Chapter –1

PRELIMINARIES AND PREREQUISITES

In this chapter ,we shall discuss some preliminaries and prerequisites of category theory , semirings, topological groups, Homotopy, semialgebra, fundamental group, covering spaces and manifolds that will give subsequent development.

1.1 Categories:

A category may be thought roughly as consists of sets, possibly with additional structures and functions, possibly preserving additional structures. More precisely, category can be defined with the following characteristics.

Definition 1.1.1

A category $\mathscr{C}$ consists of

a) a class of objects X, Y, Z,...... denoted by ob($\mathscr{C}$);
b) for each ordered pair of objects X,Y a set morphisms with domain X and rang Y denoted by $\mathscr{C}(X,Y)$;
c) for each ordered triple of objects X,Y and Z and a pair of morphisms $f : X \to Y$ and $g : Y \to Z$, their composite is denoted by $g \circ f : X \to Z$, satisfying the following two axioms:

i) associativity : if $f \in \mathscr{C}(X,Y)$, $g \in \mathscr{C}(Y,Z)$ and $h \in \mathscr{C}(Z,W)$, then $h \circ (g \circ f) = (h \circ g) \circ f \in \mathscr{C}(X,W)$;

ii) identity: for each objects Y in $\mathscr{C}$, there is a morphism $I_Y \in \mathscr{C}(X,Y)$ such that if $f \in \mathscr{C}(X,Y)$, then $I_Y \circ f = f$ and if $h \in \mathscr{C}(Y, Z)$, then $h \circ I_Y = h$.

Definition 1.1.2:

A subcategory $\mathcal{C}' \subset \mathcal{C}$ is a category such that

a) the objects of $\mathcal{C}'$ are also objects of $\mathcal{C}$;

b) for objects X′ and Y′ of $\mathcal{C}'$, $\mathcal{C}'(X',Y') \subseteq \mathcal{C}(X,Y)$;

c) if $f':X'\to Y'$ and $g':Y'\to Z'$ are morphisms of $\mathcal{C}'$,their composite in $\mathcal{C}'$ equals their composite in $\mathcal{C}$.

1.2 Functors:

Our main interest in categories is in the maps from one category to another. Those maps which have the natural properties of preserving identities and composites are called functors.

An algebraic representation of topology is a mapping from topology to algebra. Such a representation, formally called a functor, converts a toplogical problem into an algebraic one.

Definiton 1.2.1:

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A covariant functor (or contravariant functor) T from $\mathcal{C}$ to $\mathcal{D}$ consists of

i) an object function which assigns to every object X of $\mathcal{C}$ an object T(X) of $\mathcal{D}$; and

ii) a morphism function which assigns to every morphism $f:X\to Y$ in $\mathcal{C}$, a morphism $T(f):T(X)\to T(Y)$ (or $T(f):T(Y)\to T(X)$) in $\mathcal{D}$ such that

   a) $T(I_X) = I_{T(X)}$

   b) $T(g \circ f) = T(g) \circ T(f)$

   (or $T(g \circ f) = T(f) \circ T(g)$) for $g: Y\to W$ in $\mathcal{C}$.

1.3 **Natural transformations**:

In some occasions we have to compare functors with each other. We do this by mean of suitable maps between functors.

Definition 1.3.1

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Suppose T_1 and T_2 are functors of the same variance(either both covariant or both contravariant)from $\mathcal{C}$ to $\mathcal{D}$. A natural transformation Ø from T_1 to T_2 is a function from the objects of $\mathcal{C}$ to morphisms of $\mathcal{D}$ such that for every morphism $f: X \rightarrow Y$ in $\mathcal{C}$ the appropriate one of the following conditions holds: $Ø(Y)T_1(f) = T_2(f)\ Ø(X)$ (when T_1 and T_2 are both covariant functors) i.e., the diagram

$$
\begin{array}{ccc}
T_1(X) & \xrightarrow{Ø(X)} & T(X) \\
{\scriptstyle T_1(f)}\downarrow & & \downarrow{\scriptstyle T_2(f)} \\
T_1(Y) & \xrightarrow{Ø(Y)} & T_2(Y)
\end{array}
$$

is commutative.

Or $Ø(X)\ T_1(f) = T_2(f)\ Ø(Y)$ (when T_1 and T_2 are both contravariant functors) i.e., the following diagram

$$
\begin{array}{ccc}
T_1(X) & \xrightarrow{Ø(X)} & T_2(X) \\
{\scriptstyle T_1(f)}\uparrow & & \uparrow{\scriptstyle T_2(f)} \\
T_1(Y) & \xrightarrow[Ø(Y)]{} & T_2(Y)
\end{array}
$$

is commutative.

