. Fig. 3.4 narrates that as \( G \) increases the amplitude of the shear stress decreases. So we can conclude – (i) For fixed \( K \), the permeability of the porous medium the amplitude of the shear stress decreases with the increase of \( M \), the Hartmann number. (ii) For fixed \( M \) the amplitude of the shear stress decreases with the decrease of \( K \). The graph 3.5 reveals that the amplitude of the shear stress increases with the increase of \( E \), the elastic parameter and slightly decreases with the increase of \( f \), the mass concentration.
Fig. 3.1: Velocity profiles for various values of M with fixed t, E, K, f, L, m and ω.
Fig. 3.2: Velocity profiles for various values of $E$ with fixed $t$, $M$, $K$, $f$, $L$, $m$ and $\omega$. 

<table>
<thead>
<tr>
<th>$t$</th>
<th>$M$</th>
<th>$K$</th>
<th>$f$</th>
<th>$L$</th>
<th>$\omega$</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
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<td>0.1</td>
<td>0.1</td>
<td>6.67</td>
<td>0.1</td>
<td>1</td>
</tr>
</tbody>
</table>
Fig. 3.3: Velocity profiles for various values of $K$ with fixed $y$, $M$, $K$, $f$, $L$, $m$ and $\omega$. 

<table>
<thead>
<tr>
<th>$y$</th>
<th>$M$</th>
<th>$E$</th>
<th>$m$</th>
<th>$f$</th>
<th>$L$</th>
<th>$\omega$</th>
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</thead>
<tbody>
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<td>1</td>
<td>0.1</td>
<td>6.67</td>
<td>6</td>
</tr>
</tbody>
</table>
Fig. 3.4: Shear Stress at the fixed base against $t$ for various values of $G$ with fixed $E$, $f$, $L$, $m$ and $\omega$. 

<table>
<thead>
<tr>
<th>$E$</th>
<th>$f$</th>
<th>$L$</th>
<th>$\omega$</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$0.1$</td>
<td>$6.67$</td>
<td>$6$</td>
<td>$1$</td>
</tr>
</tbody>
</table>
Fig. 3.5: Shear Stress at the fixed base against \( t \) for various values of \( E \) and \( f \) with fixed \( M, K, L, m \) and \( \omega \).