1.4 Yoneda's Lemma:

Let $\mathcal{C}$ be any category and T be a contravariant functor from $\mathcal{C}$ to $\mathcal{S}$ (category of sets and functions). Then for every objects C in $\mathcal{C}$, there is an equivalence

$\theta_{C,T}$: $(h^C, T) \rightarrow T(C)$, where (h^C, T) is the set of natural transformations from the set of valued functor h^C to the set valued functor T such that $\theta_{C,T}$ is natural in C and T.

1.5 **Semirings**:

Definition 1.5.1

A semiring S is defined as an algebra $(S,+,\cdot)$ such that $(S,+)$ and $(S,\cdot)$ are semigroups, connected by

$a\cdot(b + c) = a\cdot b + a\cdot c$ and

$(b + c)\cdot a = b\cdot a + c\cdot a, \forall a, b, c \in S$.

A semiring S is said to be commutative if both $(S,+)$ and $(S,\cdot)$ are commutative semigroups.

A semirings S is said to be cancellative if $(S,+)$ and $(S,\cdot)$ satisfy both left and right cancellation laws.

If N is the set of all non-negative integers, then $(N,+,\cdot)$ is a commutative, cancellative semiring under usual addition and multiplication of integers.

Definition 1.5.2

An equivalence relation ρ on a semiring S is called a congruence relation if $(a, b)\in\rho$ and

$c\in S \Rightarrow (a + c, b + c) \in\rho$, $(ac, bc) \in\rho$ and $(ca, cb) \in\rho$.

A subset $A(\neq\emptyset)$ of a semiring S is called an ideal of S if $a + b \in A$, $sa\in A$ and $as\in A$ hold $\forall a, b\in A$ and $\forall s\in S$. An ideal A of S is called proper if $A \subset S$, where $\subset$ denotes proper inclusion. A k-ideal A of S is an ideal such that for $a\in A$ and $x\in S$, if $a + x\in A$, then $x\in A$.

Each ideal A of a semiring S defines a congruence relation ρ_A on S , called the Bourne congruence given by
$\rho_A = \{(x, y) \in S\times S : x + a_1 = y + a_2, \text{ for some } a_1, a_2 \in A\}$.
The corresponding congruence class semiring S/ρ_A consisting of classes $x\rho_A$ is also denote by S/A. A homomorphism f from a semiring S to a semiring T is a function $f : S \to T$ of the underlying sets such that
$f(a + b) = f(a) + f(b)$, $f(a \cdot b) = f(a) \cdot f(b)$ and $f(0) = 0'$,
$\forall$ a, b$\in$S. A homomorphism f is said to be an isomorphism (resp.epimorphism) if f is bijective (respectively surjective). The homomorphism $f : S \to S/A$ defined by $f(s) = s\rho_A$ is called the natural homomorphism of S onto S/A. It is clearly an epimorphism.

Definition 1.5.3

Let R be a commutative ring with identity $1(\neq 0)$. A proper ideal P of R is said to be a prime ideal iff for all a, b$\in$R, ab$\in$P $\Rightarrow$ a$\in$P or b$\in$P.

We need the following theorem:

Theorem 1.5.1

Let $(R,+,\cdot)$ and $(S,+,\cdot)$ be rings and $f : R \to S$ be a ring homomorphism. Then

i) $K = \ker f$ is an ideal of R ;
ii) the quotient ring $(R/K,+,\cdot)$ is isomorphic to the image f(R) under the mapping
$\tilde{f} : R/K \to S$ given by $\tilde{f}(r + K) = f(r)$

1.6 Topological groups:

The aim of this section is to establish certain elementary properties of topological groups which we need in order to develop and illustrate the material of **chapter 5.**

Definition 1.6.1
A pointed topological space is a nonempty topological space with a distinguished element.

Definition 1.6.2
A pointed topological group is a group G whose underlying set is equipped with a topology such that:

i) the multiplication map $\mu : G \times G \to G$, given by $(x, y) \mapsto xy$ is continuous if $G\times G$ has the product topology ;

ii) the inversion map $i : G\to G$, given by $x\mapsto x^{-1}$ is continuous.

Then (G, e) is a pointed topological space, where e is the identity element.

Definition 1.6.3

Let G be a topological group and let X be a set (topological space). Then G acts on X if there is a continuous function

$G \times X\to X$, denoted by, $(g, x) \mapsto g \circ x$

such that $(g \circ g')(x) = g \circ (g' \circ x)$ and $1 \circ x = x, \forall x \in X$ and $g, g' \in G$ (1 is the identity element of G). We call X a G-set if G acts transitively on X if for each $x, x'\in X$, there exists $g \in G$ with $g \circ x = x'$.

Let a group G act on a set X . For $g \in G$, the function on X defined by $x\mapsto g \circ x$ is a permutation on X; moreover, if G acts on a topological space X, then $x\mapsto g \circ y$ is a homeomorphism .

Definition 1.6.4
Let a group G act on set X, and let $x\in X$. Then the orbit of x is $O(x) = \{ g \circ x : g\in G \} \subset X$, and the stabilizer of x (also called the isotropy subgroup of x) is

$G_x = \{ g\in G : g \circ x = x \} \subset G$.

Definition 1.6.5

Let G be group and let Y and Z be G-sets. A function $\emptyset : Y \to Z$ is a G-map if $\emptyset(g \circ y) = g\, \emptyset(y)$, $\forall\, g \in G$ and $y \in Y$. A G-isomorphism is a G-map that is also a bijection.

1.7 **Homotopy** :

Definition 1.7.1

Let $A \subset X$ and let $f_0, f_1 : X \to Y$ be base point preserving continuous maps with $f_0 | A = f_1 | A$. We write $f_0 \simeq f_1$ rel.A , if there is a continuous map $F : X \times I \to Y$ with $F : f_0 \simeq f_1$ and $F(a, t) = f_0(a) = f_1(a), \forall\, a \in A$ and all $t \in I$. Such a map F is called a homotopy relative to A from f_0 to f_1 and is denoted by $F : f_0 \simeq f_1$ rel. A.

Definition 1.7.2

If $f : X \to Y$ is base point preserving continuous maps , its homotopy class is the equivalence class

$[f] = \{g \in C(X,Y) : f \simeq g\}$, where $C(X,Y)$ denotes the set of all base point preserving continuous maps from X to Y. The family of all such homotopy classes is denoted by [X;Y].

Definition 1.7.3

A base point preserving continuous map $f : X \to Y$ is a homotopy equivalence if there is a base point preserving continuous map

$g : Y \to X$ with $g \circ f \simeq I_X$ and $f \circ g \simeq I_Y$.

Two spaces X and Y have the same homotopy type denoted by $X \approx Y$ if there is a homotopy equivalence $f : X \to Y$.

Lemma 1.7.1

Homotopy is an equivalence relation on the set $C(X, Y)$ of all base point preserving continuous maps from X to Y.

Lemma 1.7.2

Let $f_i : X \to Y$ and $g_i : Y \to Z$, for $i = 0, 1$, be continuous. If $f_0 \simeq f_1$ and $g_0 \simeq g_1$ then $g_0 \circ f_0 \simeq g_1 \circ f_1$; that is $[g_0 \circ f_0] = [g_1 \circ f_1]$.

1.8 Semialgebra:

Let X be a compact Hausdorff space and let C (X) be the set of all real valued continuous functions from X to $\mathbb{R}$ i.e.,

$$C(X) = \{ f : X \to \mathbb{R}, f \text{ is continuous} \}.$$

Addition '+' and multiplication '·' on C(X) are defined by the formulas :

$$(f + g)(x) = f(x) + g(x) \text{ and } (f \cdot g)(x) = f(x) \cdot g(x),$$

$$\forall x \in X \ldots\ldots(1).$$

Where '+' and '· denote usual addition and multiplication on $\mathbb{R}$. From this definition it follows that $(C(X), +, \cdot)$ is a ring.

Definition 1.8.1

A non-empty subset A of C(X) is said to be a semialgebra if it satisfies the following conditions:

i) if $f, g \in A \Rightarrow f + g \in A$ and if $r \in \mathbb{R}^+$, $f \in A$ then $r \cdot f \in A$; where $\mathbb{R}^+$ be the set of all real numbers ;

ii) $f, -f \in A \Rightarrow f = 0$;

iii) $f, g \in A \Rightarrow f \cdot g \in A$.

Where $f + g$ and $f \cdot g$ are defined in (i) but the scalar multiplication $\mathbb{R}^+ \times A \to A, (r, f) \mapsto r \cdot f$

where $r \cdot f$ is defined by $(r \cdot f)(x) = r \cdot f(x), \forall\, x \in X$.

Since $(C(X), +, \cdot)$ is a ring and distributive property is hereditary and hence $(A, +, \cdot)$ is a semiring.

Definition 1.8.2

A non-empty subset A of C(X) is said to be unital if it contains the constant function, $c : A \to r_0 \; (\in \mathbb{R}^+)$

Definition 1.8.3

A non- empty subset A of C(X) is said to be unital semialgebra if

i) A is semialgebra and

ii) A is unital

Given a subset A of C(X) define a relation $\tilde{\rho}_A$ on X by the rule :

$x \tilde{\rho}_A y \Leftrightarrow f(x) = f(y)$, $\forall$ $f \in A$ and $\forall x, y \in X$. Clearly $\tilde{\rho}_A$ is reflexive and symmetric .

Next suppose that $x \tilde{\rho}_A y$ and $y \tilde{\rho}_A z \Leftrightarrow f(x) = f(y)$ and $f(y) = f(z)$

$$\Leftrightarrow f(x) = f(z)$$

$$\Leftrightarrow x \tilde{\rho}_A z$$

Thus $\tilde{\rho}_A$ is transitive . Hence $\tilde{\rho}_A$ is an equivalence relation.

The Stone-Weierstrass theorem tells us that if A is a closed unital subalgebra of C(X) , then $A = C(X, \tilde{\rho}_A)$, where $C(X, \tilde{\rho}_A) = \{f \in C(X) : x \tilde{\rho}_A y \Rightarrow f(x) = f(y)$

Let X be a compact real interval, then we have a sequence $\{ \mathcal{C}_n \}$ of closed unital semialgebra in C(X).
Here $\mathcal{C}_0 = C^+(X)$,the set of all non-negative valid functions in C(X) ;
$\mathcal{C}_1 = C^+(X, \rho_1)$, the set of all increasing functions in $\mathcal{C}_0$; $\mathcal{C}_2$ is the set of all convex functions in $\mathcal{C}_1$; $\mathcal{C}_3$ is the set of all functions in $\mathcal{C}_2$ with non-negative third differences ; and so on .

This shows that this sequence has the following property :

$f \in \mathcal{C}_n \Rightarrow f^n(c + f)^{-1} \in \mathcal{C}_n$, but does not imply
$f^{n-1}(c + f)^{-1} \in \mathcal{C}_n$.

Thus we have the following definition.

Definition 1.8.4

A semialgebra A is of type 0 if f , c $\in$ A $\Rightarrow (c+f)^{-1} \in$ A and for n in $\mathbb{R}$, A is of type n if f ,c $\in$ A $\Rightarrow f^n(c+f)^{-1} \in$ A ,
where $(1/(c+f))(x) = 1/(c(x) + f(x)) = 1/(r_0 + f(x))$,
$\forall\, x \in X$ and $(c+f)(x) = c(x) + f(x) = r_0 + f(x) \neq 0$,$\forall\, x \in X$.

From this definition, we have if A is of type 0 , then A is of type 1 and if a semialgebra A is of type n ($n \in \mathbb{Z}^+$), then A is of type n+1.

Definition 1.8.5

A relation of quasiorder on X is a binary relation ρ_1 on X such that :

i) $x \rho_1 y$ and $y \rho_1 z \Rightarrow x \rho_1 z$.
ii) $x \rho_1 x$ ($x \in X$) .

A quasiorder becomes a partial order if also

iii) $x \rho_1 y$ and $y \rho_1 z \Rightarrow x = y$.

Definition 1.8.6

A subset A of C(X) is said to be admit Δ if $f \Delta g \in A$, whenever f and g belongs to A and is said to be admit ∇ if $f \nabla g$ belongs to A,

where $(f \Delta g)(x) = \max \{ f(x), g(x)\}$ and
$(f \nabla g)(x) = \min \{f(x), g(x)\}, \forall x \in X$ and $f, g \in C(X)$

1.9 Fundamental group :

Definition 1.9.1

If $f : I \to X$ is a path from x_0 to x_1, call x_0 the origin of f and write $x_0 = \alpha(f)$; call x_1 the end of f and write $x_1 = w(f)$. A path f in X is closed at x_0 if $\alpha(f) = x_0 = w(f)$.

Definition 1.9.2

Fix a point $x_0 \in X$ and call it the base point . The fundamental group with base point x_0 is

$\pi_1 (X, x_0) = \{ [f] : [f]$ is a path class in X with $\alpha([f]) = x_0 = w([f])\}$ with binary operation

$$[f] * [g] = [f \circ g].$$

1.10 Covering Spaces

Definiton 1.10.1

A covering space of a space (X, x_0) is a triple $(\tilde{X}, \tilde{x}, p)$ consisting of a pointed space $(\tilde{X}, \tilde{x}_0)$ and a continuous surjective map $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ such that the following condition is satisfied : each point $x \in X$ has an arcwise connected open neighbourhood U such that each arc comoponent $p^{-1}(U)$ is mapped homeomorphically onto U by p.

Definition 1.10.2

Let $(\tilde{X}_1, \tilde{x}_1, p_1)$ and $(\tilde{X}_2, \tilde{x}_2, p_2)$ be two covering spaces of (X, x_0). A homomorphism of $(\tilde{X}_1, \tilde{x}_1, p_1)$ into $(\tilde{X}_2, \tilde{x}_2, p_2)$ is a base point preserving continuous map $f : (\tilde{X}_1, \tilde{x}_1) \rightarrow (\tilde{X}_2, \tilde{x}_2)$ such that $p_2 \circ f = p_1$. If in pariticular, f is a homeomorphism, then the coverings $(\tilde{X}_1, \tilde{x}_1, p_1)$ and $(\tilde{X}_2, \tilde{x}_2, p_2)$ are said to be isomorphic.

Let $\mathscr{C}(X)$ denote the set of all covering spaces $(\tilde{X}, \tilde{x}, p)$ of (X, x_0). Then for each $(\tilde{X}, \tilde{x}_0, p) \in \mathscr{C}(X)$, the continuous map $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ induces a monomorphism $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$ in the corresponding fundamental groups [1.9.2]. The image group $H = p_* \, \pi_1(\tilde{X}, \tilde{x}_0)$ depends on the choice of base points $\tilde{x}_0 \in p^{-1}(x_0)$.

Defination 1.10.3

Let (X, x_0) be the pointed space, let $(\tilde{X}, \tilde{x}, p)$ be a covering space of (X, x_0) and $Y = p^{-1}(x_0)$. Let $\theta : \pi_1(X, x_0) \rightarrow S_y$ (where S_y is the symmetric group on Y) be the homomorphism corresponding to the action of $\pi_1(X, x_0)$ on the fibre, namely, $\theta([f]) : \tilde{x} \mapsto \tilde{x}([f])$.

Then the quotient $\pi_1(X, x_0) / \ker\theta$ is called the monodromy group of $(\tilde{X}, \tilde{x}, p)$, where $\ker\theta = \bigcap_{\tilde{x} \in Y} p_* \pi_1(\tilde{X}, \tilde{x})$.

1.11 Manifolds

Definition 1.11.1

An n-manifold is a Hausdorff space M such that each point in M has a neighbourhood homeomorphic to $\mathbb{R}^n$.

Definition 1.11.2

For any two immersions or maps $f : M \rightarrow E^m$ and $g : M \rightarrow E^{m'}$ of an n-dimensional manifold M, we define the tensor product

$(f \otimes g) : M \rightarrow E^m \otimes E^{m'} = E^{mm'}$ of f and g by
$(f \otimes g)(p) = f(p) \otimes g(p) \in E^m \otimes E^{m'}$ for any $p \in M$

Definition 1.11.4

Two immersions $f : M \rightarrow E^m$ and $g : M \rightarrow E^{m'}$ of an n-dimentional manifold M are said to be equivalent if there exists a Euclidean space E^N containing E^m and $E^{m'}$ as linear subspaces and if there exists an isometry $\varnothing$ on E^N which carries f into g. We identify two equivalent immersions f and g of M; and denote it by $f = g$.

Let M and M′ be two differentiable manifolds and $f : M \rightarrow M'$ is a C^1-mapping. Also let $T_p(M)$ denote the tangent space at the point p of the manifolds M.

Then the differentiable f_* and f at a point $p \in M$ is the linear map $f_* : T_p(M) \rightarrow T_{f(p)}(M')$ satisfying the following condition :

for each $X \in T_p(M)$, choose a curve $x(t)$ in M such that X is the vector tangent to $x(t)$ at $p = x(t_0)$, then $f_*(x)$ is the vector tangent to the curve $f(x(t))$ at $f(p) = f(x(t_0))$.

We call f on immersion if $(f_*)p$ is injective for every point p of M.